*Positive Solutions of Some Nonlinear Eigenvalue Problems**

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1. Introduction. We consider nonlinear eigenvalue problems of the form

$$(1.1a) \quad Lu + \lambda r(x)u = f(\lambda, x, u), \quad x \in D;$$

$$(1.1b) \quad Bu \equiv \beta_0(x)u + \beta_1(x) \frac{\partial u}{\partial \nu} = 0, \quad x \in \partial D;$$

where L is a second order self-adjoint elliptic operator. We seek positive solutions, $u(x) > 0$ on D , and investigate the set of values of λ for which such solutions exist. The nonlinearity is required to satisfy, for appropriate values of λ ,

$$f(\lambda, x, 0) \leq 0 \quad \text{on } D.$$

The case $f(\lambda, x, 0) \equiv 0$, which we include, has the trivial solution, $u(x) \equiv 0$, for all values of λ and thus leads to a bifurcation problem. Under various monotonicity conditions on $f(\lambda, x, z)$ and $f_z(\lambda, x, z)$ for appropriate λ and z we prove uniqueness of positive solutions, existence and non-existence of positive solutions, characterize the bifurcation points and show that positive solutions increase with λ and possibly other parameters. Closely related results are contained in [7] and [8].

The second order operator L is uniformly elliptic self-adjoint, say:

$$(1.2a) \quad L\varphi \equiv \sum_{i,j=1}^m \partial_i a_{ij}(x) \partial_j \varphi - a_0(x) \varphi;$$

with

$$(1.2b) \quad \sum_{i,j=1}^m a_{ij}(x) \xi_i \xi_j \geq a^2 \sum_{i=1}^m \xi_i^2, \quad a_0(x) \geq 0, \quad r(x) > 0 \quad \text{on } \bar{D}.$$

The conormal derivative $\partial/\partial \nu$ is

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$$(1.2c) \quad \frac{\partial \varphi}{\partial \nu} \equiv \sum_{i,j=1}^m n_i(x) a_{ij}(x) \partial_j \varphi, \quad \text{on } \partial D;$$

where $\mathbf{n}(x) \equiv (n_1(x), \dots, n_m(x))$ is the outward unit normal to ∂D at x . The coefficients in B are required to satisfy

$$(1.2d) \quad \begin{aligned} \beta_0(x) &> 0, & \beta_1(x) &\equiv 0 & \text{on } \partial D_1; \\ \beta_0(x) &\geq 0, & \beta_1(x) &> 0 & \text{on } \partial D_2; \quad \partial D_1 \cup \partial D_2 = \partial D. \end{aligned}$$

By $C^{n+\alpha}(G)$ we denote the class of functions with continuous derivatives of all orders up to n on the set G and whose n^{th} derivatives are uniformly Hölder continuous with exponent $\alpha \in (0, 1)$. The smoothness conditions on the coefficients are

$$(1.3a) \quad a_{ij}(x) \in C^{1+\alpha}(\bar{D}), \quad a_0(x) \in C^\alpha(\bar{D}), \quad r(x) \in C^\alpha(\bar{D}), \quad \beta_i(x) \in C^{1+\alpha}(\partial D).$$

On the $m + 1$ dimensional half cylinder

$$\bar{D}_+ \equiv \{x, z \mid x \in \bar{D}, z \geq 0\}$$

and for some set of real numbers S we require that the nonlinearity satisfy

$$(1.3b) \quad f(\lambda, x, z) \in C^\alpha(\bar{D}_+), \quad f_z(\lambda, x, z) \in C^\alpha(\bar{D}_+) \quad \text{for all } \lambda \in S.$$

The set S , in most applications, is usually an interval; in fact a half line or even all the reals. But we only introduce such conditions and continuity requirements with respect to λ as needed. Finally the boundary ∂D is required to be of class $C^{2+\alpha}$ (i.e. locally it can be mapped 1-1 onto an $m - 1$ dimensional hyperplane by $C^{2+\alpha}$ mappings). For many applications the smoothness conditions imposed above can be considerably relaxed. In particular, various spherically symmetric problems yielding ordinary differential equations with singular endpoints are easily included. We do not supply the rather clear extension to such cases here.

A basic result that we employ, from [1], is the

Positivity Lemma. *Let $\varphi(x) \in C^2(D)$ and satisfy*

$$(1.4) \quad \begin{aligned} L\varphi + \lambda r(x)\varphi &< 0 & \text{on } D; \\ B\varphi &= 0 & \text{on } \partial D. \end{aligned}$$

Then $\varphi(x) > 0$ on D if and only if $\lambda < \mu_1$, the least eigenvalue of:

$$(1.5) \quad \begin{aligned} L\psi + \mu r(x)\psi &= 0 & \text{on } D; \\ B\psi &= 0 & \text{on } \partial D. \end{aligned}$$

With rather obvious modifications in the necessity and sufficiency proofs of the above we obtain a somewhat weaker and essentially well-known result which can be stated as the

Weak Positivity Lemma. *Let $\varphi(x) \in C^2(D)$ and satisfy*

$$(1.6) \quad \begin{aligned} L\varphi + \lambda r(x)\varphi &\leq 0 \quad \text{on } D; \\ B\varphi &= 0 \quad \text{on } \partial D. \end{aligned}$$

- A. If $\varphi(x) > 0$ on D then $\lambda \leq \mu_1$.
 B. If $\lambda < \mu_1$ then $\varphi(x) \geq 0$ on D .

2. Uniqueness. We shall present a uniqueness result which is a slight extension of Theorem 2 in [2]. The techniques employed in the proof recur throughout this paper. First we introduce some basic notation.

If $\varphi(x)$ and $\psi(x)$ are nonnegative continuous functions on $\bar{D}$ then by (1.3b) we have for any $\lambda \in S$:

$$(2.1a) \quad f(\lambda, x, \varphi(x)) - f(\lambda, x, \psi(x)) = q(\lambda, x, \varphi(x), \psi(x))[\varphi(x) - \psi(x)],$$

where

$$(2.1b) \quad q(\lambda, x, \varphi(x), \psi(x)) \equiv \int_0^1 f_s(\lambda, x, s\varphi(x) + (1-s)\psi(x)) ds.$$

We define the quantity (functional) $\mu_1\{g(x)\}$ for each $g(x) \in C(\bar{D})$ to be the least eigenvalue of:

$$(2.2) \quad \begin{aligned} L\varphi + [\mu r(x) - g(x)]\varphi &= 0 \quad \text{on } D, \\ B\varphi &= 0 \quad \text{on } \partial D. \end{aligned}$$

This quantity can also be characterized by the variational principle:

$$(2.3a) \quad \mu_1\{g(x)\} = \min_{\varphi \in \mathfrak{A}} \left[\frac{Q[\varphi] + \int_D g(x)\varphi^2(x) dx}{H[\varphi]} \right].$$

