Linear Functionals and the Cauchy-Kowalewski Theorem

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1. Introduction. The purpose of this note is to show how the classical Cauchy-Kowalewski theorem, the Cauchy integral formula, and the calculus of residues in several complex variables can all three be combined to yield, in a constructive fashion, formulas which represent the solution of the linear non-characteristic Cauchy problem as a linear functional of its Cauchy data. The formal construction is based on an extremely simple idea. Its rigorous justification relies on the method of majorants and uses essentially the same bounds that occur in the proof of the Cauchy-Kowalewski theorem. In many special cases where it can be calculated explicitly the general representation we obtain here reduces to well known classical formulas.

Recall that the non-characteristic Cauchy problem for an arbitrary linear system can be reduced to a Cauchy problem for an $N \times N$ linear first-order system in $n + 1$ variables of the form

$$(1) \quad \frac{\partial u}{\partial t} - \sum_{k=1}^n a^k(t, x) \frac{\partial u}{\partial x_k} - b(t, x)u = g(t, x)$$

$$u(0, x) = f(x).$$

Here everything is complex: $t \in \mathbf{C}$, $x = (x_1, x_2, \dots, x_n) \in \mathbf{C}^n$, the a^k, b are $N \times N$ matrices, and the u, g, f are N -tuples. We assume that the a^k, b, f, g are holomorphic functions of all of their arguments in some polydisc. Then according to the Cauchy-Kowalewski theorem there is some smaller polydisc in which there exists a unique holomorphic solution u of (1).

We claim that u has a representation of the form

$$(2) \quad u(t, x) = \text{Res} \int_{\partial(0)} K(t, x; 0, \xi) f(\xi) d\xi$$

$$+ \int_0^t \text{Res} \int_{\partial(\tau)} K(t, x; \tau, \xi) g(\tau, \xi) d\xi d\tau.$$

The precise meaning of (2) will be explained below. When $g \equiv 0$ equation (2)

represents the solution of the homogeneous problem corresponding to (1) as a linear functional of the Cauchy data f . When $f \equiv 0$ equation (2) represents the solution of the inhomogeneous problem with homogeneous initial conditions as a linear functional of the inhomogeneous term g .

Our analysis was motivated by the work of Garabedian [1], [2] and by the similar techniques used in [4], [5]. In [1] the case of a single equation of the second order was discussed from a different point of view. There the point of departure was the divergence theorem and the constructive part of the analysis relied on the fundamental solution constructed for such equations by Hadamard [3]. Here we treat instead the general case and base our analysis on the Cauchy integral formula and the Cauchy–Kowalewski theorem. It should be pointed out that Hadamard’s construction was itself based on arguments similar to those used to prove the Cauchy–Kowalewski theorem. For other related work and details on the calculus of residues in several complex variables see [6] and [7].

2. The formal construction. In this section we proceed with a purely formal derivation and postpone until Section 3 a rigorous justification of the method.

By means of Duhamel’s principle one can reduce the general problem (1) to the problem of solving a homogeneous system

$$(3) \quad \frac{\partial u}{\partial t} - \sum_{k=1}^n a^k(t, x) \frac{\partial u}{\partial x_k} - b(t, x)u = 0$$

$$u(\tau, x) = f(x).$$

Hence we shall concentrate on (3) and represent its solution as

$$(4) \quad u(t, x) = \text{Res} \int_{\partial(\tau)} K(t, x; \tau, \xi) f(\xi) d\xi.$$

In what follows $t, \tau \in \mathbf{C}; x, \xi \in \mathbf{C}^n; z = (t - \tau, x) \in \mathbf{C}^{n+1}; \alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ where the α_i, β_j are non-negative integers. We use the customary notation $\alpha! = \alpha_0! \alpha_1! \dots \alpha_n!$, $|\alpha| = \alpha_0 + \alpha_1 + \dots + \alpha_n$,

$$z^\alpha = (t - \tau)^{\alpha_0} x_1^{\alpha_1} \dots x_n^{\alpha_n}, \quad D^\alpha = \partial_t^{\alpha_0} \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n},$$

$$D_x^\beta = \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2} \dots \partial_{x_n}^{\beta_n}, \quad \text{etc.}$$

