On Certain Fractional Area Integrals

C. SEGOVIA & R. L. WHEEDEN*

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§0. Introduction. Let $F(x, t)$ be the gradient of a function $U(x, t)$ which is defined and harmonic in $E_{n+1}^+ = \{(x, t) : x = (x_1, \dots, x_n) \in E_n, t > 0\}$ —i.e.,

$$F(x, t) = \left(\frac{\partial U}{\partial x_1}, \dots, \frac{\partial U}{\partial x_n}, \frac{\partial U}{\partial t} \right)(x, t).$$

Following [9], we say F belongs to the Hardy class H^p if there exists a constant $K > 0$ such that

$$\int_{E_n} |F(x, t)|^p dx \leq K^p$$

for every $t > 0$. We will always assume that $p > (n-1)/n$. Denoting $F'(x, t) = (\partial/\partial t)F(x, t)$, we set

$$g^{(F)}(x) = \left(\int_0^\infty t |F'(x, t)|^2 dt \right)^{1/2},$$

$$S(F)(x) = S(F)(x; \epsilon) = \left(\iint_{|x-y| < \epsilon t} |F'(y, t)|^2 \frac{dy dt}{t^{n-1}} \right)^{1/2}$$

and

$$g^*(F)(x) = g^*(F)(x; \lambda) = \left(\iint_{t>0} \frac{t^\lambda}{(|x-y|+t)^\lambda} |F'(y, t)|^2 \frac{dy dt}{t^{n-1}} \right)^{1/2}$$

for $\lambda > 1$.

These well known functions of Littlewood–Paley, Lusin, and Littlewood–Paley–Zygmund respectively have been studied extensively and share many important properties with the vector F . They satisfy the inequalities

$$g(F)(x) \leq cS(F)(x; \epsilon) \leq c_{\epsilon, \lambda} g^*(F)(x; \lambda)$$

(see [4], p. 447).

Our first goal is to generalize these integrals as follows. Given $\alpha > 0$, let

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m be the smallest integer which strictly exceeds α and set $m = \alpha + \beta$. Thus $0 < \beta \leq 1$. We define the α -harmonic derivative $F^{(\alpha)}$ of F by

$$F^{(\alpha)}(x, t) = \frac{e^{-i\beta\pi}}{\Gamma(\beta)} \int_0^\infty \frac{\partial^m}{\partial t^m} F(x, t+s) s^{\beta-1} ds.$$

Since $F \in H^p$, we have $|(\partial^m/\partial t^m)F(x, t)| \leq c_{m,K} t^{-m-n/p}$ (which follows easily from the case $m = 0$ proved in [9], p. 59) and we see the integral defining $F^{(\alpha)}$ converges absolutely. Moreover, $F^{(\alpha)}$ is a harmonic vector with the properties $[F^{(\alpha_1)}]^{(\alpha_2)} = F^{(\alpha_1+\alpha_2)}$ and $F^{(m)}(x, t) = (\partial^m/\partial t^m)F(x, t)$ for integral m .

The integrals we want to consider are then

$$g_\alpha(F)(x) = \left(\int_0^\infty t^{1+2(\alpha-1)} |F^{(\alpha)}(x, t)|^2 dt \right)^{1/2},$$

$$S_\alpha(F)(x; \epsilon) = \left(\iint_{|x-y| < \epsilon t} t^{2(\alpha-1)} |F^{(\alpha)}(y, t)|^2 \frac{dy dt}{t^{n-1}} \right)^{1/2}$$

and

$$g_\alpha^*(F)(x; \lambda) = \left(\iint_{t>0} \frac{t^{\lambda n+2(\alpha-1)}}{(|x-y|+t)^{\lambda n}} |F^{(\alpha)}(y, t)|^2 \frac{dy dt}{t^{n-1}} \right)^{1/2}$$

for $\lambda > 1$. As usual,

$$g_\alpha(F)(x) \leq c_\alpha S_\alpha(F)(x; \epsilon) \leq c_{\epsilon, \lambda} g_\alpha^*(F)(x; \lambda),$$

and we note that $g = g_1$, $S = S_1$ and $g^* = g_1^*$.

For integral m and $n = 1$, the functions $S_m(F)$ were studied by A. P. Calderón in [3], p. 27. Calderón shows essentially that $S_m(F)$ is equivalent pointwise to $S_1(F)$, and we wish to prove a similar statement for both $S_\alpha(F)$ and $g_\alpha^*(F)$.