Here we have used the quadratic functionals

$$(2.3b) \quad \begin{aligned} Q[\varphi] &\equiv \int_D \left[\sum_{i,j=1}^m a_{ij}(x) \partial_i \varphi(x) \partial_j \varphi(x) + a_0(x) \varphi^2(x) \right] dx + \oint_{\partial D} \frac{\beta_0(x)}{\beta_1(x)} \varphi^2(x) dx, \\ H[\varphi] &\equiv \int_D r(x) \varphi^2(x) dx. \end{aligned}$$

The admissible functions in (2.3a) can be taken as

$$(2.3c) \quad \mathfrak{A} = \{\varphi(x) \mid \varphi(x) \in C(\bar{D}) \cap C^1(D), \varphi(x) = 0 \text{ on } \partial D_1\}.$$

Theorem 2.1. Let $f(\lambda, x, z)$ satisfy

$$H_0 : f(\lambda, x, 0) \leq 0 \quad \text{on } D \quad \text{for all } \lambda \in S;$$

$$H_1 : f_z(\lambda, x, z) > f_z(\lambda, x, z') \quad \text{on } D \quad \text{for all } z > z' > 0 \quad \text{and } \lambda \in S.$$

Then positive solutions of (1.1) are unique for all $\lambda \in S$.

Proof. Assume that $u(\lambda, x) \equiv u(x)$ and $v(\lambda, x) \equiv v(x)$ are distinct positive solutions of (1.1) for the same $\lambda \in S$. Then with $w(x) \equiv u(x) - v(x)$ we obtain, using (2.1),

$$(2.4) \quad \begin{aligned} Lw + [\lambda r(x) - q(\lambda, x, u(x), v(x))]w &= 0 \quad \text{on } D, \\ Bw &= 0 \quad \text{on } \partial D. \end{aligned}$$

Noting that

$$f(\lambda, x, u(x)) - f(\lambda, x, 0) = q(\lambda, x, u(x), 0)u(x)$$

we can write (1.1) for u as

$$(2.5) \quad \begin{aligned} Lu + [\lambda r(x) - q(\lambda, x, u(x), 0)]u &= f(\lambda, x, 0) \quad \text{on } D; \\ Bu &= 0 \quad \text{on } \partial D. \end{aligned}$$

By H_1 and the assumption that $v(x) > 0$ on D we have

$$q(\lambda, x, u, v) > q(\lambda, x, u, 0) \quad \text{on } D.$$

But then from (2.3) it follows that

$$\mu_1\{q(\lambda, x, u, v)\} > \mu_1\{q(\lambda, x, u, 0)\}.$$

Since $w(x) \not\equiv 0$ by assumption we see from (2.4) that λ must be some eigenvalue of the problem (2.2) with $g(x) \equiv q(\lambda, x, u, v)$. However as $\mu_1\{q(\lambda, x, u, v)\}$ is the minimum eigenvalue of this problem we have that

$$\lambda \geq \mu_1\{q(\lambda, x, u, v)\} > \mu_1\{q(\lambda, x, u, 0)\} \equiv \tau_1.$$

Now write (2.5) in the form

$$\begin{aligned} Lu + [\tau_1 r(x) - q(\lambda, x, u, 0)]u &= f(\lambda, x, 0) + (\tau_1 - \lambda)r(x)u(x) \quad \text{on } D; \\ Bu &= 0 \quad \text{on } \partial D. \end{aligned}$$

As τ_1 is the least eigenvalue of (2.2) with $g(x) \equiv q(\lambda, x, u, 0)$ its corresponding eigenfunction, $\varphi_1(x)$, is of one sign on D and it must satisfy

$$\int_D \varphi_1(x)[f(\lambda, x, 0) + (\tau_1 - \lambda)r(x)u(x)] dx = 0.$$

But this is impossible as the bracketed term is negative on D by H_0 and the assumption that $u(x) > 0$ on D . The contradiction implies $w(x) \equiv 0$. Q.E.D.

The above proof remains valid if we relax the monotonicity hypothesis H_1 to require only

$$f_z(\lambda, x, z) \geq f_z(\lambda, x, z') \quad \text{on } D \quad \text{for all } z > z' > 0 \quad \text{and } \lambda \in S,$$

but strengthen hypothesis H_0 to

$$f(\lambda, x, 0) < 0 \quad \text{on } D \quad \text{for all } \lambda \in S.$$

3. Existence and nonexistence theorems. The nonlinear term is said to be "forced" under the hypothesis

$$H_{oF} : f(\lambda, x, 0) < 0 \quad \text{on } D \quad \text{for all } \lambda \in S;$$

and it is said to be "unforced" under the hypothesis

$$H_{oU} : f(\lambda, x, 0) = 0 \quad \text{on } D \quad \text{for all } \lambda \in S.$$

Clearly the Uniqueness Theorem 2.1 holds if H_o is replaced by either H_{oF} or H_{oU} . However an important contrast holds between the two cases concerning existence and non-existence of positive solutions. To state these results we define the quantity

$$(3.1) \quad \mu_1(\lambda) \equiv \mu_1\{f_z(\lambda, x, 0)\} = \mu_1\{q(\lambda, x, 0, 0)\},$$

and the subset S_1 of S by

$$(3.2) \quad S_1 \equiv \{\lambda \mid \lambda \in S, \lambda < \mu_1(\lambda)\}.$$

Now we proceed to show that, under a slightly stronger form of H_1 , the forced case has positive solutions for all $\lambda \in S_1$ while the unforced case has none. First we present the non-existence result which holds for a weaker form of H_1 .

Theorem 3.1. *Let $f(\lambda, x, z)$ satisfy H_{oU} and*

$$H_1^0 : f_z(\lambda, x, z) \geq f_z(\lambda, x, 0) \quad \text{on } D \quad \text{for all } z > 0 \quad \text{and } \lambda \in S.$$

Then problem (1.1) has no positive solution for any $\lambda \in S_1$.

Proof. Assume that (1.1) does have a positive solution, $u(x) \equiv u(\lambda, x) > 0$ on D , for some $\lambda \in S_1$. Since f is unforced the equivalent form (2.5) of (1.1) can be written as: $Bu = 0$ on ∂D and

$$\begin{aligned} Lu + [\mu_1(\lambda)r(x) - f_z(\lambda, x, 0)]u &= [\mu_1(\lambda) - \lambda]r(x)u \\ &+ [q(\lambda, x, u(x), 0) - f_z(\lambda, x, 0)]u \quad \text{on } D. \end{aligned}$$

From H_1^0 and $u(x) > 0$ on D it follows that $[q(\lambda, x, u(x), 0) - f_z(\lambda, x, 0)] \geq 0$ on D . Then as $\lambda \in S_1$ implies $\mu_1(\lambda) - \lambda > 0$ we have

$$[\mu_1(\lambda) - \lambda]r(x)u(x) + [q(\lambda, x, u(x), 0) - f_z(\lambda, x, 0)]u(x) > 0 \quad \text{on } D.$$

But the left hand side of this inequality must be orthogonal to $\varphi_1(x)$, the eigenfunction belonging to $\mu_1(\lambda)$. Since this eigenfunction is of one sign on D we get a contradiction and hence $\lambda \notin S_1$. Q.E.D.

