*The deRham Theorem for Function Spaces*¹

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1. Introduction. The classical theorem of deRham states that there is a degree preserving isomorphism from the real singular cohomology of a paracompact smooth (C^∞) manifold onto the cohomology of smooth forms via the integral.

In this note we prove that this theorem is valid for the group of diffeomorphisms of a compact smooth manifold and for the space of smooth maps from a compact smooth manifold into a paracompact smooth manifold with C^∞ -topologies. The essential difficulty is the definition of differential forms.

One approach to such a theorem is given by Eells in [1]. Such a method is not feasible in our case since the function spaces we are concerned with are locally homeomorphic to Montel spaces and hence cannot be normed.

Thus in order to avoid the theory of differentiation in locally convex topological vector spaces we have been inspired by an idea of J. W. Smith [5]. The point of departure here is a new form of differentiability.

To provide some continuity we briefly summarize [2] and [5]. Let $\mathfrak{M}$ be the category of smooth manifolds and smooth maps. The smooth maps of U into V are denoted by $C^\infty(U, V)$ for $U, V \in \mathfrak{M}$; the continuous maps of X into Y are denoted by $C(X, Y)$ for X, Y topological spaces.

Let X be a topological space and $\mathfrak{F}$ a subsheaf of $C(X, \mathbf{R})$. For any manifold U and $\alpha \in C(U, X)$, α is called differentiable (of class C^∞) iff $\alpha^*\mathfrak{F} \subset C^\infty(U, \mathbf{R})$ where α^* is the family of functions of the form $\alpha^*f = f \circ \alpha$, $f \in \mathfrak{F}$. In a similar manner for any $V \in \mathfrak{M}$, $\beta \in C(X, V)$ is called differentiable iff $\beta^*(C^\infty(V, \mathbf{R})) \subset \mathfrak{F}$. If $\mathfrak{F}$ is to consist of differentiable functions one needs the following condition: If for all differentiable $\alpha \in C(U, X)$, $\beta \circ \alpha \in C^\infty(U, V)$ where $\beta \in C(X, V)$ then β is differentiable. A subsheaf $\mathfrak{F} \subset C(X, \mathbf{R})$ satisfying the condition is called an inductive differentiable structure or simply a differentiable structure.

Remarks. 1. If $\mathfrak{W}$ is a functor which makes correspond to any manifold U , $\mathfrak{W}(U, X) \subset C(U, X)$, and to any $f \in C^\infty(U', U)$, $f_*: \mathfrak{W}(U, X) \rightarrow \mathfrak{W}(U', X)$ where $f_*\alpha = f\alpha$, $\alpha \in \mathfrak{W}(U, X)$ then there is a uniquely determined $\mathfrak{F}$ such that

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every $\alpha \in \mathfrak{W}(U, X)$ is differentiable. The family $\mathfrak{F}$ is defined by requiring that $\beta \in \mathcal{C}(X, V)$, V a manifold, is differentiable iff $\beta \circ \alpha \in C^\infty(U, V)$ for all $\alpha \in \mathfrak{W}(U, X)$.

2. This construction allows us to define differential forms. For α a differential k -cube, $\alpha: I^k \rightarrow X$, $f_0 df_1 \wedge \cdots \wedge df_k$ is defined by:

$$\langle \sigma, f_0 df_1 \wedge \cdots \wedge df_k \rangle = \int_{I^k} \tilde{f}_0 d\tilde{f}_1 \wedge \cdots \wedge d\tilde{f}_k$$

where $\tilde{f}_i = f_i \circ \alpha$, $f_i \in \mathfrak{F}$.

The main theorem of [5] states that if X is locally differentiably contractible and the sheaf of germs of functions from $\mathfrak{F}$ is soft then deRham's theorem is valid.