Let us illustrate the situation with the simplest example. We will show in Lemma 2 below that for $\alpha > 0$ and $\tau > 0$

$$F^{(\alpha)}(x, t) = c \int_0^\infty F^{(\alpha+\tau)}(x, t+s) s^{\tau-1} ds.$$

(Here, as elsewhere, c denotes a constant depending on various parameters but not on the function F or the variables x, t .) Taking this for granted, we have

$$\begin{aligned} |F^{(\alpha)}(x, t)|^2 &\leq \left(c \int_t^\infty |F^{(\alpha+\tau)}(x, s)| (s-t)^{(\tau-1)/2} s^{\gamma/2} s^{-\gamma/2} (s-t)^{(\tau-1)/2} ds \right)^2 \\ &\leq c \int_t^\infty |F^{(\alpha+\tau)}(x, s)|^2 (s-t)^{\tau-1} s^\gamma ds \int_t^\infty s^{-\gamma} (s-t)^{\tau-1} ds. \end{aligned}$$

Choosing $\gamma > \tau$ and putting $s = tu$ in the last integral we obtain

$$|F^{(\alpha)}(x, t)|^2 \leq ct^{\tau-\gamma} \int_t^\infty |F^{(\alpha+\tau)}(x, s)|^2 (s-t)^{\tau-1} s^\gamma ds.$$

Therefore

$$[g_\alpha(F)(x)]^2 \leq c \int_0^\infty t^{2\alpha-1+\tau-\gamma} dt \int_t^\infty |F^{(\alpha+\tau)}(x, s)|^2 (s-t)^{\tau-1} s^\gamma ds.$$

Interchanging the order of integration and changing variables in the inner integral, we find

$$[g_\alpha(F)(x)]^2 \leq c \int_0^\infty s^{1+2(\alpha+\tau-1)} |F^{(\alpha+\tau)}(x, s)|^2 ds \int_0^1 t^{2\alpha-1+\tau-\gamma} (1-t)^{\tau-1} dt.$$

If $2\alpha - 1 + \tau - \gamma > -1$, we obtain

$$[g_\alpha(F)(x)]^2 \leq c[g_{\alpha+\tau}(F)(x)]^2, \quad \tau > 0.$$

Finally, we observe that the conditions on γ , namely, $2\alpha + \tau > \gamma > \tau$, are compatible for $\alpha > 0$.

For the larger functions S_α and g_α^* we will prove the following two theorems.

Theorem 1. For $\alpha > 0$ there exist two positive constants c_1 and c_2 such that

$$c_1 S_1(F)(x; \epsilon') \leq S_\alpha(F)(x; \epsilon) \leq c_2 S_1(F)(x; \epsilon'')$$

for all $x \in E_n$ and $\epsilon' < \epsilon < \epsilon''$. The constants c_1 and c_2 depend only on α , ϵ , ϵ' , ϵ'' and n .

Theorem 2. For $\alpha > 0$ and $\lambda > 1$ there exist two positive constants c_1 and c_2 such that

$$c_1 g_1^*(F)(x; \lambda) \leq g_\alpha^*(F)(x; \lambda) \leq c_2 g_1^*(F)(x; \lambda)$$

for all $x \in E_n$. The constants c_1 and c_2 depend only on α , λ and n .

These theorems will be proved in Sections 1 and 2 below. The remainder of the paper will be devoted to an application of the theorems. The application concerns the Lebesgue spaces L_α^p . We recall (see [1]) that L_α^p , $1 < p < \infty$, $\alpha > 0$, is the space of functions $f \in L^p$ whose Fourier transform in the sense of tempered distributions satisfies

$$\hat{f}(x) = (1 + |x|^2)^{-\alpha/2} \hat{\psi}(x)$$

for some $\psi \in L^p$. Here $\|f\|_{p, \alpha} = \|\psi\|_p$ and f has the representation $f = G_\alpha * \psi$ where $G_\alpha \in L^1$, $\hat{G}_\alpha(x) = (1 + |x|^2)^{-\alpha/2}$. L_1^p of course coincides with the Sobolev space of functions whose first order distributional derivatives belong to L^p , $1 < p < \infty$.

Our application is a characterization of L_α^p .

Theorem 3. Let k be a positive integer and $f \in L^p$, $1 < p < \infty$. Let $f(x, t)$, $t > 0$, denote the Poisson integral of f and for $0 < \alpha < k$ set

$$S_{\alpha, k}(f)(x; \epsilon) = \left(\iint_{|x-y| < \epsilon t} t^{2(k-\alpha-1)} \left| \frac{\partial^k}{\partial t^k} f(y, t) \right|^2 \frac{dy dt}{t^{n-1}} \right)^{1/2}.$$

Then if $f \in L_\alpha^p$,

$$c_1 \|f\|_{p,\alpha} \leq \|S_{\alpha,k}(f)\|_p + \|f\|_p \leq c_2 \|f\|_{p,\alpha}.$$

Moreover, if f is any L^p function, $1 < p < \infty$, and $S_{\alpha,k}(f) \in L^p$ then $f \in L_\alpha^p$.

In Section 3 we will give similar characterizations in terms of the g and g^* functions. We wish to emphasize that part of our method of proof is taken from the proof of Theorem 2 of [8].

§1. The Lusin area integral of order α . To prove theorem 1 we shall need the following 3 Lemmas. Lemma 3 in particular is crucial.