Note that replacing H_1^0 by H_1 allows the roots of $\lambda = \mu_1(\lambda)$ to be in S_1 . This is also the case if H_1^0 is strengthened to require the strict inequality.

In many applications $S \equiv \{\lambda \mid \lambda \geq 0\}$ while $f_z(\lambda, x, 0) \geq -a_0(x)$ on D and is continuous in λ for all $\lambda \in S$. Then, from the variational characterization (2.3), $\mu_1(\lambda) > 0$ for all $\lambda \geq 0$ and we are assured that S_1 contains a nonde-

generate interval of the form $(0, \lambda_1)$ where λ_1 is the first positive root of $\lambda = \mu_1(\lambda)$. Extending these ideas slightly we can characterize λ -intervals on which the unforced problem cannot bifurcate. Specifically we have

Corollary 3.1. *Let S and λ_0 be such that*

$$\lambda_0 < \mu_1(\lambda_0), \quad [\lambda_0, \infty) \subset S.$$

Let $f(\lambda, x, z)$ satisfy H_{0U} , H_1^0 with the strict inequality and $f_z(\lambda, x, 0)$ be continuous in λ on $[\lambda_0, \infty)$ uniformly for $x \in D$. Then positive solutions of problem (1.1) cannot bifurcate in $[\lambda_0, \lambda_1]$ where λ_1 is the first odd root in $[\lambda_0, \infty)$ of $\lambda = \mu_1(\lambda)$.

Proof. From the continuity of $f_z(\lambda, x, 0)$ it follows that $\mu_1(\lambda)$ is continuous on $[\lambda_0, \infty)$. Then since $\mu_1(\lambda_0) > \lambda_0$ this continuity and Theorem 3.1 imply that problem (1.1) has no positive solutions for some interval to the right of λ_0 . Since the strong form of H_1^0 is satisfied Theorem 3.1 applies with S_1 replaced by

$$\bar{S}_1 \equiv \{\lambda \mid \lambda \in S, \lambda \leq \mu_1(\lambda)\},$$

as observed above. Thus $[\lambda_0, \lambda_1] \subset \bar{S}_1$.

Q.E.D.

Under additional smoothness and monotonicity conditions on f we can show that the odd roots of $\lambda = \mu_1(\lambda)$ are indeed bifurcation points. This is done by perturbation methods using contracting maps and will be reported elsewhere [7].

If $f_z(\lambda, x, z) > 0$ on D for $z \geq 0$ then a non-existence theorem in [2] can be applied and compared with Theorem 3.1. Specifically with the definition

$$F(\lambda, x, z) \equiv f(\lambda, x, z) - \lambda r(x)z$$

we have that

$$\begin{aligned} F(\lambda, x, z) + \tilde{\mu}_1 r(x)z &= q(\lambda, x, z, 0)z + (\tilde{\mu}_1 - \lambda)r(x)z, \\ &> 0 \quad \text{on } D, \end{aligned}$$

if $z > 0$ and $\lambda \leq \tilde{\mu}_1$. Thus no positive solution of (1.1) exists, by Theorem 1 of [2], for any $\lambda \leq \tilde{\mu}_1$ when $\tilde{\mu}_1$ is the least eigenvalue of

$$\begin{aligned} L\varphi + \tilde{\mu} r(x)\varphi &= 0 \quad \text{on } D, \\ B\varphi &= 0 \quad \text{on } \partial D. \end{aligned}$$

However $\mu_1(\lambda) > \tilde{\mu}_1$ since $f_z(\lambda, x, 0) > 0$ and we see that the result of Theorem 3.1 is sharper than that of Theorem 1 in [2].

Returning to the case of forced nonlinearities we now show that under a strengthened form of H_1 positive solutions exist for all $\lambda \in S_1$.

Theorem 3.2. *Let $f(\lambda, x, z)$ satisfy H_{0F} and*

$$H_1^1 : f_z(\lambda, x, z) > f_z(\lambda, x, z') \geq -a_0(x) \text{ on } D \text{ for all } z > z' \geq 0 \text{ and } \lambda \in S.$$

Then problem (1.1) has a unique positive solution for each $\lambda \in S_1$. The solution,

for each $\lambda \in S_1$, is the limit of the monotone sequence of iterates, $\{u_n(x)\}_{n=1}^\infty$, defined by Newton's method:

$$Lu_{n+1} + [\lambda r(x) - f_z(\lambda, x, u_n(x))]u_{n+1} = f(\lambda, x, u_n(x)) - f_z(\lambda, x, u_n(x))u_n(x) \quad \text{on } D;$$

$$(3.3) \quad Bu_{n+1} = 0 \quad \text{on } \partial D; \quad n = 0, 1, \dots$$

with initial iterate $u_0(x) \equiv 0$.

Proof. Under the assumption, to be verified later, that all the iterates exist and are in $C^{2+\alpha}(D)$ we will show by induction that

$$(3.4) \quad u_0(x) \equiv 0 < \dots < u_n(x) < u_{n-1}(x) < \dots < u_1(x).$$

For $n = 0$ in (3.3) we get, recalling H_{0F} ,

$$Lu_1 + [\lambda r(x) - f_z(\lambda, x, 0)]u_1 < 0 \quad \text{on } D,$$

$$Bu_1 = 0 \quad \text{on } \partial D.$$

Since $\lambda \in S_1$ we have $\lambda < \mu_1\{f_z(\lambda, x, 0)\}$ and H_1^1 insures $f_z(\lambda, x, 0) + a_0(x) \geq 0$ on D . Thus the Positivity Lemma is applicable above and implies that $u_1(x) > 0$ on D . For an induction assume that $u_n(x) > 0$ on D and then recalling (2.1), H_{0F} and H_1^1 we obtain

$$\begin{aligned} Lu_{n+1} + [\lambda r(x) - f_z(\lambda, x, u_n)]u_{n+1} \\ &= f(\lambda, x, u_n) - f_z(\lambda, x, u_n)u_n \\ &= f(\lambda, x, 0) + [q(\lambda, x, u_n(x), 0) - f_z(\lambda, x, u_n)]u_n \\ &< 0 \quad \text{on } D. \end{aligned}$$

But H_1^1 and the variational characterization of $\mu_1\{g(x)\}$ yield

$$\mu_1\{f_z(\lambda, x, 0)\} \leq \mu_1\{f_z(\lambda, x, u_n)\}.$$

So the Positivity Lemma is again applicable and implies $u_{n+1}(x) > 0$ on D , concluding the proof that $u_\nu(x) > 0$ on D for $\nu = 1, 2, \dots$.

