On Weinstein's Intermediate Problems for Integral Equations with Difference Kernels

IVAR STAKGOLD*

Communicated by MURRAY ROSENBLATT

1. Introduction. The Ritz-Rayleigh procedure for estimating the eigenvalues of a compact, self-adjoint operator A leads to a finite-dimensional problem having, in general, some positive and negative eigenvalues which, with the usual ordering, provide lower bounds to the positive eigenvalues of A and upper bounds to the negative eigenvalues of A , respectively.

If A is definite, say positive, it seems natural to use Weinstein's method of intermediate problems (see [2], [4] for a detailed description) to obtain complementary bounds to the eigenvalues. The method relies on a base operator B , with known spectrum, whose eigenvalues are larger than those of A . If B is itself compact and self-adjoint, the eigenvalues of successive intermediate problems decrease monotonically to those of A . On the other hand, Fichera [3] has shown that a non-compact base operator B may fail to yield suitable intermediate problems. When A is a definite integral operator, Weinstein [10] has constructed, in a simple and elegant manner, a non-compact base operator suitable for the application of the method of intermediate problems. This base operator must be modified in an elementary way at each stage in the succession of intermediate problems.

The purpose of the present note is to construct a *compact* base operator whenever A is a compact, self-adjoint, integral operator generated by a difference kernel. The base operator remains fixed for the various intermediate problems. Moreover, the usual restriction of A being definite is unnecessary. The procedure yields bounds complementary to the Ritz-Rayleigh bounds; thus one obtains upper bounds to the positive eigenvalues and lower bounds to the negative eigenvalues. The base operator is constructed by extending the kernel to a larger interval over which it has an appropriate periodicity. The spectrum of the base operator can be calculated explicitly by Fourier expansion. The same kind of periodic extension has been used by Rosenblatt [5] in discussing the asymptotic behavior of eigenvalues. An application to the estimation of functionals associated with inhomogeneous equations was given in [6].

* Research supported under NSF grant GP6616.

2. Formulation of problem. Let $k(t)$ be an even, real function of the real variable t , $-a < t < a$, and assume that $k(t) \in L_2(0, a)$. It follows that $k(s - t)$ is symmetric on the square $0 < s < a$, $0 < t < a$, and that

$$\int_0^a k(s - t)u(t) dt$$

defines a self-adjoint Hilbert-Schmidt transformation from $L_2(0, a)$ into itself. We shall consider the eigenvalue problem

$$(1) \quad \int_0^a k(s - t)u(t) dt = \lambda u(s), \quad 0 < s < a.$$

There is no general method for solving (1) explicitly, but a Fourier expansion yields an explicit solution if $k(t)$ has period a , that is, if $k(t) = k(t - a)$ for $0 < t < a$. Under this periodicity assumption, the representation

$$k(t) = \sum_{n=0}^{\infty} k_n \cos 2\pi nt/a$$

is valid on $-a < t < a$. Setting

$$u(t) = \sum_{n=0}^{\infty} a_n \cos 2\pi nt/a + \sum_{n=1}^{\infty} b_n \sin 2\pi nt/a$$

and substituting in (1), we find that the eigenfunctions are

$$1, \cos 2\pi t/a, \sin 2\pi t/a, \cos 4\pi t/a, \sin 4\pi t/a, \dots$$

with corresponding eigenvalues

$$ak_0, ak_1/2, ak_1/2, ak_2/2, ak_2/2, \dots$$

In the general case, we extend the definition of the kernel to a larger interval $-b < t < b$, so that the extended kernel (still labeled k) has period b . Obviously this is always possible by choosing $b = 2a$. However, if the original kernel is the section of a function having a natural period larger than a , it will be preferable to set b equal to the natural period. (For instance, in a typical boundary value problem for the two-dimensional region exterior to the circular arc $0 < \theta < a$, $r = R$, the integral equation formulation leads to a kernel (defined on $-a < \theta < a$) which is the section of a function having period 2π .) Taking $H = L_2(0, b)$ as our fundamental Hilbert space, we define the self-adjoint Hilbert-Schmidt transformation of H into itself:

$$Bz = \int_0^b k(s - t)z(t) dt.$$

By the previous discussion, the eigenvalues and eigenfunctions of B are known.

We will now express (1) as an operator equation in H . Let $M \subset H$ be the subspace of functions vanishing identically on the interval (a, b) ; clearly $M^\perp$ is the subspace of functions vanishing on $(0, a)$. We denote by Q and P the pro-

jection operators on M and $M^\perp$, respectively. With $\lambda \neq 0$, any eigenpair $\{\lambda, u\}$ of

$$(2) \quad QBQu = \lambda u$$

satisfies equation (1), and any eigenpair of (1)—with u extended to be identically 0 in the interval (a, b) —satisfies (2). For our purposes, we can therefore study (2) instead of (1).