Lemma 1. Let $v(x)$, $x \in E_n$, be harmonic in a ball with center x_0 and radius r_0 , and let $P(D)$ be a homogeneous differential operator of order m with constant coefficients. For every $0 < r \leq r_0$ we have

$$|P(D)(v)(x_0)| \leq c_P r^{-m-n/2} \left(\int_{|x-x_0| < r} |v(x)|^2 dx \right)^{1/2},$$

with c_P independent of r and v .

For a proof see [8], p. 159.

Lemma 2. For $\sigma > 0$, $\tau > 0$ and $F \in H^p$,

$$F^{(\sigma)}(x, t) = c \int_0^\infty F^{(\sigma+\tau)}(x, t+s) s^{\tau-1} ds.$$

Proof. By definition of $F^{(\sigma+\tau)}$ we have with $m = (\sigma + \tau) + \beta$, $0 < \beta \leq 1$ and m integral,

$$F^{(\sigma+\tau)}(x, t+s) = c \int_0^\infty F^{(m)}(x, t+s+\rho) \rho^{\beta-1} d\rho.$$

Since $F \in H^p$, $|F^{(m)}(x, t)| \leq ct^{-m-n/p}$. Therefore

$$\begin{aligned} \int_0^\infty s^{\tau-1} ds \int_0^\infty |F^{(m)}(x, t+s+\rho)| \rho^{\beta-1} d\rho \\ \leq c \int_0^\infty \int_0^\infty \frac{s^{\tau-1} \rho^{\beta-1}}{(t+s+\rho)^{m+n/p}} ds d\rho < \infty \end{aligned}$$

for $t > 0$. Hence we may interchange the order of integration to obtain

$$\begin{aligned} \int_0^\infty F^{(\sigma+\tau)}(x, t+s) s^{\tau-1} ds &= \int_0^\infty F^{(m)}(x, t+u) du \int_0^u s^{\tau-1} (u-s)^{\beta-1} ds \\ &= c \int_0^\infty F^{(m)}(x, t+u) u^{\tau+\beta-1} du. \end{aligned}$$

Integrating by parts k times, $k < \tau + \beta \leq k + 1$, we obtain a constant times

$$\int_0^\infty F^{(m-k)}(x, t+u) u^{(\tau+\beta-k)-1} du.$$

This completes the proof since $m - k = \sigma + (\tau + \beta - k)$, $0 < \tau + \beta - k \leq 1$.

Lemma 3. Let $\mu > -1$, $b > 0$ and

$$|G(x, t)| \leq \int_t^\infty s^{a-(n+1)/2} (s-t)^{b-1} J(x, s)^{1/2} ds$$

where

$$J(x, s) = \iint_{\Sigma(x, s)} |H(\xi, \eta)|^2 d\xi d\eta,$$

$\Sigma(x, s)$ being the ball with center (x, s) and radius rs , $0 < r < 1$. Given ϵ and δ , $\epsilon < \delta$, there exists $r = r(\epsilon, \delta)$ such that

$$\iint_{|x| < \epsilon t} t^{2\mu} |G(x, t)|^2 \frac{dx dt}{t^{n-1}} \leq c \iint_{|x| < \delta t} t^{2(\mu+a+b)} |H(x, t)|^2 \frac{dx dt}{t^{n-1}}.$$

Proof. If $|x| < \epsilon t$ then $|x| < \epsilon s$ for $s \geq t$. Hence given $\epsilon < \delta$ we can choose r so that $|\xi| < \delta \eta$ for $(\xi, \eta) \in \Sigma(x, s)$. If $\chi(\xi, \eta; s)$ denotes the characteristic function of the strip in (ξ, η) defined by $|\eta - s| < rs$ then

$$J(x, s) \leq M_s = \iint_{|\xi| < \delta \eta} \chi(\xi, \eta; s) |H(\xi, \eta)|^2 d\xi d\eta.$$

Thus

$$|G(x, t)| \leq \int_t^\infty s^{a-(n+1)/2} (s-t)^{b-1} M_s^{1/2} ds$$

and, by Schwarz's inequality,

$$|G(x, t)|^2 \leq \int_t^\infty s^{2a-n-1+\gamma} (s-t)^{b-1} M_s ds \int_t^\infty s^{-\gamma} (s-t)^{b-1} ds.$$

Choosing $\gamma > b$, this gives

$$|G(x, t)|^2 \leq ct^{b-\gamma} \int_t^\infty s^{2a-n-1+\gamma} (s-t)^{b-1} M_s ds.$$

