Absolute Continuity and Generalized Reynolds Operators

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Communicated by GIAN-CARLO ROTA

1. Introduction. Let (S, Σ, μ) denote a totally σ -finite measure space and let R denote a bounded self-adjoint operator on the complex Hilbert space $L^2(S) = L^2(S, \Sigma, \mu)$. Suppose that g, h are fixed elements satisfying

$$(1.1) \quad g \in L^2(S) \quad \text{and} \quad g, h, Rg \in L^\infty(S),$$

and that R satisfies the generalized "Reynolds equation" (see Section 2 below):

$$(1.2) \quad R(fRg + gRf) = RfRg + cR(RfRg), \quad c = \text{const.},$$

where

$$(1.3) \quad f = \bar{h}f', \quad f' \text{ arbitrary in } L^2(S).$$

Finally, suppose that

$$(1.4) \quad hR \operatorname{Im} [(I - cR)g]R\bar{h} \geq 0,$$

where $h, \bar{h}$ and $\operatorname{Im} [(I - cR)g]$ are regarded as multiplication operators. It will be shown how some results concerning the absolute continuity of the bounded self-adjoint operator Q on $L^2(S)$, defined by

$$(1.5) \quad Q = hR\bar{h},$$

can be obtained using the methods of commutators.

First, recall that the absolutely continuous part of a self-adjoint operator A with spectral resolution $A = \int \lambda \, dE_\lambda$ on a Hilbert space $\mathfrak{H}$ is the subspace $\mathfrak{H}_a$ of elements f for which $\|E_\lambda f\|^2$ is an absolutely continuous function of λ on $(-\infty, \infty)$. The operator A is said to be absolutely continuous if $\mathfrak{H}_a = \mathfrak{H}$. The following result is sometimes useful for determining the space $\mathfrak{H}_a(A)$ in case A is bounded:

(*) *Let A, B and C denote bounded self-adjoint operators on a Hilbert space $\mathfrak{H}$ satisfying*

$$(1.6) \quad AB - BA = -iC, \quad C \geq 0 \quad \text{or} \quad C \leq 0.$$

This work was supported by a National Science Foundation research grant.

Then the space $\mathfrak{S}_a(A)$ contains the smallest space reducing both A and B and containing the range of C . In particular, if 0 is not in the point spectrum of C , then $\mathfrak{S}_a(A) = \mathfrak{S}$, so that A is absolutely continuous.

For a proof of (*), see Putnam [4], p. 20.

Consider the commutator $Q(Rg) - (Rg)Q = -D$, where, by (1.2) and (1.3), $Df = hR[(I - cR)g]R(\bar{h}f)$, for $f \in L^2(S)$. On taking adjoints, one obtains

$$(1.7) \quad Q \operatorname{Re} (Rg) - \operatorname{Re} (Rg)Q = -i \operatorname{Im} (D),$$

where, by (1.4),

$$(1.8) \quad (\operatorname{Im} (D)f, f) = \int_S \operatorname{Im} [(I - cR)g] |R(\bar{h}f)|^2 d\mu \geq 0.$$

Thus, Q , $\operatorname{Re} (Rg)$ and $\operatorname{Im} (D)$ can be identified with A , B and C respectively of (*).

Let E be a measurable subset of S of positive measure (possibly $E = S$) with the property that

$$(1.9) \quad h \neq 0 \quad \text{on } E \quad \text{and} \quad h = 0 \quad \text{on } S - E.$$

Then $L^2(E)$ reduces both Q and the multiplication operator Rg , and (1.7) can be regarded as an operator equation on $L^2(E)$. (It is to be noted that $L^2(E)$ need not reduce R so that in (1.8) the integration domain S cannot in general be replaced by E .)

It may be noted that a sufficient condition for the validity of (1.4) is that

$$(1.10) \quad \operatorname{Im} [(I - cR)g] \geq 0$$

hold a.e. on S and that in this case, from the point of view of an application of (*), it is desirable that the strict inequality hold on as large a set as possible. On the other hand, it is possible that (1.4) is satisfied provided only that (1.10) holds on some measurable subset F of S ; this is the case, for instance, if $R(L^2(E)) \subset L^2(F)$.