3. Poincaré extremal principle. Let A be a compact, self-adjoint operator on H . Let λ_k^+ , $k = 1, 2, \dots$, be the positive eigenvalues of A , listed in non-increasing order with due regard to multiplicity. If the sequence λ_k^+ consists only of a finite number of terms, we artificially turn it into an infinite sequence by adjoining zeros at its tail. With a similar convention, we let λ_k^- be the negative eigenvalues in nondecreasing order. Under this definition, both of the sequences λ_k^+ and λ_k^- are infinite.

The inner product and norm on H are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. The Rayleigh quotient for an element u is

$$R(u) = \frac{\langle Au, u \rangle}{\|u\|^2},$$

where it is always understood that $u \neq 0$.

Extremal principles of the “minimax” type are based on the Rayleigh quotient and use either trial elements orthogonal to a finite-dimensional subspace (Courant) or trial elements belonging to such a subspace (Poincaré). A thorough discussion of these principles can be found in Stenger [7, 8] (see also Diaz and Metcalf [1]).

Here we shall find the Poincaré principle more convenient. We include the proof for the indefinite case since it does not seem to be usually proved completely in the literature (Stenger considers a more general case where the lower part of the spectrum is discrete, but then only characterizes the eigenvalues in that part of the spectrum, whereas we wish to characterize all eigenvalues).

Theorem 1. *Let E_k be any k -dimensional subspace of H . Define*

$$\nu^+(E_k) = \min_{u \in E_k} R(u),$$

$$\nu^-(E_k) = \max_{u \in E_k} R(u).$$

Then

$$(3) \quad \lambda_k^+ = \sup_{E_k} \nu^+(E_k),$$

$$(4) \quad \lambda_k^- = \inf_{E_k} \nu^-(E_k).$$

Remark. Since E_k is finite-dimensional, the definitions of $\nu^+(E_k)$ and $\nu^-(E_k)$ always make sense. On the other hand, λ_k^+ can be characterized for all k as $\max$

instead of sup only if the union of the null space of A and the space spanned by all the eigenvectors corresponding to positive eigenvalues is infinite-dimensional. A similar remark holds for λ_k^- .

Proof of Theorem 1. a) Suppose $\lambda_k^+ > 0$ (that is, A has at least k positive eigenvalues). Let M_k^+ be the linear manifold spanned by k orthonormal eigenvectors corresponding to $\lambda_1^+, \dots, \lambda_k^+$. Then, for $u \in M_k^+$, it is easily seen that $R(u) \geq \lambda_k^+$. Moreover M_k^+ contains an eigenvector u_k^+ corresponding to λ_k^+ , and $R(u_k^+) = \lambda_k^+$. Therefore $\nu^+(M_k^+) = \lambda_k^+$ and

$$\sup_{E_k} \nu^+(E_k) \geq \lambda_k^+.$$

Since the arbitrary subspace E_k is of higher dimension than M_{k-1}^+ , it must contain a nonzero element v orthogonal to M_{k-1}^+ . Expanding v in the eigenvectors of A , we find $R(v) \leq \lambda_k^+$. Therefore, for any E_k , $\nu^+(E_k) \leq \lambda_k^+$ and

$$\sup_{E_k} \nu^+(E_k) \leq \lambda_k^+.$$

Together with the previous inequality, this proves (3).

b) Suppose $\lambda_k^+ = 0$ and $k \leq p$, where p is the dimension of the union of the null space and the space spanned by all the eigenvectors corresponding to positive eigenvalues. The proof is then nearly identical with that of part (a) the only difference being that M_k^+ is now chosen to contain also some elements of the null space of A .

c) Suppose $\lambda_k^+ = 0$ and $k > p$ (which implies that p is finite). Then 0 is a limit-point of negative eigenvalues of A . Let $\epsilon > 0$ be an arbitrary positive number and let M_k be the span of k orthonormal eigenvectors corresponding to the eigenvalues $\lambda_{j+1}^-, \dots, \lambda_{j+k}^-$, where j is chosen sufficiently large so that $R(u_{j+1}^-) \geq -\epsilon$, u_{j+1}^- being an eigenvector corresponding to λ_{j+1}^- . An easy calculation shows that $R(u) \geq -\epsilon$ for any $u \in M_k$. Hence $\nu^+(M_k) \geq -\epsilon$, and

$$\sup_{E_k} \nu^+(E_k) \geq -\epsilon.$$

By assumption, the dimension of the span of eigenvectors corresponding to positive eigenvalues does not exceed $k - 1$. Therefore any k -dimensional space E_k contains a nonzero element, say v , orthogonal to the span of eigenvectors corresponding to positive eigenvalues. Thus, $R(v) \leq 0$, $\nu^+(E_k) \leq 0$, and

$$\sup_{E_k} \nu^+(E_k) \leq 0.$$

In conjunction with the previous inequality and the fact that ϵ is arbitrary, (3) is proved.

d) Similar reasoning can be used to prove (4).