We now change to polar coordinates $(x, t) = (\rho, \sigma)$, $t = \rho \cos \theta$, $x = |x| x' = \rho \sin \theta x'$, $dx dt = \rho^n d\rho d\sigma$, and we obtain

$$\iint_{|x| < \epsilon t} t^{2\mu} |G(x, t)|^2 \frac{dx dt}{t^{n-1}} = \int (\cos \theta)^{2\mu+1-n} d\sigma \int_0^\infty \rho^{2\mu+1} |G(\rho, \sigma)|^2 d\rho,$$

where the integration in σ is extended only over the intersection of the $(n+1)$ -dimensional unit sphere with the cone $|x| < \epsilon t$. Since $\cos \theta$ is bounded below

by a positive constant in this domain, we have only to show that

$$\int_0^\infty \rho^{2\mu+1} |G(\rho, \sigma)|^2 d\rho \leq c \iint_{|\xi| < \delta\eta} \eta^{2(\mu+a+b)} |H(\xi, \eta)|^2 \frac{d\xi d\eta}{\eta^{n-1}},$$

with c independent of σ . Integration with respect to σ will then complete the proof. Since

$$|G(\rho, \sigma)|^2 \leq c(\rho \cos \theta)^{b-\gamma} \int_{\rho \cos \theta}^\infty s^{2a-n-1+\gamma} (s - \rho \cos \theta)^{b-1} M_\bullet ds$$

and $\cos \theta$ is bounded from zero, we have

$$\int_0^\infty \rho^{2\mu+1} |G(\rho, \sigma)|^2 d\rho \leq c \int_0^\infty \rho^{2\mu+1+b-\gamma} d\rho \int_{\rho \cos \theta}^\infty s^{2a-n-1+\gamma} (s - \rho \cos \theta)^{b-1} M_\bullet ds.$$

Changing the order of integration and then changing variables in the inner integral, we have

$$\begin{aligned} \int_0^\infty \rho^{2\mu+1} |G(\rho, \sigma)|^2 d\rho &\leq c \int_0^\infty s^{2(\mu+a+b)-n} M_\bullet ds \int_0^1 \rho^{2\mu+1+b-\gamma} (1 - \rho)^{b-1} d\rho \\ &= c \int_0^\infty s^{2(\mu+a+b)-n} M_\bullet ds \end{aligned}$$

if $2\mu + 2 + b > \gamma$. This is compatible with the condition $\gamma > b$ if $\mu > -1$. However,

$$\begin{aligned} \int_0^\infty s^{2(\mu+a+b)-n} M_\bullet ds &= \int_0^\infty s^{2(\mu+a+b)-n} ds \iint_{|\xi| < \delta\eta} \chi(\xi, \eta; s) |H(\xi, \eta)|^2 d\xi d\eta \\ &= \iint_{|\xi| < \delta\eta} |H(\xi, \eta)|^2 d\xi d\eta \int_0^\infty s^{2(\mu+a+b)-n} \chi(\xi, \eta; s) ds \\ &= c \iint_{|\xi| < \delta\eta} \eta^{2(\mu+a+b)} |H(\xi, \eta)|^2 \frac{d\xi d\eta}{\eta^{n-1}}. \end{aligned}$$

This completes the proof of Lemma 3.

Let us now prove the second inequality in Theorem 1, namely, $S_\alpha(F)(x; \epsilon) \leq c_2 S_1(F)(x; \epsilon'')$ for $\epsilon < \epsilon''$. Without loss of generality we may assume $x = 0$. By definition of $F^{(\alpha)}$ we have

$$F^{(\alpha)}(x, t) = c \int_t^\infty F^{(m)}(x, s) (s - t)^{\beta-1} ds,$$

$m = \alpha + \beta$, $0 < \beta \leq 1$, and by Lemma 1 we have

$$|F^{(m)}(x, s)| \leq cs^{-(m-1)-(n+1)/2} \left(\iint_{\Sigma(x, s)} |F'(\xi, \eta)|^2 d\xi d\eta \right)^{1/2}.$$

If we now apply Lemma 3 with $G = F^{(\alpha)}$, $H = F'$, $a = 1 - m$, $b = \beta > 0$,

$\mu = \alpha - 1 > -1$, we obtain immediately that

$$[S_\alpha(F)(0; \epsilon)]^2 \leq c[S_1(F)(0; \epsilon')]^2.$$

The proof of the first inequality in Theorem 1 is equally simple. For by Lemma 2 we have

$$F'(x, t) = c \int_t^\infty F^{(\alpha+1)}(x, s)(s - t)^{\alpha-1} ds,$$

and by Lemma 1 we have

$$|F^{(\alpha+1)}(x, s)| \leq cs^{-1-(n+1)/2} \left(\iint_{\Sigma(x, s)} |F^{(\alpha)}(\xi, \eta)|^2 d\xi d\eta \right)^{1/2}.$$

Hence with $\epsilon' < \epsilon$, Lemma 3 implies

$$[S_1(F)(0; \epsilon')]^2 \leq c[S_\alpha(F)(0; \epsilon)]^2,$$

which completes the proof of Theorem 1.

We will now add a remark which will be useful later.

Lemma 4. *Let $\mu > -1$ and let $u(x, t)$ and $v(x, t)$ be two harmonic functions related by*

$$\frac{\partial^k}{\partial t^k} u(x, t) = P(D)v(x, t)$$

where $P(D)$ is a homogeneous differential operator of order k with constant coefficients. Then for $\epsilon < \delta$

$$\iint_{|x-y| < \epsilon t} t^{2\mu} |u(y, t)|^2 \frac{dy dt}{t^{n-1}} \leq c \iint_{|x-y| < \delta t} t^{2\mu} |v(y, t)|^2 \frac{dy dt}{t^{n-1}}$$

for all $x \in E_n$ with c independent of x , u and v .