If R satisfies (1.2) for some c then bR ($b \neq 0$) satisfies (1.2) with c replaced by c/b . From this it is clear that there exist two essentially different relations (1.2), corresponding to $c = 0$ or $c = 1$, that is,

$$(1.11) \quad R(fRg + gRf) = RfRg$$

or

$$(1.12) \quad R(fRg + gRf) = RfRg + R(RfRg).$$

Examples of operators satisfying (1.11) or (1.12) will be discussed below. In Section 2 some background material dealing with these relations will be given. In Section 3 there will be considered the Hilbert transform which, under conditions stated there, satisfies (1.11). In Section 4, operators satisfying (1.12) will be considered. In particular, certain self-adjoint operators R arising in connection with measure preserving transformations $\{T^n\}$ will be discussed in

relation to the Theorem (*). Section 5 will deal with similar operators which, though not self-adjoint, are normal, and which can be investigated by means of a Theorem (**) related to (*). Connections between commutators and the measures of the spectra of the above self-adjoint operators R associated with $\{T'\}$, or of the infinitesimal generators of the related unitary groups, will be noted in Section 6. Finally, Section 7 will be devoted to a discussion of the applicability of the methods considered earlier.

2. Some background and comparative remarks. If, for the moment, R is regarded as a linear mapping of an algebra into itself, then the equation (1.11), where f, g are arbitrary elements of the algebra, is a special case of an identity considered by Baxter [2], namely,

$$(2.1) \quad T(fTg + gTf) = TfTg + T(cfg),$$

where T is a linear mapping of an algebra of elements $f, g, \dots$, into itself and where c is either a fixed element of the algebra or a scalar. Atkinson [1] has also investigated (2.1) as well as some similar identities. Relation (1.12) is known as the Reynolds identity and R a Reynolds operator. For a history of such operators and an extensive bibliography on the subject, see Rota [10, 11]. These latter papers contain results on the structure of Reynolds operators when the underlying space is the algebra $L^\infty(S)$, or the Hilbert space $L^2(S)$, when the domain of R can be thus extended. In addition, R is subject to (some of) certain other conditions such as $R1 = 1$, where 1 is the identity element, or $Rf \geq 0$ if $f \geq 0$, or $\int_S |Rf|^2 d\mu \leq \int_S |f|^2 d\mu$.

In the present paper, these latter restrictions are not made, but are replaced by the crucial condition (1.4) (or, in the discussion of Section 5, by the related condition (5.8)). Further, in [10, 11], in case R can be regarded as an operator on $L^2(S)$, it is not supposed, as has been done so far in the present paper (but not in Section 5), that R is self-adjoint. The severity of this latter restriction is counterbalanced however by the relaxation of (1.12), as well as of (1.11), from a requirement on arbitrary f and g to one in which only f is "arbitrary" while g is fixed. Concerning a restricted form of (2.1), where f and g belong to a subset of the underlying algebra on which T is an operator, see the remarks of Atkinson [1], pp. 3, 25.

Some connections between certain integral operators and relations like (1.11), (1.12), (2.1) or commutators may be noted. As was indicated earlier, the Hilbert transform satisfies (1.11) for certain f and g (see Section 3). Atkinson [1] has pointed out a connection between (2.1) and Wiener-Hopf integral operators on the half-line $[0, \infty)$. It may be further observed that (2.1) is related to commutators, in a manner similar to that of (1.2), through the expression $T(fTg) - TfTg$, where Tg is regarded as a multiplication operator. In addition, commutator methods were used in Putnam [5, 6] to obtain spectral properties of certain self-adjoint Wiener-Hopf operators on $L^2(0, \infty)$. Cf. also Pincus [3] and Rosenblum [8].

3. The Hilbert transform. Concerning (1.11), let H denote the Hilbert transform mapping $L^2(-\infty, \infty)$ onto itself defined by the Cauchy principal value integral

$$(3.1) \quad (Hf)(x) = (i\pi)^{-1} \int_{-\infty}^{\infty} (t-x)^{-1} f(t) dt.$$

Then H is both self-adjoint and unitary and, in fact, the spectrum of H consists of ± 1 each of infinite multiplicity; see Titchmarsh [13].