Consider the power series expansion of u about the point $(\tau, 0)$:

$$(5) \quad u(t, x) = \sum_{\alpha} \frac{1}{\alpha!} D^\alpha u(\tau, 0) z^\alpha.$$

According to the construction used in the Cauchy–Kowalewski Theorem, the coefficients in (5) are determined recursively through tangential differentiation D_x^β and use of the relations

$$\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \dots \partial_{x_n}^{\alpha_n} u = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \dots \partial_{x_n}^{\alpha_n} f$$

$$\partial_t^{\alpha_0+1} \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} u = \partial_t^{\alpha_0} \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} \left\{ \sum_{k=1}^n a^k u_{x_k} + bu \right\}$$

evaluated at $(\tau, 0)$. Since the coefficients a^k , b do not depend on u it follows that each $D^\alpha u(\tau, 0)$ depends *linearly* on the derivatives of f and has the form

$$D^\alpha u(\tau, 0) = \sum_{|\beta| \leq |\alpha|} C_{\alpha\beta}(\tau) D_x^\beta f(0).$$

This is an important point, for in general the $D^\alpha u(0, \tau)$ would depend nonlinearly on the $D_x^\beta f(0)$ if the partial differential equation in (3) were nonlinear. However even in the linear case the $N \times N$ matrices $C_{\alpha\beta}(\tau)$ are nonlinear functions of the derivatives of the coefficient matrices a^k , b evaluated at $(\tau, 0)$. Thus the series (5) can be written as

$$(6) \quad u(t, x) = \sum_{\alpha} \sum_{|\beta| \leq |\alpha|} \frac{1}{\alpha!} C_{\alpha\beta}(\tau) D_x^\beta f(0) z^\alpha.$$

Next consider the Cauchy integral formula

$$(7) \quad D_x^\beta f(0) = \frac{\beta!}{(2\pi i)^n} \int_{\partial(\tau)} \frac{f(\xi)}{\xi^{\beta+1}} d\xi.$$

Here $\xi^{\beta+1} = \xi_1^{\beta_1+1} \xi_2^{\beta_2+1} \cdots \xi_n^{\beta_n+1}$ and $d\xi = d\xi_1 \wedge d\xi_2 \wedge \cdots \wedge d\xi_n$. The integration in (7) is over the distinguished boundary $\partial(\tau)$ of some polydisc which is centered at $(\tau, 0)$ and lies in the hyperplane $t = \tau$. Now we insert (7) into (6) and interchange the order of the summation and the integration. The result is

$$(8) \quad u(t, x) = \int_{\partial(\tau)} K(t, x; \tau, \xi) f(\xi) d\xi,$$

where the $N \times N$ matrix K is defined by

$$(9) \quad K(t, x; \tau, \xi) = \frac{1}{(2\pi i)^n} \sum_{\alpha} \sum_{|\beta| \leq |\alpha|} \frac{\beta!}{\alpha!} C_{\alpha\beta}(\tau) \frac{z^\alpha}{\xi^{\beta+1}}.$$

Formula (8) provides a representation of the solution of (3) as a linear functional of its data; the representation (4) is obtained from (8) by computing a residue.

Let t, x, τ be fixed and suppose K is holomorphic in ξ . Since f is holomorphic, the integrand in (8) is a closed form of type $(n, 0)$. Therefore by Stokes' Theorem the validity of (8) is unchanged if the original path of integration $\partial(\tau)$ is replaced by any other cycle $\partial'(\tau)$ which lies in the hyperplane $t = \tau$ and is homologous to $\partial(\tau)$ in a region where K remains holomorphic in ξ . It turns out that K is holomorphic in ξ provided $\min_{1 \leq i \leq n} |\xi_i|$ is sufficiently large, but K has singularities for small $|\xi_i|$. Therefore $\partial(\tau)$ can be taken large at the outset in (8) and then shrunk down around the singularities of K . This is the meaning of the residue indicated by (4).