In case $\mu = 0$, Lemma 4 is a known result due to E. M. Stein (see [8], Theorem 2). For $\mu > -1$ the lemma is an immediate consequence of Lemma 3. For

$$u(x, t) = c \int_t^\infty P(D)v(x, s)(s - t)^{k-1} ds$$

and by Lemma 1,

$$|P(D)v(x, s)| \leq cs^{-k-(n+1)/2} \left(\iint_{\Sigma(x, s)} |v(\xi, \eta)|^2 d\xi d\eta \right)^{1/2}.$$

Applying Lemma 3 with $G = u$, $H = v$, $a = -k$, $b = k$ and $\mu > -1$, we obtain Lemma 4.

§2. The Littlewood–Paley–Zygmund integral of order α . To prove Theorem 2 we will use a refinement of the proof of Theorem 1.

Lemma 5. *Let $\mu > -1$, $b > 0$ and*

$$|G(x, t)| \leq \int_t^\infty s^{a-(n+1)/2} (s-t)^{b-1} J(x, s)^{1/2} ds$$

where

$$J(x, s) = \iint_{\Sigma(x, s)} |H(\xi, \eta)|^2 d\xi d\eta,$$

$\Sigma(x, s)$ now denoting the ball with center (x, s) and radius $s/2$. Then

$$\iint_{t>0} \frac{t^{\lambda n+2\mu}}{(|x|+t)^{\lambda n}} |G(x, t)|^2 \frac{dx dt}{t^{n-1}} \leq c \iint_{t>0} \frac{t^{\lambda n+2(\mu+a+b)}}{(|x|+t)^{\lambda n}} |H(x, t)|^2 \frac{dx dt}{t^{n-1}}$$

for $\lambda > 1$.

Proof. Choosing $\gamma > b$ and applying Schwarz's inequality as usual, we have

$$|G(x, t)|^2 \leq ct^{b-\gamma} \int_t^\infty s^{2a-n-1+\gamma} (s-t)^{b-1} J(x, s) ds.$$

Passing to polar coordinates $(x, t) = (\rho, \sigma)$, $t = \rho \cos \theta$, $x = |x| x' = \rho \sin \theta x'$, $0 \leq \theta < \pi/2$, $dx dt = \rho^n d\rho d\sigma$ and $d\sigma = (\sin \theta)^{n-1} dx' d\theta$, we obtain

$$\begin{aligned} & \iint_{t>0} \frac{t^{\lambda n+2\mu}}{(|x|+t)^{\lambda n}} |G(x, t)|^2 \frac{dx dt}{t^{n-1}} \\ & \leq c \iint_{t>0} \frac{t^{\lambda n+2\mu+b-\gamma+1-n}}{(|x|+t)^{\lambda n}} dx dt \int_t^\infty s^{2a-n-1+\gamma} (s-t)^{b-1} J(x, s) ds \\ & = c \int \frac{(\cos \theta)^{\lambda n+2\mu+b-\gamma+1-n}}{(\sin \theta + \cos \theta)^{\lambda n}} d\sigma \int_0^\infty \rho^{2\mu+b-\gamma+1} d\rho \\ & \quad \cdot \int_{\rho \cos \theta}^\infty s^{2a-n+1+\gamma} (s-\rho \cos \theta)^{b-1} J(\rho \sin \theta x', s) ds. \end{aligned}$$

If we observe that $\sin \theta + \cos \theta$ exceeds a positive constant and replace ρ by $\rho/\cos \theta$, we see the last expression is majorized by a constant times

$$\int (\cos \theta)^{\lambda n-n-1} d\sigma \int_0^\infty \rho^{2\mu+b-\gamma+1} d\rho \int_\rho^\infty s^{2a-n-1+\gamma} (s-\rho)^{b-1} J(\rho \tan \theta x', s) ds.$$

Changing the order of integration in ρ and s and then replacing ρ by $s\rho$, we obtain

$$(2.1) \quad \int (\cos \theta)^{\lambda n-n-1} d\sigma \int_0^\infty s^{2(\mu+a+b)-n} ds \cdot \int_0^1 \rho^{2\mu+b-\gamma+1} (1-\rho)^{b-1} J(s\rho \tan \theta x', s) d\rho.$$

Here

$$\begin{aligned} J(s\rho \tan \theta x', s) &\leq \iint_{\substack{|s\rho \tan \theta x' - \xi| < s/2 \\ |s - \eta| < s/2}} \frac{|H(\xi, \eta)|^2}{(|\xi| + \eta)^{\lambda n}} (|\xi| + \eta)^{\lambda n} d\xi d\eta \\ &\leq c(s\rho |\tan \theta| + s)^{\lambda n} M(s, \rho, \theta, x'), \end{aligned}$$

where

$$M(s, \rho, \theta, x') = \iint_{\substack{|s\rho \tan \theta x' - \xi| < s/2 \\ |s - \eta| < s/2}} \frac{|H(\xi, \eta)|^2}{(|\xi| + \eta)^{\lambda n}} d\xi d\eta.$$