2. Extending differentiable functions. Let X, Y be smooth manifolds, let $\text{Diff}(X)$ be the group of smooth diffeomorphisms of X with the C^∞ -topology and let $\mathcal{C}(X, Y)$ be the space of smooth maps from X into Y with the C^∞ -topology. There are evaluation maps $\text{ev}: \text{Diff}(X) \times X \rightarrow X$ and $\text{ev}: \mathcal{C}(X, Y) \times X \rightarrow Y$ defined by $\text{ev}(f, x)$ which are continuous. For any $\alpha: U \times X \rightarrow X$ define $\hat{\alpha}(u) \cdot x = \alpha(u, x)$, where $u \in U$, $x \in X$. Now the differentiable maps of any manifold U into $\text{Diff}(X)$ are maps $\alpha \in C^\infty(U \times X, X)$ such that $\hat{\alpha}(U) \subset \text{Diff}(X)$. By Remark 1 of the introduction this definition generates an inductive differentiable structure on $\text{Diff}(X)$, denoted by $\mathcal{E}$, and called the evaluation structures. Note that the same construction may be carried out for $\mathcal{C}(X, Y)$.

We first show that the structure $\mathcal{E}$ on $\text{Diff}(X)$ is soft.

Lemma. *The triple $\text{ev}_x: \text{Diff}(X) \rightarrow X$ has local smooth cross sections.*

Proof. It suffices to show that there is a neighborhood U of x and a map $\phi: U \rightarrow \text{Diff}(X)$ such that $\text{ev}_x \phi = \text{id}_U$. Let (h, U) be a coordinate neighborhood of x such that $h(\bar{U})$ is the closed unit ball, B^n in $\mathbb{R}^n$ and $h(x) = 0$. Let T be the diffeomorphism of B^n defined by $T(y) \cdot p = q$ where q is determined by:

where pq is parallel to $Oh(y)$. Next define $\phi(y) = h^{-1} \circ T(y) \circ h$ for $y \in U$ and observe that $\phi(y) \mid \partial U$ is the identity and $\phi(y)$ easily extends to a diffeomorphism of X . Since $\phi(y) \cdot x = h^{-1}(T(y) \cdot h(x)) = y$ for $y \in U$, the map $y \rightarrow \phi(y)$ is a smooth cross section of $\text{Diff}(X)$.

Theorem 1. *The sheaf $\mathcal{E}$ on $\text{Diff}(X)$ is soft.*

Proof. It suffices to show that $\text{ev}_x^* C^\infty(X, \mathbf{R})$ is soft by Theorem 3.7.1 in [3]. In turn it is enough to show that $\text{ev}_x^* C^\infty(X, \mathbf{R})$ is soft for small neighborhoods by Theorem 3.7.2 in [3]. Let F be a neighborhood of id_x such that $\text{ev}_x(F) \subset V \subset \bar{V} \subset U$, where U is a trivializing neighborhood from the lemma and V is open. Now let H be closed in F and consider $y_n \rightarrow y$ where $y_n \in \text{ev}_x(H)$; thus $y_n = f_n(x)$ with $f_n \in H$ and $y = f(x) \in U$. Using the lemma there is a cross section ϕ on U and since $y_n \rightarrow y$ in U we have $\phi(y_n) \rightarrow \phi(y)$. As a result $f_n \rightarrow f(\phi(y_n) = f_n, \phi(y) = f)$ but H is closed so $f \in H$ and $y \in \text{ev}_x(H)$.

Theorem 2. *The sheaf $\mathcal{E}$ on $\mathcal{C}(X, Y)$ is soft.*

Proof. As in the lemma to Theorem 1 the triple $\text{ev}_x : \mathcal{C}(X, Y) \rightarrow Y$ has local cross sections in the sense that for any $y \in Y$ there is a neighborhood V of y and a map $\psi : V \rightarrow \mathcal{C}(X, Y)$ such that the following diagram commutes:

$$\begin{array}{ccc} & \mathcal{C}(X, Y) & \\ \psi \nearrow & \downarrow \text{ev}_x & \\ V & \longrightarrow & Y \\ \iota_v \end{array}$$

Choose the neighborhood V so that there is a map $\phi : V \rightarrow \text{Diff}(Y)$ such that

$$\begin{array}{ccc} & \text{Diff}(Y) & \\ \phi \nearrow & \downarrow \text{ev}_v & \\ V & \longrightarrow & Y \\ \iota_v \end{array}$$

Let $f \in \mathcal{C}(X, Y)$ such that $f(x) = y$ and define $\psi(z) = \phi(z) \cdot f$. Clearly $z \rightarrow \psi(z)$ is continuous and $\text{ev}_x \psi(z) = \phi(z) \cdot f(x) = \phi(z) \cdot y = z$ for all $z \in V$. The proof now proceeds as in Theorem 1.