From (3.3) for $n = 0, 1$ we now find

$$\begin{aligned} L(u_2 - u_1) + [\lambda r(x) - f_z(\lambda, x, u_1)](u_2 - u_1) \\ &= f(\lambda, x, u_1) - f(\lambda, x, 0) - f_z(\lambda, x, 0)u_1, \\ &= [q(\lambda, x, u_1(x), 0) - f_z(\lambda, x, 0)]u_1, \\ &> 0 \quad \text{on } D. \end{aligned}$$

Thus by the Positivity Lemma $u_2(x) < u_1(x)$ on D and an obvious induction concludes the proof of (3.4).

Since the sequence $\{u_n(x)\}_{n=1}^\infty$ is monotone decreasing and bounded below it converges, say

$$\lim_{n \rightarrow \infty} u_n(x) = u(\lambda, x).$$

We proceed to show that the limit $u(\lambda, x)$ is a positive solution of (1.1). Clearly $u(\lambda, x) \geq 0$ on D since $u_n(x) > 0$ on D for all $n > 0$. The case $u(\lambda, x) \equiv 0$ is ruled out if $u(\lambda, x)$ is a solution of (1.1) since f is forced. If $u(\lambda, x) \in C^2(D)$ and is a solution of (1.1) and for some $x^0 \in D$ we have $u(\lambda, x^0) = 0$ then

$$Lu(\lambda, x^0) = f(\lambda, x^0, 0) < 0.$$

But x^0 must be at a relative minimum of $u(\lambda, x)$ and we get a contradiction as in the proof of the standard minimum principle. Thus it only remains to show that $u(\lambda, x)$ is a solution of (1.1) in $C^2(D)$. We shall do this by applying the compactness Theorem 12.2 of Agmon, Douglas and Nirenberg [3].

By induction it easily follows from the smoothness conditions (1.3), the smoothness of ∂D , the result (3.4) hypothesis H_1^1 and Theorem 3.2, p. 137, of Ladyženskaja and Ural'tseva [4] that the solutions of (3.3) exist and $u_n(x) \in C^{2+\alpha}(D)$ for $n = 1, 2, \dots$.

Denote any of the iterates by $v(x)$ and let $x, \xi \in D$ be any points with $|x - \xi| = \rho$. Since the ball $B_\rho(\bar{x})$ with center $\bar{x} = (x + \xi)/2$ and radius ρ is not always in D we extend $v(x)$ outside D in such a way that

$$\left(\int_{B_\rho(\bar{x})} |\nabla v(y)|^p dy \right)^{1/p} \leq K_0 \|v\|_{p,D}^1$$

for all such spheres of radius ρ . That this can be done for any $p > 1$, with constant $K_0 \geq 1$ independent of v is assured by the Calderon Extension Theorem (see Theorem 3.4.3 of [5]). For any point $\eta \in B_\rho(\bar{x})$ we have

$$[v(\eta) - v(\xi)] = [\eta - \xi] \cdot \int_0^1 \nabla v(\xi + t[\eta - \xi]) dt.$$

Take absolute values and average with respect to η over $B_\rho(\bar{x})$ to get:

$$\begin{aligned} \frac{1}{V_\rho} \int_{B_\rho(\bar{x})} |v(\eta) - v(\xi)| d\eta &\leq \frac{2\rho}{V_\rho} \int_{B_\rho(\bar{x})} \int_0^1 |\nabla v(\xi + t[\eta - \xi])| dt d\eta, \\ &\leq \frac{2\rho}{V_\rho} \int_0^1 \left(\int_{B_{t\rho}(\bar{x}(t))} |\nabla v(y)| dy \right) t^{-m} dt. \end{aligned}$$

Here V_ρ is the volume of $B_\rho(\bar{x})$ and $\bar{x}(t) = \xi + t[x - \xi]$. Using Hölder's Inequality in the form

$$\int_B |\nabla v(y)| dy \leq \left(\int_B |\nabla v(y)|^p dy \right)^{1/p} \left(\int_B 1^q dy \right)^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

but with $p > m$ we obtain

$$\begin{aligned} \frac{1}{V_\rho} \int_{B_\rho(\bar{x})} |v(\eta) - v(\xi)| d\eta &\leq K_1 \rho^{1-m/p} \left(\int_{B_\rho(\bar{x})} |\nabla v(y)|^p dy \right)^{1/p}, \\ &\leq K_2 \rho^{1-m/p} \|v\|_{p,D}^1. \end{aligned}$$

From this result with ξ and x interchanged and the triangle inequality we finally get

$$(3.5) \quad |v(x) - v(\xi)| \leq K_3 \|v\|_{p,D}^1 |x - \xi|^\gamma, \quad \gamma = 1 - \frac{m}{p}.$$

(This derivation, based on an estimate of Morrey [5], is due to R. B. Simpson.) The constant K_3 is independent of v (it depends only on m , p and K_0 in the Calderon extension).

It is well known that for $v(x) \in C^2(D)$

$$\|v\|_{p,D}^1 \leq \|v\|_{p,D}^2.$$

Further Agmon, Douglas and Nirenberg have established an inequality, (15.5) of [3], which in our case becomes, since $Bv = 0$ on ∂D ,

$$\|v\|_{p,D}^2 \leq K_4 \{ \|Lv\|_{p,D}^0 + \|v\|_{p,D}^0 \},$$

with K_4 independent of v . If we employ these results in (3.5) and take $v(x) \equiv u_n(x)$ for any $n > 0$ it follows that

$$(3.6) \quad |u_n(x) - u_n(\xi)| \leq K_5 |x - \xi|^\gamma, \quad n = 1, 2, \dots; \quad \gamma \in (0, 1).$$

Here K_5 is independent of u_n (but not of λ) and we have used the uniform boundedness of the $u_n(x)$ on $\bar{D}$ which by (1.3) implies the uniform boundedness of $f(\lambda, x, u_n(x))$ and $f_z(\lambda, x, u_n(x))$ in (3.3).

From (3.6) the equicontinuity and hence uniform convergence of the monotone sequence $\{u_n(x)\}_{n=1}^\infty$ follow. Then by (1.3) the uniform convergence of

$$f(\lambda, x, u_n(x)) \rightarrow f(\lambda, x, u(\lambda, x)) \quad \text{and} \quad f_z(\lambda, x, u_n(x)) \rightarrow f_z(\lambda, x, u(\lambda, x))$$

also follow and hence Theorem 12.2 of [3] is applicable. Taking the limit in (3.3) is now valid and it follows that $u(\lambda, x)$ is a solution of (1.1) in $C^{2+\alpha}(D)$.
Q.E.D.