It was shown in [7] that (1.11) holds with $R = H$ of (3.1) whenever $f, g \in L^2(-\infty, \infty)$, $f(x)g(x) = 0$ a.e., and both g and Hg belong to $L^\infty(-\infty, \infty)$. If $g = ik$, where k is real and $k \in L^2(-\infty, \infty)$, then $Rg = iHk$ is real. If E is a measurable subset of $(-\infty, \infty)$ such that both E and $(-\infty, \infty) - E$ have positive measures, then, as was shown in [7], k can be chosen so as to satisfy also $k = 0$ on E and $k > 0$ on $(-\infty, \infty) - E$, and so that $k, Hk \in L^\infty(-\infty, \infty)$. Since $\text{Im} [(I - cR)g] = \text{Im} (g) > 0$ on $(-\infty, \infty) - E$ and $= 0$ on E , it is seen that g satisfies (1.1) and that (1.4) holds, with $c = 0$. If $L^2(E)$ is regarded in the natural way as a subspace of $L^2(-\infty, \infty)$, then $f(x)g(x) = 0$ a.e. on $(-\infty, \infty)$ for $f \in L^2(E)$ and $g = ik$. Consequently, if h satisfies (1.1) and (1.9), with $S = (-\infty, \infty)$, then (1.11) (that is, (1.2) with $c = 0$) and (1.3) hold. The Theorem (*) can be used to show that $Q = hRh$, as an operator on $L^2(E)$, is absolutely continuous; see [7]. (Rosenblum [9] has established the absolute continuity of Q by other methods.)

4. Reynolds operators. In order to obtain an R satisfying (1.12), consider a continuous one-parameter group of measure preserving transformations $\{T^t\}$, $-\infty < t < \infty$, of (S, Σ, μ) into itself. Let $\{U^t\}$ denote the corresponding unitary group on $L^2(S)$, so that for $f \in L^2(S)$, $U^t : f(x) \rightarrow f(T^t x)$, and let R_a be defined by

$$(4.1) \quad R_a = a \int_0^\infty e^{-at} U^t dt, \quad \text{Re } (a) > 0.$$

(See Rota [11], pp. 72-73, where in fact $\{T^t\}$ is assumed only to be a semi-group.) Then R_a can be regarded as an operator either on $L^2(S)$ or on $L^\infty(S)$. Moreover, it is easily verified that $d(U^t R_a)/dt = aU^t(R_a - I)$ and that, if $f \in L^2(S)$ and g (hence Rg) $\in L^\infty(S)$, one has $U^t(fg) = (U^t f)(U^t g)$. Using these facts, one shows on integrating

$$(4.2) \quad a \int_0^\infty e^{-at} U^t (R_a f R_a g) dt$$

by parts (cf. [11]) that R_a satisfies (1.12) for arbitrary f in $L^2(S)$ and g in $L^\infty(S)$.

If H denotes the infinitesimal self-adjoint generator of the group $\{U^t\}$, so that $U^t = e^{itH}$, then clearly

$$(4.3) \quad R_a = a(aI - iH)^{-1}.$$

Although R_a is normal, it is not self-adjoint for $\text{Re } (a) > 0$. But if $a = \alpha + i\beta$,

where α, β are real and $\alpha > 0$, it is seen, on letting $\alpha \rightarrow 0$, that, if β is not in the spectrum of H , then

$$(4.4) \quad R = R_{i\beta} = \beta(I - H)^{-1}$$

is self-adjoint. (Cf. Rota [10], p. 863). Further, it is clear that R of (4.4) satisfies (1.12) (that is, (1.2) with $c = 1$) for all $f \in L^2(S)$ and for any g satisfying (1.1). Hence, if g can be chosen so as to satisfy also (1.4), for $c = 1$, that is

$$(4.5) \quad hR \operatorname{Im} [(I - R)g]R\bar{h} \geq 0,$$

then, in particular, relations (1.1)–(1.4) hold (with $c = 1$) and (*) is again applicable to the operator $hR\bar{h}$.

5. Non self-adjoint Reynolds operators. The concepts of absolute continuity defined in section 1, preceding the statement of (*), apply equally well to a unitary operator U having the spectral family $\{E_\lambda\}$. A theorem related to (*) is the following:

(**) *Let U denote a unitary operator and B, C bounded self-adjoint operators on a Hilbert space $\mathfrak{H}$ satisfying*

$$(5.1) \quad UBU^* - B = C, \quad C \geq 0 \quad \text{or} \quad C \leq 0.$$

Then $\mathfrak{S}_a(U)$ contains the smallest space reducing U and containing the range of C . In particular, if 0 is not in the point spectrum of C then $\mathfrak{S}_a(U) = \mathfrak{H}$, so that U is absolutely continuous. Moreover, if

$$(5.2) \quad \operatorname{meas} \operatorname{sp} (U) < 2\pi,$$

where “meas” refers to ordinary Lebesgue measure, then $\mathfrak{S}_a(U)$ contains the smallest space reducing both U and B and containing the range of C .