The following lemma has an elementary proof which is omitted.

Lemma 1. *Let T be a compact, self-adjoint operator on H and let P be a projection. Then PTP is a compact self-adjoint operator.*

Theorem 2. *Let T and P be as in Lemma 1. Let the eigenvalues of T be σ_k^+ , σ_k^- and those of PTP be λ_k^+ , λ_k^- . Then*

$$(5) \quad \lambda_k^+ \leq \sigma_k^+, \quad k = 1, 2, \dots,$$

$$(6) \quad \lambda_k^- \geq \sigma_k^-, \quad k = 1, 2, \dots.$$

Proof. We prove (5). The result is obviously true if $\lambda_k^+ = 0$; therefore we shall assume $\lambda_k^+ > 0$. Let u_k^+ be an eigenvector of PTP corresponding to λ_k^+ . Then, $u_k^+ = Pu_k^+$. Now, from (3) applied to T , we have, for any E_k ,

$$\sigma_k^+ \geq \min_{u \in E_k} \frac{\langle Tu, u \rangle}{\|u\|^2}.$$

Choose $E_k = N_k^+$, where N_k^+ is the span of k orthonormal eigenvectors of PTP corresponding to the eigenvalues λ_1^+ , $\dots$, λ_k^+ . If $u \in N_k^+$, then $Pu = u$, and

$$\begin{aligned} \sigma_k^+ &\geq \min_{u \in N_k^+} \frac{\langle Tu, u \rangle}{\|u\|^2} = \min_{u \in N_k^+} \frac{\langle TPu, Pu \rangle}{\|u\|^2} \\ &= \min_{u \in N_k^+} \frac{\langle PTPu, u \rangle}{\|u\|^2} = \lambda_k^+. \end{aligned}$$

The argument is easily modified to prove (6).

Theorem 3. (see Fichera [2]). *Let T be a compact operator and let $\{P_n\}$ be a sequence of projections such that*

$$\lim_{n \rightarrow \infty} \|P_n u - Pu\| = 0, \quad \text{for any } u \in H.$$

Then P is necessarily a projection and,

$$\lim_{n \rightarrow \infty} \|P_n T P_n - PTP\| = 0.$$

4. Weinstein's intermediate problems. Let B , P , Q , M be defined as in Section 2. The eigenvalue problem

$$Bu = \lambda u$$

is known as the *base problem*; its eigenvalues are denoted by $\lambda_{0,k}^+$, $\lambda_{0,k}^-$.

We are interested in obtaining upper bounds to the positive eigenvalues λ_k^+ of (2) and lower bounds to the negative eigenvalues λ_k^- of (2). We construct *intermediate problems* as follows: Let $\{v_1, \dots, v_n, \dots\}$ be a complete independent set in M , and let P_n be the projection operator on the span of $v_1, \dots, v_n$. For convenience we also define P_0 to be the zero operator. The n th intermediate problem is

$$(I - P_n)B(I - P_n)u = \lambda u,$$

where I is the identity operator. The eigenvalues of the n th intermediate problem are denoted by $\lambda_{n,k}^+$, $\lambda_{n,k}^-$. Note that the zeroth intermediate problem

is just the base problem. We introduce the additional terminology:

$$Q_n = I - P_n ,$$

$$A_n = Q_n B Q_n ,$$

$$A = B Q .$$

We now prove the principal theorem about the convergence of the eigenvalues of the intermediate problems.