Remark. Theorem 1 has the following technical implication: a closed subgroup of $\text{Diff}(X)$ is a differentiable subgroup, [2]. It is to be emphasized, that this is a faint echo of the fact that a closed subgroup of a Lie group is a Lie subgroup.

3. Local contractibility of $\text{Diff}(X)$, $\mathcal{C}(X, Y)$. Consider spaces X, Y with differentiable structures $\mathfrak{F}, \mathfrak{G}$. The product differentiable structure $\mathcal{H}$ on $X \times Y$ is obtained by requiring that for any manifold U , $h \in \mathcal{C}(U, X \times Y)$ is dif-

ferentiable iff $\pi_x \circ h$ and $\pi_y \circ h$ are differentiable. It is easily seen that the projection maps $\pi_x : X \times Y \rightarrow X$, $\pi_y : X \times Y \rightarrow Y$ are differentiable [2].

Suppose now that $f_1, \dots, f_n \in \mathcal{F}$, $g_1, \dots, g_m \in \mathcal{G}$ and let $\pi_x \circ f_i = f'_i$, $\pi_y \circ g_i = g'_i$. Clearly $f'_i, g'_i \in \mathcal{H}$ and for any $\phi \in C^\infty(\mathbf{R}^{n+m}, \mathbf{R})$:

$$(*) \quad h = \phi(f'_1, \dots, f'_n, g'_1, \dots, g'_m)$$

is in $\mathcal{H}$. The functions in $\mathcal{H}$ of the form $(*)$ are called proper by J. W. Smith [5] and a differentiable map $f: (X \times Y, \mathcal{H}) \rightarrow (W, \mathcal{G})$ is called proper iff the pullback of any map in $\mathcal{G}$ by f is proper.

Further a space $(X, \mathcal{F})$ is called differentially contractible if there is a proper map $h: \mathbf{I} \times X \rightarrow X$, where $\mathbf{I}$ is the closed unit interval with the usual differentiable structure such that $h(1, x) = x$ for all $x \in X$ and $h(0, x) = x_0$ for all $x \in X$.

In [2] it was shown that if X is compact then $\text{Diff}(X)$ is locally homeomorphic to its tangent spaces.

Theorem 3. *If X is compact then $\text{Diff}(X)$ is locally differentially contractible.*

Proof. It suffices to consider W a neighborhood of zero in $T_{\text{id}}(\text{Diff}(X)) = \Gamma(X)$, the space of smooth vector fields on X . In the neighborhood (U, A) of zero where U is a chart and A is open in TX consider $f: (U, A) \rightarrow \mathbf{R}$ a differentiable function. If e_i , $1 \leq i \leq \dim X$, is a local basis for $\Gamma(U)$ over $C^\infty(U, \mathbf{R})$ then for any $v \in \Gamma(U)$, $t \in [0, 1]$

$$f(tv) = f\left(\sum_i e_i tw^i\right) = \bar{f}(tw^1, \dots, tw^n)$$

where $\bar{f} \in C^\infty(\mathbf{R}^n, \mathbf{R})$. Thus $h(t, v) = tv$ is a proper map and $\text{Diff}(X)$ is a locally differentially contractible space.

Theorem 4. *The space $\mathcal{C}(X, Y)$ where X is compact is a locally differentially contractible space.*

Proof. Similar to that of Theorem 3.

Remarks.

1. That our cohomology agrees with that of Eells in the case $\mathcal{C}^\infty(X, Y)$ follows from the theorem on the invariance of homotopy type in Palais [4].

2. Any knowledge of an invariant integral on $\text{Diff}(X)$ would lead to applications of our method to gain information on the cohomology of $\text{Diff}(X)$.

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