For a proof, see [4], pp. 21–22.

It was noted above that R_a of (4.3) is not a bounded self-adjoint operator unless $a = i\beta$, where β is real and does not belong to the spectrum of H . It will be shown below how certain results concerning the absolute continuity of H can be obtained through a consideration of the normal operator R_a of (4.3) when a is real and positive. It is seen that in this case,

$$(5.3) \quad R + R^* = 2RR^* (=2R^*R), \quad R = R_a \quad \text{and} \quad a > 0.$$

Further, Rg , that is

$$(5.4) \quad (Rg)(x) = a \int_0^\infty e^{-at} g(T^t x) dt, \quad a > 0,$$

is defined not only for $g \in L^2(S)$ but also for any $g \in L^\infty(S)$, and, moreover, in this latter case, $Rg \in L^\infty(S)$. Also, as noted in Section 4, the relation (1.12) holds for arbitrary $f \in L^2(S)$ and $g \in L^\infty(S)$. As in Section 1,

$$(5.5) \quad R(Rg) - (Rg)R = -R(I - R)gR, \quad g \in L^\infty(S),$$

where $(I - R)g$ is a multiplication operator.

In view of (4.3), one has

$$(5.6) \quad U \equiv I - 2R = (H - iaI)(H + iaI)^{-1},$$

so that U is the (unitary) Cayley transform of H/a . It follows from (5.5) that $Rn - nR = RmR$, where $m = (I - R)g$, $n = -Rg$, so, by (5.6), $Un - nU = -2RmR$, that is, $UnU^* - n = -2RmRU^*$. But $RU^* = R(I - 2R^*) = R - 2RR^* = -R^*$, by (5.3), and so

$$(5.7) \quad UnU^* - n = 2RmR^*, \quad m = (I - R)g \quad \text{and} \quad n = -Rg.$$

It will now be supposed that g of $L^\infty(S)$ is also real. Clearly Rg is also real (and in $L^\infty(S)$) and hence m and n are bounded self-adjoint multiplication operators on $L^2(S)$.

In order to be able to apply (**) to U , the condition that RmR^* , that is, since R is non-singular, that m be semi-definite, say

$$(5.8) \quad m = (I - R)g \geq 0 \quad \text{a.e. on } S, \quad g \in L^\infty(S),$$

must be imposed. Further, it is clear that if

$$(5.9) \quad m = (I - R)g > 0 \quad \text{a.e. on } S, \quad g \in L^\infty(S),$$

then 0 is not in the point spectrum of RmR^* , and hence, by (5.7) and (**), U is absolutely continuous. In view of the fact that the absolutely continuous parts of H and of U coincide (cf. (5.6)), it is seen that (5.9) implies the absolute continuity of H .

As an example, let $S = (-\infty, \infty)$ and let μ denote ordinary Lebesgue measure, so that $L^2(S, \Sigma, \mu) = L^2(-\infty, \infty)$. If $T^t = x + t$, then the unitary group $U^t : f(x) \rightarrow f(x + t)$ has the self-adjoint generator ($U^t = e^{itH}$)

$$(5.10) \quad H = -id/dx;$$

cf. Stone [12], p. 441. If, for instance, one puts $g = g(x) = -\text{Arc tan } x$, then (cf. (5.4))

$$(5.11) \quad m(x) = (I - R)g(x) = a \int_0^\infty e^{-at}[g(x) - g(x + t)] dt > 0,$$

for $-\infty < x < \infty$, and so (5.9) holds. It follows that H of (5.10) on $L^2(-\infty, \infty)$ is absolutely continuous (as is well-known). See also [4], p.30.