We will consider separately the parts of (2.1) arising from $s^{\lambda n} M(s, \rho, \theta, x')$ and $(s\rho |\tan \theta|)^{\lambda n} M(s, \rho, \theta, x')$. The first of these is at most

$$\begin{aligned} &\int (\cos \theta)^{\lambda n - n - 1} d\sigma \int_0^\infty s^{2(\mu + a + b) + \lambda n - n} ds \\ &\quad \cdot \int_0^\infty \rho^{2\mu + b - \gamma + 1} (1 - \rho)^{b-1} d\rho \iint_{|s - \eta| < s/2} \frac{|H(\xi, \eta)|^2}{(|\xi| + \eta)^{\lambda n}} d\xi d\eta. \end{aligned}$$

Observing that $|s - \eta| < s/2$ implies $\frac{2}{3}\eta < s < 2\eta$ and then carrying out the integration in s , we obtain at most

$$c \int (\cos \theta)^{\lambda n - n - 1} d\sigma \int_0^\infty \rho^{2\mu + b - \gamma + 1} (1 - \rho)^{b-1} d\rho \iint_{\eta > 0} \frac{\eta^{\lambda n + 2(\mu + a + b)}}{(|\xi| + \eta)^{\lambda n}} |H(\xi, \eta)|^2 \frac{d\xi d\eta}{\eta^{n-1}}.$$

The integration over the unit sphere converges since $\lambda n - n = (\lambda - 1)n > 0$. The integration in ρ converges if $2\mu + 2 + b > \gamma$. This is compatible with the condition $\gamma > b$ if $\mu > -1$. Thus we obtain our result in this case.

Since $|\tan \theta| \leq 1/\cos \theta$, the term of (2.1) arising from $(s\rho |\tan \theta|)^{\lambda n} M(s, \rho, \theta, x')$ is at most

$$\int (\cos \theta)^{-n-1} d\sigma \int_0^\infty s^{2(\mu + a + b) + \lambda n - n} ds \int_0^1 \rho^{2\mu + b - \gamma + 1 + \lambda n} (1 - \rho)^{b-1} M(s, \rho, \theta, x') d\rho.$$

Carrying out the integration in σ we obtain

$$\begin{aligned} &\int_0^\infty s^{2(\mu + a + b) + \lambda n - n} ds \int_0^1 \rho^{2\mu + b - \gamma + 1 + \lambda n} (1 - \rho)^{b-1} d\rho \\ &\quad \cdot \int_{|s\rho \tan \theta x' - \xi| < s/2} (\cos \theta)^{-n-1} d\sigma \iint_{|s - \eta| < s/2} \frac{|H(\xi, \eta)|^2}{(|\xi| + \eta)^{\lambda n}} d\xi d\eta. \end{aligned}$$

We claim that $\int_{|s\rho \tan \theta x' - \xi| < s/2} (\cos \theta)^{-n-1} d\sigma$ is majorized by a constant independent of s, ρ, ξ and η times a sum of terms ρ^{-k} for $k \leq n$. Assuming for the moment that this is true and carrying out the integration in s we obtain at

most a sum of terms

$$c \int_0^1 \rho^{2\mu+b-\gamma+1+\lambda n-k} (1-\rho)^{b-1} d\rho \iint_{\eta>0} \frac{\eta^{\lambda n+2(\mu+a+b)}}{(|\xi|+\eta)^{\lambda n}} |H(\xi, \eta)|^2 \frac{d\xi d\eta}{\eta^{n-1}}.$$

For convergence of the ρ -integration we need $2\mu + 2 + b + \lambda n - k > \gamma$. For $\mu > -1$ and $k \leq n$ this is compatible with the condition $\gamma > b$.

To prove our claim, we write

$$\int_{|s\rho \tan \theta x' - \xi| < s/2} (\cos \theta)^{-n-1} d\sigma = \int dx' \int_{|s\rho u \tan \theta x' - \xi| < s/2} (\cos \theta)^{-n-1} (\sin \theta)^{n-1} d\theta.$$

Putting $u = \tan \theta$, $\cos \theta = (u^2 + 1)^{-1/2}$ and $du = (\cos \theta)^{-2} d\theta$ we obtain at most a constant times

$$\int dx' \int_{|s\rho u x' - \xi| < s/2} (u^{n-1} + 1) du.$$

Consider first the part $\int dx' \int_{|s\rho u x' - \xi| < s/2} du$ and write $\xi = \tau x' + \omega x''$ where x'' is a unit vector orthogonal to x' . The condition $|(s\rho u - \tau)x' - \omega x''| < s/2$ implies $|s\rho u - \tau| < s/2$, or

$$\frac{\tau}{s\rho} - \frac{1}{2\rho} < u < \frac{\tau}{s\rho} + \frac{1}{2\rho}.$$