4. Variation of solutions with parameters. If $f(\lambda, x, z)$ does not increase too fast with λ the positive solutions will increase with λ (for both forced and unforced nonlinearities). Specifically we have

Theorem 4.1. *Let $f(\lambda, x, z)$ satisfy H_0 , H_1^1 and*

$$H_2 : [f(\lambda, x, z) - f(\lambda', x, z)]/(\lambda - \lambda') < r(x)z$$

on D for all $z > 0$ and $\lambda > \lambda'$ in S . If $u(\lambda, x)$ and $u(\lambda', x)$ are positive solutions of (1.1) for $\lambda > \lambda'$ in S then:

$$u(\lambda, x) > u(\lambda', x) \text{ on } D.$$

Proof. With the notation $u(x) \equiv u(\lambda, x)$, $v(x) \equiv u(\lambda', x)$, $w(x) \equiv u(x) - v(x)$

we obtain from (1.1), using (2.1), that (2.5) holds and

$$Lw + [\lambda r(x) - q(\lambda, x, u, v)]w = f(\lambda, x, v) - f(\lambda', x, v) - (\lambda - \lambda')r(x)v \quad \text{on } D; \quad (4.1)$$

$$Bw = 0 \quad \text{on } \partial D.$$

Since $f(\lambda, x, 0) \leq 0$ on D by H_0 and $u(x) > 0$ on D by assumption, part A of the Weak Positivity Lemma implies by (2.5) and H_1^1 that $\lambda \leq \mu_1\{q(\lambda, x, u, 0)\}$. Since $v(x) > 0$ on D the variational characterization of $\mu_1\{\cdot\}$ and H_1^1 imply: $\lambda < \mu_1\{q(\lambda, x, u, v)\}$. Thus using H_2 and the Positivity Lemma in (4.1) yields $w(x) > 0$ on D . Q.E.D.

Note that condition H_2 is much stronger than is required to show that $u(\lambda, x) > u(\lambda', x)$ on D for a *specific* pair of values $\lambda > \lambda' \in S$. That is, the bound on the difference quotient is only required for z on $0 < z \leq \max_D u(\lambda', x)$ and for the specific values λ and λ' . Thus if one positive solution is known, call it $u(\lambda', x)$, we can determine those values of $\lambda > \lambda'$ for which the weakened form of H_2 holds and hence for which positive solutions exceed $u(\lambda', x)$ if they exist.

Finally we point out that if the strong inequality in H_2 is replaced by $\leq$ then the conclusion: $u(\lambda, x) \geq u(\lambda', x)$ on D , is valid by an application of part B of the Weak Positivity Lemma in the final step of the proof.

Positive solutions of (1.1) for different nonlinearities can frequently be compared. In particular this analysis is used to show how the positive solutions for $\lambda \in S_1$ of problems with forced nonlinearities are related to the trivial solutions for $\lambda \in S_1$ of problems with corresponding unforced nonlinearities. More generally we consider a one parameter family of nonlinearities of the form

$$f(t, \lambda, x, z)$$

where the parameter t ranges over some real interval, say $T \equiv [t_0, t_{\max}]$. The smoothness conditions (1.3b) are imposed for all $t \in T$. The notation (2.1) is extended in the obvious way, i.e.,

$$q(t, \lambda, \phi, \psi) \equiv \int_0^1 f_z(t, \lambda, s\phi + (1-s)\psi) ds,$$

and (3.1) is replaced by the definition

$$(4.2) \quad \mu_1(t, \lambda) \equiv \mu_1\{f_z(t, \lambda, x, 0)\}.$$

In place of (3.2) we define for each $t \in T$ the set of λ values:

$$(4.3) \quad S_1(t) \equiv \{\lambda \mid \lambda \in S, \lambda < \mu_1(t, \lambda)\}.$$

Our basic comparison result can now be stated as

Theorem 4.2. *Let $f(t, \lambda, x, z)$ satisfy H_0 and H_1^1 for all $t \in T$ and also*

$$H_T : f(t_1, \lambda, x, z) > f(t_2, \lambda, x, z) \text{ for } (\lambda, x, z) \in S \times \bar{D}_+ \text{ if } t_2 < t_1 \leq t_{\max}.$$

Then the problem

$$(4.4) \quad \begin{aligned} Lu + \lambda r(x)u &= f(t, \lambda, x, u), \quad \text{on } D, \\ Bu &= 0, \quad \text{on } \partial D, \end{aligned}$$

has a unique positive solution, $u(t, \lambda, x)$, for all $\lambda \in S_1(t)$ for each $t \in [t_0, t_{\max})$. If for some $t_2 < t_1 < t_{\max}$ we have $S_1(t_1) \cap S_1(t_2) \neq \emptyset$ then

$$(4.5) \quad u(t_1, \lambda, x) < u(t_2, \lambda, x) \text{ on } D$$

for all $\lambda \in S_1(t_1) \cap S_1(t_2)$. If $f(t_{\max}, \lambda, x, z)$ is forced or unforced we may let $t_1 = t_{\max}$ in the above.

Proof. By H_0 and H_T it follows that $f(t, \lambda, x, z)$ is forced for all $t \in [t_0, t_{\max})$. Then Theorem 3.2 applies for each t in this interval. If $f(t_{\max}, \lambda, x, z)$ is forced the same applies while if it is unforced Theorem 3.1 applies and we may then set $u(t_{\max}, \lambda, x) \equiv 0$.

Now assume $\lambda \in S_1(t_1) \cap S_1(t_2)$ with $t_2 < t_1 \leq t_{\max}$ so that $u_1(x) \equiv u(t_1, \lambda, x) \geq 0$ on D and $u_2(x) \equiv u(t_2, \lambda, x) > 0$ on D . With $w(x) \equiv u_2(x) - u_1(x)$ we obtain, on subtraction of the relevant equations, $Bw(x) = 0$ on ∂D and

$$(4.6) \quad Lw + [\lambda r(x) - q(t_2, \lambda, x, u_1, u_2)]w = f(t_2, \lambda, x, u_1) - f(t_1, \lambda, x, u_1), \\ < 0, \quad \text{on } D.$$

Here we have used H_T . Since $\lambda \in S_1(t_2)$ we have $\lambda < \mu_1(t_2, \lambda)$. But by H_1^1 and the variational characterization (2.3) of $\mu_1\{\cdot\}$ it follows from $u_1 \geq 0$ and $u_2 > 0$ that

$$\lambda < \mu_1\{f_z(t_2, \lambda, x, 0)\} < \mu_1\{q(t_2, \lambda, u_1, u_2)\}.$$

This and the fact, again from H_1^1 , that $q(t_2, \lambda, x, u_1, u_2) + a_0(x) \geq 0$ imply that the Positivity Lemma is applicable to (4.6). Thus we get $w(x) > 0$ on D which is (4.5). Q.E.D.