6. Measure of spectrum. It may be noted that if A, B, C are bounded self-adjoint operators satisfying (1.6) then

$$(6.1) \quad \|C\| \leq (1/2\pi)[\max \text{sp}(B) - \min \text{sp}(B)] \text{ meas sp}(A).$$

Also, if U is unitary and if B and C are bounded self-adjoint operators satisfying

(5.1), then

$$(6.2) \quad \|C\| \leq (1/2\pi) [\max \operatorname{sp}(B) - \min \operatorname{sp}(B)] \operatorname{meas} \operatorname{sp}(U).$$

See [4], p. 16. Thus, these formulas can be used to obtain estimates for the measures of the spectra of R , where R is the self-adjoint operator of (4.4), and of the Cayley transform U of H/a , given in (5.6). Incidentally, it is seen that in this latter case, $\operatorname{meas} \operatorname{sp}(U) = 2\pi$ if and only if $\operatorname{sp}(H) = (-\infty, \infty)$.

7. Remarks. It was seen in Section 5 that (*) can be used to yield information about the operator $Q = hR\bar{h}$ when R is the self-adjoint operator defined by (4.4). However, it was necessary to suppose the existence of a real β not in the spectrum of the self-adjoint infinitesimal generator H , that is,

$$(7.1) \quad \operatorname{sp}(H) \neq (-\infty, \infty).$$

Further, in order to be able to apply (*), it was supposed that g satisfies (1.1) and (1.4) (for $c = 1$), that is, (4.5).

This situation is to be compared with that discussed in Section 5 where the restriction (7.1) was not imposed and where the properties of the generator H were obtained via the non self-adjoint operator R_a of (4.3) with $a > 0$. Moreover, condition (4.5) was replaced by the condition (5.8). If one chooses g to be a purely imaginary element of $L^\infty(S)$ then (5.8) becomes $\operatorname{Im} (I - R)g \geq 0$, which, since R is non-singular, is the same as $R \operatorname{Im} [(I - R)g]R^* \geq 0$. This last condition is seen to correspond to (4.5) for $h = 1$. (Note that in (4.5) however, R is defined by (4.4) and is self-adjoint.) Further, the element g of (5.8), R being defined by $R = R_a$ with $a > 0$, need not belong to $L^2(S)$.

Thus, if $h \equiv 1$, it would appear that the "non self-adjoint method" is more useful for investigating the absolute continuity properties of H , through $R = a(aI - iH)^{-1}$, than is the "self-adjoint method," through $R = \beta(H - \beta I)^{-1}$. For example, the operator $H = -id/dx$ on $L^2(-\infty, \infty)$ considered at the end of Section 5 was shown to be amenable to the non self-adjoint method. However, it is known that $\operatorname{sp}(H) = (-\infty, \infty)$, so that (7.1) is violated, and hence the self-adjoint method does not apply. On the other hand, the self-adjoint method, when it is applicable, can be used to study the operator $hR\bar{h}$ when R is self-adjoint (for example, $R = \beta(\beta I - H)^{-1}$ of (4.4)) for any h in $L^\infty(S)$, not only for the special case $h \equiv 1$.

For instance, let E be any measurable subset of S of positive measure and let h denote the characteristic function of E . Then, if R is self-adjoint, $hR\bar{h} = P_E R P_E$, where P_E denotes the orthogonal projection

$$(7.2) \quad P_E : L^2(S) \rightarrow L^2(E),$$

and the self-adjoint method can be used to investigate the absolute continuity properties of $P_E R P_E$ for various sets E . The property of being absolutely continuous can depend of course on the set E , as the example of the Hilbert transform in Section 3 shows. Thus, as noted there, if H is defined by (3.1),

then $P_E H P_E$ is absolutely continuous if and only if the set $(-\infty, \infty) - E$ has positive measure.

Incidentally, it may be observed that the methods of this paper, when applied to the operators R_a and H of (4.3), deal primarily with the case where $\mu(S) = \infty$. In fact, if $\mu(S) < \infty$, then $L^\infty(S) \subset L^2(S)$ and hence if $R = R_a$ is defined by (4.3) (that is, (4.1)), then $R1 = R^*1 = 1$, equivalently, $H1 = 0$, and consequently H cannot be absolutely continuous (although the possibility of an absolutely continuous part is not ruled out). Also, it is seen that if m is defined as in (5.8) (where $R = R_a$, $a > 0$), then $(m, 1) = 0$ and hence the inequality of (5.8) can hold only in the trivial case $m(x) \equiv 0$ a.e.

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Date communicated: FEBRUARY 21, 1969