Thus u belongs to an interval of length $1/\rho$ and

$$\int dx' \int_{|s\rho u x' - \xi| < s/2} du < c/\rho.$$

Consider finally $\int dx' \int_{|s\rho u x' - \xi| < s/2} u^{n-1} du$. Given ξ , we choose for spherical coordinates x' the system induced by the usual n -dimensional polar coordinates having ξ for north pole vector. The polar angle ϕ , $0 \leq \phi \leq \pi$, is the acute angle between x' and ξ . Thus there exists a unit vector x'' orthogonal to x' such that $\xi = \tau x' + \omega x''$, $\tau = \xi \cdot x' = |\xi| \cos \phi$, $\omega = \xi \cdot x'' = |\xi| \sin \phi \geq 0$. Hence it is enough to show that

$$\int (\sin \phi)^{n-2} d\phi \int_{(s\rho u - \tau)^2 + \omega^2 < s^2/4} u^{n-1} du \leq c/\rho^n$$

or

$$\int (\sin \phi)^{n-2} \left\{ \frac{\left[\tau + \left(\frac{s^2}{4} - \omega^2 \right)^{1/2} \right]^n - \left[\tau - \left(\frac{s^2}{4} - \omega^2 \right)^{1/2} \right]^n}{s^n \rho^n} \right\} d\phi \leq c/\rho^n,$$

integration being extended over the part of $0 \leq \phi \leq \pi$ where $s \geq 2\omega$. Using the binomial expansion to express the term in braces, we obtain a sum of terms of the form

$$\frac{1}{s^n \rho^n} \int (|\xi| \cos \phi)^k \left(\frac{s^2}{4} - |\xi|^2 \sin^2 \phi \right)^{(n-k)/2} (\sin \phi)^{n-2} d\phi$$

where $0 \leq k \leq n-1$. Hence we must show that

$$\frac{|\xi|^n}{s^n} \int (\cos \phi)^k \left(\frac{s^2}{4|\xi|^2} - \sin^2 \phi \right)^{(n-k)/2} (\sin \phi)^{n-2} d\phi \leq c.$$

If $s/|\xi| > \frac{1}{2}$, the expression above is at most

$$c \frac{|\xi|^n}{s^n} \int_0^\pi \left(\frac{s}{|\xi|} \right)^{n-k} d\phi = c \frac{|\xi|^k}{s^k} < c.$$

If $s/|\xi| \leq \frac{1}{2}$ then since $s \geq 2\omega = 2|\xi| \sin \phi$, we have $\sin \phi \leq \frac{1}{4}$ so that $\cos \phi$ is bounded below by a positive constant. Thus

$$\begin{aligned} \frac{|\xi|^n}{s^n} \int \left(\frac{s^2}{4|\xi|^2} - \sin^2 \phi \right)^{(n-k)/2} (\sin \phi)^{n-2} d\phi \\ \leq c \frac{|\xi|^n}{s^n} \int \left(\frac{s^2}{4|\xi|^2} - \sin^2 \phi \right)^{(n-k)/2} (\sin \phi)^{n-2} \cos \phi d\phi. \end{aligned}$$

Putting $\sin \phi = (s/2|\xi|)t$, $\cos \phi d\phi = (s/2|\xi|) dt$, we obtain at most a constant times $(|\xi|/s)^n (s/|\xi|)^{n+(n-1-k)} = (s/|\xi|)^{n-1-k}$, which is bounded since $n-1-k \geq 0$ and $s/|\xi| \leq \frac{1}{2}$. This completes the proof of Lemma 5.

The proof of Theorem 2 is now exactly the same as the proof of Theorem 1 if we use Lemma 5 instead of Lemma 3.

Again we add a remark which will be useful later.

Lemma 6. Let $\mu > -1$ and let $u(x, t)$ and $v(x, t)$ be two harmonic functions related by

$$\frac{\partial^k}{\partial t^k} u(x, t) = P(D)v(x, t)$$

where $P(D)$ is a homogeneous differential operator of order k with constant coefficients. Then for $\lambda > 1$, and $\mu > -1$,

$$\iint_{t>0} \frac{t^{\lambda n+2\mu}}{(|x-y|+t)^{\lambda n}} |u(y, t)|^2 \frac{dy dt}{t^{n-1}} \leq c \iint_{t>0} \frac{t^{\lambda n+2\mu}}{(|x-y|+t)^{\lambda n}} |v(y, t)|^2 \frac{dy dt}{t^{n-1}}$$

for all $x \in E_n$ with c independent of x, u and v .

The proof of Lemma 6 is the same as that of Lemma 4 if we use Lemma 5 instead of Lemma 3.

§3. Characterizations of L_α^p . In this section, we will prove Theorem 3 and its analogues for g and g^* . We begin with two lemmas.