To show continuity in t of the positive solutions of (4.4) we note

Corollary 4.2. *Let $f(t, \lambda, x, z)$ satisfy the hypothesis of Theorem 4.2, with $f_t(t, \lambda, x, z)$ continuous in t on T for all $(\lambda, x, z) \in S \times \bar{D}_+$ and in $C^\alpha(\bar{D}_+)$ with respect to (x, z) uniformly for all $(t, \lambda) \in T \times S$. Finally let $f_{tz}(t, \lambda, x, 0) \geq 0$ on $T \times S \times D$. Then (4.4) has a unique nonnegative solution $u(t, \lambda, x)$ for each $\lambda \in S_1(t_0)$ and $t \in T$. For each fixed $\lambda \in S_1(t_0)$ the solution and its first and second x -derivatives are Lipschitz continuous in t on T .*

Proof. By Theorem 4.2 we know that unique positive solutions of (4.4) exist for all $\lambda \in S_1(t)$ for each $t \in [t_0, t_{\max})$ and that the trivial solution or a unique positive solution exists for all $\lambda \in S_1(t_{\max})$ when $t = t_{\max}$. But $f_{tz}(t, \lambda, x, 0) \geq 0$ implies that $f_z(t, \lambda, x, 0) \geq f_z(t_0, \lambda, x, 0)$ for $t \geq t_0$ and so $S_1(t_0) \subset S_1(t)$ for all $t \in T$. Then by Theorem 4.2 we have the uniform bound $0 \leq u(t, \lambda, x) \leq u(t_0, \lambda, x)$ on D for all $\lambda \in S_1(t_0)$ and $t \in T$.

Now let $\lambda \in S_1(t_0)$, take $t_1, t_2 \in T$ and set $u_1(x) \equiv u(t_1, \lambda, x)$, $u_2(x) \equiv u(t_2, \lambda, x)$. With $w(x) \equiv u_1(x) - u_2(x)$ we obtain the equation in (4.6) for w . Since $u_1(x)$ and $u_2(x)$ are bounded by $u_0(x) \equiv u(t_0, \lambda, x)$ it follows that $q(t_2, \lambda, x, u_1, u_2) \leq f_z(t_2, \lambda, x, u_0) \leq f_z(t_{\max}, \lambda, x, u_0)$, which yields a uniform bound on q . We now apply the estimate in Theorem 7.3 of [3] (recalling Remark 2) to (4.6) to get

$$|w|_{2,\alpha}^D \leq \text{const } |f(t_1, \lambda, x, u_1) - f(t_2, \lambda, x, u_1)|_\alpha^D.$$

Here, in the notation of [3], we employ the norms:

$$|w|_{2,\alpha}^D = \sum_{|i| \leq 2} \sup_D |\partial^i w| + |\partial^2 w|_\alpha^D \quad \text{and} \quad |g|_\alpha^D \equiv \sup_{\substack{P, Q \in D \\ P \neq Q}} \frac{|g(P) - g(Q)|}{|P - Q|^\alpha}.$$

However from the smoothness of f_t we have that

$$f(t_1, \lambda, x, u_1) - f(t_2, \lambda, x, u_1) = \int_0^1 f_t(st_1 + [1-s]t_2, \lambda, x, u_1) ds (t_1 - t_2)$$

and so

$$|f(t_1, \lambda, x, u_1(x)) - f(t_2, \lambda, x, u_1(x))|_\alpha^D \leq \sup_{t \in T} |f_t|_\alpha^D \cdot |t_1 - t_2|.$$

The result clearly follows.

Q.E.D.

Suppose $f_1(\lambda, x, z)$ is an *unforced* nonlinearity and $f_2(\lambda, x, z)$ is a *forced* nonlinearity, satisfying H_1^1 for all λ in a common set S . In addition let $f_1(\lambda, x, z) > f_2(\lambda, x, z)$ on $S \times \bar{D}_+$ and $f_{1,z}(\lambda, x, 0) \geq f_{2,z}(\lambda, x, 0)$ on $S \times D$. Then the family

$$(4.7) \quad f(t, \lambda, x, z) \equiv tf_1(\lambda, x, z) + (1-t)f_2(\lambda, x, z), \quad t \in [0, 1] \equiv T$$

satisfies the hypothesis of Theorem 4.2 and its corollary so that $S_1(0) \subset S_1(t)$ for all $t \in T$. Clearly problem (1.1) with $f \equiv f_1$ has only the trivial solution $u_1(\lambda, x) \equiv 0$ for all $\lambda \in S_1(0)$ by Theorem 3.1. Also problem (1.1) with $f \equiv f_2$ has a positive solution, $u_2(\lambda, x) > 0$, for all $\lambda \in S_1(0)$ by Theorem 3.2. But by Theorem 4.2 the problem (4.4) with f given in (4.7) has positive solutions $u(t, \lambda, x)$ for all $t \in [0, 1]$. These solutions are monotone decreasing functions of t with $u(1, \lambda, x) = u_1(\lambda, x) \equiv 0$ and $u(0, \lambda, x) = u_2(\lambda, x)$. Further we see from Corollary 4.2 that $u(t, \lambda, x)$ is continuous in t on T . Thus the positive solutions for $\lambda \in S_1(0)$ continuously deform to the trivial solution as $f_2(\lambda, x, z)$ is transformed into $f_1(\lambda, x, z)$.

5. Extension of existence range. If it is known that a positive solution exists for some $\lambda_0 \in S$ we can show, under appropriate conditions, that positive solutions also exist for λ in an interval above λ_0 and/or below λ_0 . These results apply to both forced and unforced cases but require additional monotonicity conditions on f_z with respect to λ . Specifically for solutions above λ_0 we have:

Theorem 5.1. *Let $f(\lambda, x, z)$ satisfy H_0 , H_1^1 , H_2 and*

$$H_3 : f_z(\lambda, x, z) \geq f_z(\lambda', x, z) \text{ on } D \text{ for all } z \geq 0 \text{ and } \lambda > \lambda' \text{ in } S.$$

Then if (1.1) has a positive solution, $u(\lambda_0, x)$, for some $\lambda_0 \in S$ it also has a positive solution, $u(\lambda, x)$, for every $\lambda \in S_+(\lambda_0) \equiv \{\lambda \mid \lambda \in S, \lambda_0 \leq \lambda < \lambda_0 + \delta\lambda_0\}$ where

$$(5.1) \quad \delta\lambda_0 = \frac{\int_D \int_0^1 [f_z(\lambda_0, x, u(\lambda_0, x)) - f_z(\lambda_0, x, tu(\lambda_0, x))] dt \varphi_1^2(x) dx}{\int_D f(x) \varphi_1^2(x) dx}$$

and $\varphi_1(x)$ is the eigenfunction belonging to $\mu_1\{f_z(\lambda_0, x, u(\lambda_0, x))\}$ in (2.2). Further $u(\lambda, x) = \lim_{n \rightarrow \infty} u_n(x)$ where $\{u_n(x)\}_{n=1}^\infty$ is the monotone decreasing sequence of Newton iterates defined by (3.3) with initial iterate $u_0(x) \equiv u(\lambda_0, x)$.