When $t = \tau$ the series (5) must reduce to the Taylor series for $f(x)$. Hence in that case (8) must reduce to the Cauchy integral formula for f . Thus

$$(10) \quad K(\tau, x; \tau, \xi) = \frac{I}{(2\pi i)^n (\xi - x)},$$

where $(\xi - x) = (\xi_1 - x_1)(\xi_2 - x_2) \cdots (\xi_n - x_n)$ and I is the $N \times N$ unit matrix. It follows that for holomorphic functions f , K has the reproducing property

$$(11) \quad \int_{\partial(\tau)} K(\tau, x; \tau, \xi) f(\xi) d\xi = f(x).$$

3. Convergence. In this section we use the method of majorants to justify the formal construction described in Section 2. In particular we shall prove that the series (9) defining K converges absolutely and uniformly provided that $\min_{1 \leq i \leq n} |\xi_i|$ is sufficiently large relative to $|t - \tau|$ and $\max_{1 \leq i \leq n} |x_i|$. In that case the interchange of summation and integration leading to (8) is justified. Hence the representation (8) is valid provided $\partial(\tau)$ is taken large enough.

In what follows let $a^k = [a_{ij}^k]$, $b = [b_{ij}]$, and assume that the a_{ij}^k , b_{ij} are holomorphic in all of their variables in some polydisc

$$\Delta(a) = \{(t, x) \mid |t - \tau| \leq a, |x_i| \leq a, i = 1, 2, \dots, n\}.$$

M denotes an upper bound for the maximum modulus of all the a_{ij}^k , b_{ij} on $\Delta(a)$ and δ is a number $0 < \delta < 1$ to be chosen later. For convenience in notation set

$$\zeta = a - [(t - \tau)\delta^{-1} + x_1 + x_2 + \cdots + x_n].$$

Then the function

$$(12) \quad m(t, x) \equiv \frac{aM}{a - \zeta}$$

is a majorant for all the coefficients a_{ij}^k , b_{ij} . This means that at the point $(\tau, 0)$ any derivative of m is a positive quantity which forms an upper bound for the modulus of the same derivative of any of the a_{ij}^k , b_{ij} . If

$$(13) \quad \min_{1 \leq i \leq n} |\xi_i| \geq b > 0$$

then the relation

$$(14) \quad \left| \frac{\beta!}{\xi^\beta} \right| \leq \frac{|\beta|!}{b^{|\beta|}} = D_x^\beta F(0)$$

holds for the function

$$F(x) = \frac{b}{b - (x_1 + x_2 + \cdots + x_n)}.$$

Since only differentiations and substitutions were involved in the derivation of the series (9), no minus signs were introduced by that construction. Hence if in (3) the coefficients a_{ij}^k , b_{ij} are replaced by their majorants (12), f is replaced by F , and condition (13) holds, then the corresponding series (6) for this new problem will dominate (9). Actually we shall consider the majorizing problem

$$(15) \quad \frac{\partial u_i}{\partial t} = m \left[\sum_{j,k} \frac{\partial u_j}{\partial x_k} + \frac{M'}{M} \sum_i u_i \right]$$

$$u_i(\tau, x) = F(x); \quad i = 1, 2, \dots, N$$

for some $M' \geq M$. Thus in order to establish the absolute convergence of the series (9) under condition (13) it suffices to show, by solving (15) explicitly, that (15) has a holomorphic solution in a neighborhood of $(\tau, 0)$.

Let ρ be fixed such that

$$0 < \rho < \frac{a}{n+1}$$

and choose

$$0 < \delta < \min \left\{ 1, \frac{1}{aNM}, \frac{a - (n+1)\rho}{anNM} \right\}.$$

Setting

$$M' = (\delta a N)^{-1}$$

$$b = a - \delta a n N M,$$

it follows that $M' > M$ and $(n+1)\rho < b < a$. With this choice of δ , M' , b it is easily verified by direct calculation that the solution of (15) is given by

$$(16) \quad u_i(t, x) = \frac{b}{b - \xi}$$

for $i = 1, 2, \dots, N$. The functions (16) are holomorphic and their Taylor expansions about $(\tau, 0)$ corresponding to (6) converge in the polydisc

$$\Delta(\delta, \rho) = \{(t, x) \mid |t - \tau| \leq \delta\rho, |x_i| \leq \rho, i = 1, 2, \dots, n\}.$$

The conclusion is the following: For $(t, x) \in \Delta(\delta, \rho)$ and ξ satisfying (13) the series (9) converges absolutely and uniformly. Hence in that range $K(t, x; \tau, \xi)$ is holomorphic in all of its variables and the representation (8) is valid.

