On Existence, Uniqueness and Convergence of Successive Approximations for Hyperbolic Partial Differential Equations

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In [1] the author generalized the Krasnoselskii–Krein criteria for ordinary differential equations, to hyperbolic partial differential equations in normal form. Also, examples were provided to show optimality of the conditions imposed. Following the same line of thought, an extension of Kooi's condition [2] will be given. Since similar techniques and a similar approach will be used, we don't provide the proofs but only state the final results.

Definition. Let $f(x, y, \dots, p_{ii}, \dots)$ be defined in

$$D = \begin{cases} (x, y) \in R \equiv [0, a] \times [0, b] \\ -\infty < p_{ii} < +\infty \\ 0 \leq i \leq r, \quad 0 \leq j \leq s, \quad r + s = n, \quad i + j \leq r + s - 1 \end{cases}$$

where it is also continuous. We say that f satisfies in D a *generalized Kooi condition of order (r, s)* if, for $(x, y, \dots, p_{ii}, \dots), (x, y, \dots, \bar{p}_{ii}, \dots) \in D$:

$$\begin{aligned} \text{A)} \quad x^\alpha y^\beta |f(x, y, \dots, p_{ii}, \dots) - f(x, y, \dots, \bar{p}_{ii}, \dots)| \\ \leq \sum_{(i,j)} K_{ij}(x, y) x^i y^j |p_{ii} - \bar{p}_{ii}|^\alpha \end{aligned}$$

$$\text{B)} \quad |f| \leq H x^\alpha y^\beta$$

where:

$$\text{a)} \quad \{K_{ij}(x, y)\} \subset C(R), \quad K_{ij}(x, y) \geq 0 \quad \forall (i, j)$$

$$K = [\max_R (\max_{(i,j)} K_{ij}(x, y))] > 0$$

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- b) $q \geq 1, \quad a > -1, \quad b > -1$
- c) $q(r + a) - \alpha = a$
 $q(s + b) - \beta = b$
- d) $\rho = K(2H)^{q-1}A^q < 1$, with A
 $= \sum_{(i,j)} ((a+1) \cdots (a+r-i)(b+1) \cdots (b+s-j))^{-1}.$

Theorem. Let f be defined in D , where it satisfies a generalized Kooi condition of order (r, s) . Then, the problem:

$$\frac{\partial^n u}{\partial x^r \partial y^s} = f\left(x, y, \cdots, \frac{\partial^{i+j} u}{\partial x^i \partial y^j}, \cdots\right), \quad \text{where } r + s = n,$$

$$0 \leq i \leq r, \quad 0 \leq j \leq s, \quad i + j \leq n - 1, \quad \text{with data}$$

$$\frac{\partial^i u(0, y)}{\partial x^i} = \phi_{1,i}(y), \quad \frac{\partial^i u(x, 0)}{\partial y^i} = \phi_{2,i}(x),$$

$$\frac{\partial^i \phi_{1,i}(0)}{\partial y^j} = \frac{\partial^i \phi_{2,j}(0)}{\partial x^i}, \quad 0 \leq i \leq r - 1, \quad 0 \leq j \leq s - 1,$$

has one and only one solution in $C^*(R)$, which can be obtained by successive approximations.

$$\left(C^*(R) = \{u(x, y) \in C(R) \mid \frac{\partial^{i+j} u}{\partial x^i \partial y^j} \in C(R), \quad 0 \leq i \leq r, \quad 0 \leq j \leq s, \quad i + j \leq n - 1\} \right)$$

The proof uses Luxemburg's fixed point theorem contained in [3] (i.e. a "local" fixed point theorem) in the same way that the fixed point theorem of [4] was used in [1].

Remarks. 1. We observe that the special case $K_{i,j}(x, y) \equiv K^{1/q}$, $b = 0$, $n = r = 1$, $s = 0$ is precisely the original Kooi condition [2].

2. For similar definitions and results concerning generalized Nagumo conditions, the reader is referred to [5].

3. When $q = 1$, $a = b = 0$, we have a special case of the generalized Nagumo condition, precisely with $K < 1/A$.

4. In [6], [7], the particular case $n = 2$, $r = s = 1$, $i = j = 0$ is treated, with conditions $a = b$, $\alpha = \beta$. In that case, where for us $A = (a + 1)^{-2}$, Wong gets $A = (a + 1)^{-1}$. Again, we find distinctions for the cases with "incomplete" and "complete" second member (see [1]).

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