Proof. From (2.3) using $\varphi \equiv \varphi_1(x)$ we easily find that

$$\mu_1\{f_z(\lambda_0, x, u(\lambda_0, x))\} \geq \mu_1\{q(\lambda_0, x, u(\lambda_0, x), 0)\} + \delta\lambda_0.$$

But just as in the proof of Theorem 4.1 we have $\lambda_0 \leq \mu_1\{q(\lambda_0, x, u(\lambda_0, x), 0)\}$. Thus it follows that all $\lambda \in S_+(\lambda_0)$ satisfy

$$\lambda < \mu_1\{f_z(\lambda_0, x, u(\lambda_0, x))\}.$$

Now let $\lambda \in S_+(\lambda_0)$ and use the fact that $u_0(x) \equiv u(\lambda_0, x)$ is a solution of (1.1) to get from (3.3) with $n = 0$:

$$\begin{aligned} L(u_1 - u_0) + [\lambda r(x) - f_z(\lambda, x, u_0)](u_1 - u_0) \\ = f(\lambda, x, u_0) - f(\lambda_0, x, u_0) - (\lambda - \lambda_0)r(x)u_0, \\ < 0 \quad \text{on } D. \end{aligned}$$

The inequality follows from H_2 since $\lambda > \lambda_0$ and $u_0(x) > 0$ and D . From H_3 we have $f_z(\lambda, x, u_0) \geq f_z(\lambda_0, x, u_0)$ and so using (2.3) again gives $\lambda < \mu_1\{f_z(\lambda, x, u_0)\}$. Thus the Positivity Lemma is applicable above and implies $u_1(x) > u_0(x)$ on D . By induction it can now be shown that $u_n(x) > u_0(x)$ on D for all $n = 1, 2, \dots$.

From (3.3) we have for $n \geq 1$:

$$\begin{aligned} L(u_{n+1} - u_n) + [\lambda r(x) - f_z(\lambda, x, u_n)](u_{n+1} - u_n) \\ = [q(\lambda, x, u_n, u_{n-1}) - f_z(\lambda, x, u_{n-1})](u_n - u_{n-1}). \end{aligned}$$

However since $u_1(x) > u_0(x) > 0$ on D it follows that

$$[q(\lambda, x, u_1, u_0) - f_z(\lambda, x, u_0)](u_1 - u_0) > 0.$$

Then the Positivity Lemma, as previously justified, yields $u_2(x) < u_1(x)$ on D . If $0 < u_n(x) < u_{n-1}(x)$ on D we also get that

$$[q(\lambda, x, u_n, u_{n-1}) - f_z(\lambda, x, u_{n-1})](u_n - u_{n-1}) > 0 \quad \text{on } D$$

and so by induction it is concluded that $u_{n+1}(x) < u_n(x)$ on D for $n = 1, 2, \dots$.

Thus we have shown that the sequence of Newton iterates $\{u_n(x)\}_{n=1}^\infty$ is monotone decreasing and bounded below by $u(\lambda_0, x)$. The proofs of existence,

smoothness and convergence to a solution of (1.1) for these iterates are identical to the proofs of these facts for Theorem 3.2. Q.E.D.

Next we show, under the same conditions of Theorem 5.1, that positive solutions exist for $\lambda < \lambda_0$ if $\lambda > \mu_1(\lambda)$. Specifically we state

Theorem 5.2. *Let $f(\lambda, x, z)$ satisfy H_0, H_1^1, H_2 and H_3 . Then if (1.1) has a positive solution, $u(\lambda_0, x)$, for some $\lambda_0 \in S$ it also has a positive solution, $u(\lambda, x)$, for each $\lambda \in S_-(\lambda_0) \equiv \{\lambda \mid \lambda \in S, \mu_1(\lambda) < \lambda \leq \lambda_0\}$. Further $u(\lambda, x) = \lim_{n \rightarrow \infty} v_n(x)$ where $\{v_n(x)\}_{n=0}^\infty$ is the monotone increasing sequence defined by:*

$$(5.2) \quad \left. \begin{aligned} Lv_{n+1} + [\lambda r(x) - g_0(x)]v_{n+1} &= f(\lambda, x, v_n) - g_0(x)v_n \\ Bv_{n+1} &= 0 \end{aligned} \right\} \quad n = 0, 1, \dots$$

with $g_0(x) \equiv f_z(\lambda_0, x, u(\lambda_0, x))$ and $v_0(x)$ given in (5.5).

Proof. For any fixed $\lambda \in S_-(\lambda_0)$ let $\varphi_1(\lambda, x)$ be an eigenfunction for (2.2) with $g(x) \equiv f_z(\lambda, x, 0)$ belonging to $\mu_1(\lambda) \equiv \mu_1\{f_z(\lambda, x, 0)\}$. Since $\mu_1(\lambda)$ is the least eigenvalue we can make $\varphi_1(\lambda, x)$ unique by specifying that

$$(5.3a) \quad \varphi_1(\lambda, x) > 0 \quad \text{on } D, \quad \max_{x \in D} \varphi_1(\lambda, x) = 1.$$

We shall now show that there exists some $\epsilon_0 > 0$ such that

$$(5.3b) \quad \epsilon \varphi_1(\lambda, x) < u(\lambda_0, x) \quad \text{on } D \quad \text{for all } \epsilon \quad \text{in } 0 < \epsilon < \epsilon_0.$$

This clearly follows if

$$R(x) \equiv \frac{\varphi_1(\lambda, x)}{u(\lambda_0, x)}$$

is continuous on $\bar{D}$ as then we merely set $\epsilon_0^{-1} = \max_D R(x)$. But since $\varphi_1(\lambda, x)$ and $u(\lambda_0, x)$ are positive on D and continuous on $\bar{D}$ we need only study the continuity of $R(x)$ at those points $x^0 \in \partial D$ at which $u(\lambda_0, x^0) = 0$.