4. Examples. We present some simple examples of the procedure described in Section 2.

Consider the case where the $N \times N$ coefficient matrices $a^1, a^2, \dots, a^n, b$ in (1) are all *constant* and *commutative*. Then the series corresponding to (9) is easily computed. The result is

$$(17) \quad K(t, x; \tau, \xi) = \frac{1}{(2\pi i)^n} e^{(t-\tau)b} \prod_{k=1}^n [(\xi_k - x_k)I - (t - \tau)a^k]^{-1}.$$

Note that K is holomorphic except for singularities which occur along certain complex characteristics associated with (1). These are poles at $\xi_k = x_k + \lambda_k^{(j)}(t - \tau)$, where $\lambda_k^{(j)}$ is the j -th eigenvalue of a^k , of orders which depend on the spectral properties of the matrices a^k . When (17) is put into (8) and the residue indicated by (4) is computed, it is the order of these poles that determines

the number of derivatives of the Cauchy data f which appear in the final representation.

As a specific example consider the case where $n = N = 2$, $g \equiv b \equiv 0$, and

$$a^1 = a^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Setting $(x_1, x_2) = (x, y)$ we then have the system

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial y} \\ (18) \quad \frac{\partial u_2}{\partial t} &= \frac{\partial u_2}{\partial x} + \frac{\partial u_2}{\partial y} \\ u_i(0, x, y) &= f_i(x, y), \quad i = 1, 2. \end{aligned}$$

Formula (17) in this case becomes

$$K = \frac{1}{(2\pi i)^2} \cdot \begin{bmatrix} \frac{1}{(\xi - x - t)(\eta - y - t)} & \frac{t}{(\xi - x - t)^2(\eta - y - t)} + \frac{t}{(\xi - x - t)(\eta - y - t)^2} \\ 0 & \frac{1}{(\xi - x - t)(\eta - y - t)} \end{bmatrix}$$

where $(\xi_1, \xi_2) = (\xi, \eta)$. After inserting the above expression for K into (8) and computing the residue, we find

$$\begin{aligned} (19) \quad u_1(t, x, y) &= f_1(x + t, y + t) + t \partial_x f_2(x + t, y + t) + t \partial_y f_2(x + t, y + t) \\ u_2(t, x, y) &= f_2(x + t, y + t). \end{aligned}$$

It is easily checked that (19) does indeed solve (18).

As a second specific example of (17), take the trivial case $n = N = 1$ with constant a, b :

$$\begin{aligned} (20) \quad \frac{\partial u}{\partial t} - a \frac{\partial u}{\partial x} - bu &= g(t, x) \\ u(0, x) &= f(x). \end{aligned}$$

Computing K and the residues indicated by equation (2), we obtain

$$(21) \quad u(t, x) = e^{tb} f(x + at) + \int_0^t e^{(t-\tau)b} g(\tau, x + a(t - \tau)) d\tau.$$

Again it is easily verified that (21) is the solution of (20).

5. Remarks and conjectures. Now we return our attention to the general situation. It follows from the analysis of Section 3 that K , defined by (9), is holomorphic in a set of the type shown in Figure 1, but K has the singularity (10) at $t = \tau$. In the general case this is all we have actually proved. In the

special examples discussed above, however, K turned out to be holomorphic also in the top half of the region shown in Figure 1 except for singularities which occurred along characteristic varieties. The same phenomenon prevails in all other special cases the author has encountered.

We conjecture that this is true in general: That K can be continued (perhaps into the top half of the region shown in Figure 1) as a holomorphic function except for singularities which occur along certain characteristic varieties. In this connection both poles and essential singularities may be present. Essential singularities occur, for example, in the case of the heat equation [4], [5].

FIGURE 1

If K is meromorphic then the result of the residue calculation indicated in (2) or (4) should be a linear functional which involves only a finite number of derivatives of the Cauchy data. But if K has more complicated singularities, such as essential singularities, the resulting linear functional may involve an infinite number of derivatives of the data [4].

We hope to return to these questions at a later date.

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