Theorem 4. *As $n \rightarrow \infty$, $\lambda_{n,k}^+$ is a monotonically decreasing sequence with limit λ_k^+ and $\lambda_{n,k}^-$ is monotonically increasing to λ_k^- . The convergence is uniform in k .*

Proof. We have $P_{n+1}P_n = P_nP_{n+1} = P_n$, and $PP_n = P_nP = P_n$. Clearly Q_n is also a projection so that, by Lemma 1, $Q_n B Q_n$ is compact and self-adjoint. We easily check that $Q_{n+1}Q_n = Q_nQ_{n+1} = Q_{n+1}$, and $QQ_n = Q_nQ = Q$, so that

$$A_{n+1} = Q_{n+1}A_nQ_{n+1} ,$$

and

$$A = QA_{n+1}Q.$$

By Theorem 2, it follows that $\lambda_{n,k}^+ \geq \lambda_{n+1,k}^+ \geq \lambda_k^+$ and that $\lambda_{n,k}^- \leq \lambda_{n+1,k}^- \leq \lambda_k^-$ for $k = 1, 2, \dots$; $n = 0, 1, 2, \dots$. Therefore $\lambda_{n,k}^+$ and $\lambda_{n,k}^-$ converge as $n \rightarrow \infty$, and

$$(7) \quad \lim_{n \rightarrow \infty} \lambda_{n,k}^+ \geq \lambda_k^+ ,$$

$$(8) \quad \lim_{n \rightarrow \infty} \lambda_{n,k}^- \leq \lambda_k^- .$$

It remains to show that the equality sign holds in these last two formulas. Given $\epsilon > 0$ and a fixed n , Theorem 1 implies that there exists E_k such that

$$\lambda_{n,k}^+ \leq \min_{u \in E_k} \frac{\langle A_n u, u \rangle}{||u||^2} + \epsilon .$$

With E_k so chosen, we obtain by using Theorem 1 again

$$\lambda_k^+ \geq \min_{u \in E_k} \frac{\langle A u, u \rangle}{||u||^2} .$$

Now

$$\langle A_n u, u \rangle = \langle A u, u \rangle + \langle (A_n - A) u, u \rangle ,$$

and, by Schwarz's Inequality

$$|\langle (A_n - A) u, u \rangle| \leq ||A_n - A|| ||u||^2$$

so that

$$\frac{\langle A_n u, u \rangle}{||u||^2} \leq \frac{\langle A u, u \rangle}{||u||^2} + ||A_n - A|| .$$

Taking the minimum over E_k and using the previous inequalities, we find

$$\lambda_{n,k}^+ \leq \lambda_n^+ + \|A_n - A\| + \epsilon.$$

Since $\|Q_n u - Qu\| \rightarrow 0$, Theorem 3 guarantees that $\|A_n - A\| \rightarrow 0$. In conjunction with (7), we have

$$\lim_{n \rightarrow \infty} \lambda_{n,k}^+ = \lambda_k^+,$$

and the convergence is clearly uniform in k .

The proof for $\lambda_{n,k}^-$ follows similar lines and is omitted.

Remark. In numerical examples, it may be convenient to use the truncation method of Weinberger [9] in connection with intermediate problems. The present arguments can be modified to apply to Weinberger's method as well.

BIBLIOGRAPHY

- [1] J. B. DIAZ & F. T. METCALF, A functional equation for the Rayleigh quotient for eigenvalues, and some applications, *J. Math. Mech.*, **17** (1968) 623–630.
- [2] G. FICHERA, *Linear Elliptic Differential Systems and Eigenvalue Problems*, Lecture Notes in Mathematics, Springer-Verlag, New York, 1965.
- [3] G. FICHERA, On the compactness of the base operator in the theory of intermediate problems, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.*, **41** (1966) 3–7.
- [4] S. H. GOULD, *Variational Methods for Eigenvalue Problems. An Introduction to the Weinstein Method of Intermediate Problems*, second edition, University of Toronto Press, 1966.
- [5] M. ROSENBLATT, Asymptotic behavior of eigenvalues for a class of integral equations with translation kernels, *Proceedings of the Symposium on Time Series Analysis*, The SIAM Series in Applied Mathematics, John Wiley and Sons, Inc., New York, 1963.
- [6] I. STAKGOLD, Bounds for linear functionals, *Bull. Amer. Math. Soc.*, **73** (1967) 180–183.
- [7] W. STENGER, On Poincaré's bounds for higher eigenvalues, *Bull. Amer. Math. Soc.*, **72** (1966) 715–718.
- [8] W. STENGER, On the variational principles for eigenvalues for a class of unbounded operators, *J. Math. Mech.*, **17** (1968) 641–648.
- [9] H. F. WEINBERGER, A theory of lower bounds for eigenvalues, *Institute of Fluid Dynamics and Applied Mathematics*, U. of Maryland, Tech. Note BN-183, 1959.
- [10] A. WEINSTEIN, On the base problem for a compact integral operator, *J. Math. Mech.*, **17** (1967) 217–223.

Northwestern University

Date communicated: FEBRUARY 12, 1969