Employing (2.1) in (1.1) with $\lambda = \lambda_0$ we obtain

$$\begin{aligned} Lu(\lambda_0, x) - q(\lambda_0, x, u(\lambda_0, x), 0)u(\lambda_0, x) &= f(\lambda_0, x, 0) - \lambda_0 r(x)u(\lambda_0, x), \\ &< 0 \quad \text{on } D. \end{aligned}$$

Here we have used H_0 and $\lambda_0 > 0$ which follows from $\mu_1(\lambda) > 0$. Clearly also $q(\lambda_0, x, u(\lambda_0, x), 0) + a_0(x) \geq 0$ on D , by H_1^1 , and $u(\lambda_0, x) > 0$ on D so the strong maximum principal holds and implies that if $u(\lambda_0, x^0) = 0$ at $x^0 \in \partial D$ then

$$(5.4) \quad \frac{\partial u(\lambda_0, x^0)}{\partial \alpha} < 0$$

for every "outward" directional derivative, $\partial/\partial \alpha$, at x_0 (see Theorem 8, Chap. 2, of Protter and Weinberger [6]). But the conormal derivative, $\partial/\partial \nu$, is outward since

$$\sum_{i,j=1}^m n_i(x^0) a_{ij}(x^0) n_j(x^0) > 0 \quad [\text{see (1.2b, c)}].$$

Thus we have that

$$\frac{\partial u(\lambda_0, x^0)}{\partial \nu} < 0 \quad \text{at } x^0 \in \partial D$$

if $u(\lambda_0, x^0) = 0$.

However $u(\lambda_0, x)$ satisfies the boundary conditions

$$Bu \equiv \beta_0(x)u(\lambda_0, x) + \beta_1(x) \frac{\partial u(\lambda_0, x)}{\partial \nu} = 0 \quad \text{on } \partial D.$$

We recall that $\beta_1(x) = 0$ on ∂D_1 and $\beta_1(x) > 0$ on ∂D_2 . Then it follows that $u(\lambda_0, x) \neq 0$ for $x \in \partial D_2$ and we need study $R(x)$ only on ∂D_1 where (5.4) holds. From the problem (2.2) with $g(x) \equiv f_z(\lambda, x, 0)$ and $\mu = \mu_1(\lambda)$ it is clear that the above argument applies to $\varphi_1(\lambda, x)$. So if we take for the outward derivative, $\partial/\partial\alpha$, the normal derivative in (5.4) we have that

$$\frac{\partial u(\lambda_0, x)}{\partial n} < 0, \quad \frac{\partial \varphi_1(\lambda, x)}{\partial n} < 0 \quad \text{on } \partial D_1.$$

Now take any $x^0 \in \partial D_1$ and let $x \rightarrow x^0$ along an interior ray making an angle θ with the normal to ∂D_1 at x^0 . If s is the distance along the ray from x^0 to x we have:

$$\begin{aligned} \varphi_1(\lambda, x) &= s \left[\cos \theta \frac{\partial \varphi_1(\lambda, x^0)}{\partial n} + \sin \theta \frac{\partial \varphi_1(\lambda, x^0)}{\partial \tau} \right] + \mathcal{O}(s^2), \\ u(\lambda_0, x) &= s \left[\cos \theta \frac{\partial u(\lambda_0, x^0)}{\partial n} + \sin \theta \frac{\partial u(\lambda_0, x^0)}{\partial \tau} \right] + \mathcal{O}(s^2), \end{aligned}$$

where $\partial/\partial\tau$ denotes the tangential derivative at x^0 in the hyperplane containing the normal and the ray in question. However since $u(\lambda_0, x)$ and $\varphi_1(\lambda, x)$ vanish identically on ∂D_1 , both tangential derivatives vanish and we obtain, for all interior rays

$$\lim_{x \rightarrow x^0} R(x) = \frac{\partial \varphi_1(\lambda, x^0)/\partial n}{\partial u(\lambda_0, x^0)/\partial n} > 0 \quad \text{all } x^0 \in \partial D_1.$$

The continuity of $R(x)$ is thus demonstrated and (5.3b) is justified.

For some ϵ in $0 < \epsilon < \epsilon_0$ we now take for the initial iterate

$$(5.5a) \quad v_0(x) = \epsilon \varphi_1(\lambda, x) < u(\lambda_0, x) \quad \text{on } D.$$

Specifically using the continuity of $f_z(\lambda, x, z)$ in z we require that $\epsilon = \epsilon(\lambda)$ be so small that for the fixed $\lambda \in S_-(\lambda_0)$:

$$(5.5b) \quad \int_0^1 [f_z(\lambda, x, t\epsilon\varphi_1(\lambda, x)) - f_z(\lambda, x, 0)] dt < [\lambda - \mu_1(\lambda)]r(x) \quad \text{on } D.$$

We now show by induction that

$$(5.6) \quad 0 < v_0(x) < v_1(x) < \cdots < v_n(x) < \cdots < u(\lambda_0, x) \quad \text{on } D.$$

From (5.2) with $n = 0$ and the form of (2.2) satisfied by $v_0(x)$ we obtain

$$\begin{aligned} L(v_1 - v_0) + [\lambda r(x) - g_0(x)](v_1 - v_0) \\ &= f(\lambda, x, v_0) - f_z(\lambda, x, 0)v_0 + [\mu_1(\lambda) - \lambda]r(x)v_0 \\ &= f(\lambda, x, 0) + \{[q(\lambda, x, v_0, 0) - f_z(\lambda, x, 0)] + [\mu_1(\lambda) - \lambda]r(x)\}v_0 \\ &< 0 \quad \text{on } D. \end{aligned}$$

Here we have used H_0 and (5.5b). The Positivity Lemma is applicable since $\lambda \leq \lambda_0 < \mu_1\{g_0(x)\}$ and thus $v_1(x) > v_0(x)$ on D .

To complete the induction we assume that $v_{n-1}(x) < v_n(x) < u(\lambda_0, x)$ on D . Then from (5.2):

$$\begin{aligned} L(v_{n+1} - v_n) + [\lambda r(x) - g_0(x)](v_{n+1} - v_n) \\ &= f(\lambda, x, v_n) - f(\lambda, x, v_{n-1}) - g_0(x)(v_n - v_{n-1}) \\ &= [q(\lambda, x, v_n, v_{n-1}) - q(\lambda_0, x, u, u)](v_n - v_{n-1}) \\ &< 0 \quad \text{on } D \end{aligned}$$

by H_1^1 , H_3 and the inductive assumption. We thus deduce that $v_{n+1}(x) > v_n(x)$ on D . Finally since $u(\lambda_0, x) \equiv u$ is a solution of (1.1) with $\lambda = \lambda_0$ we have

$$\begin{aligned} L(u - v_{n+1}) + [\lambda r(x) - g_0(x)](u - v_{n+1}) \\ &= f(\lambda_0, x, u) - f(\lambda, x, v_n) - g_0(x)(u - v_n) - (\lambda_0 - \lambda)r(x)u, \\ &= [f(\lambda_0, x, u) - f(\lambda, x, u) - (\lambda_0 - \lambda)r(x)u] + [q(\lambda, x, u, v_n) - q_0(x)](u - v_n) \\ &< 0 \quad \text{on } D. \end{aligned}$$

Here we have used H_1^1 and H_2 . Thus $u(x) > v_{n+1}(x)$ on D and (5.6) is established.

The fact that $\lim_{n \rightarrow \infty} v_n(x) \equiv u(\lambda, x)$ is a positive solution of (1.1) now follows as in the proof of Theorem 3.2. Q.E.D.

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