Lemma 7. Let $G(x, t)$, $t > 0$, belong to H^p for some $p > (n-1)/n$ and

let $G(x) = \lim_{t \rightarrow 0} G(x, t)$. There exist two positive constants c_1 and c_2 independent of G such that

$$c_1 \|G(x)\|_p \leq \|S_\gamma(G)(x; \epsilon)\|_p \leq c_2 \|G(x)\|_p.$$

Proof. By Theorem 1, we see the lemma holds if and only if it holds for $\gamma = 1$. For $\gamma = 1$, the lemma is a simple consequence of Lemma 4 and the area theorem proved in [2].

Lemma 8. For $\alpha > 0$,

i) $|x|^\alpha = (1 + |x|^2)^{\alpha/2} d\hat{\mu}(x)$

ii) $(1 + |x|^2)^{\alpha/2} = |x|^\alpha d\hat{\sigma}(x) + d\hat{\tau}(x)$

where $d\hat{\mu}$, $d\hat{\sigma}$ and $d\hat{\tau}$ are Fourier transforms of finite measures $d\mu$, $d\sigma$ and $d\tau$.

See [5].

Let $f \in L_\alpha^p$, $1 < p < \infty$, $0 < \alpha < k$, $f = G_\alpha * \psi$. If $g = \psi * d\mu$ then $g \in L^p$ and $\psi = f * d\tau + g * d\sigma$ by Lemma 8. Denoting by $f_i(x) = \text{p.v. } (f * (x_i/|x|^{n+1}))$ and $g_i(x) = \text{p.v. } (g * (x_i/|x|^{n+1}))$ the i^{th} M. Riesz transforms of f and g and by $f_i(x, t)$ and $g_i(x, t)$ their Poisson integrals, we know the vectors

$$F(x, t) = (f(x, t), f_1(x, t), \dots, f_n(x, t))$$

and

$$G(x, t) = (g(x, t), g_1(x, t), \dots, g_n(x, t))$$

belong to H^p . We claim that

$$(3.1) \quad F^{(k)}(x, t) = cG^{(k-\alpha)}(x, t).$$

If so, then by Lemma 7 with $\gamma = k - \alpha$ we have

$$(3.2) \quad c_1 \|G\|_p \leq \left\| \left(\iint_{|x-y| < \epsilon t} t^{2(k-\alpha-1)} |F^{(k)}(y, t)|^2 \frac{dy dt}{t^{n-1}} \right)^{1/2} \right\|_p \leq c_2 \|G\|_p$$

and the proof of the first part of Theorem 3 will be nearly complete.

To prove (3.1), we first observe that

$$G^{(k-\alpha)}(x, t) = c \int_0^\infty G^{(k)}(x, s+t) s^{\alpha-1} ds$$

by Lemma 2. Hence (3.1) will follow if we prove that

$$F^{(k)}(x, t) = c \int_0^\infty G^{(k)}(x, s+t) s^{\alpha-1} ds$$

too, or what is the same, that

$$(3.3) \quad f_i^{(k)}(x, t) = c \int_0^\infty g_i^{(k)}(x, s+t) s^{\alpha-1} ds$$

for each i , with c independent of i . To accomplish this, we observe that for

smooth ψ with compact support

$$\hat{f}(x) = (1 + |x|^2)^{-\alpha/2} \hat{\psi}(x) = |x|^{-\alpha} \hat{\psi}(x) d\mu(x) = |x|^{-\alpha} g(x).$$

Moreover, the right side of (3.3) is an integrable function of x since

$$\int_0^\infty s^{\alpha-1} ds \int_{E_n} |g_i^{(k)}(x, s+t)| dx \leq c \|g\|_1 \int_0^\infty \frac{s^{\alpha-1}}{(s+t)^{k+n}} ds.$$

Taking Fourier transforms in x , we obtain for the left side of (3.3) a constant independent of i times $(-|x|)^k x'_i e^{-t|x|} \hat{f}(x)$, while for the right side we obtain a constant times

$$(-|x|)^k x'_i e^{-t|x|} g(x) \int_0^\infty s^{\alpha-1} e^{-s|x|} ds.$$

Since the last integral is $c|x|^{-\alpha}$, we obtain (3.3) for smooth ψ with compact support. For an arbitrary $\psi \in L^p$, $1 < p < \infty$, (3.3) follows from a simple approximation argument.

Clearly,

$$S_{\alpha,k}(f)(x, \epsilon) \leq \left(\iint_{|x-y| < \epsilon t} t^{2(k-\alpha-1)} |F^{(k)}(y, t)|^2 \frac{dy dt}{t^{n-1}} \right)^{1/2}.$$

However, Lemma 4 and the fact that F satisfies the Cauchy–Riemann equations show the opposite inequality holds, provided we replace the cone $|x - y| < \epsilon t$ on the right above by a larger cone. Hence we obtain from (3.2)

$$c_1 \|G\|_p \leq \|S_{\alpha,k}(f)\|_p \leq c_2 \|G\|_p.$$

From the well known behavior in L^p , $1 < p < \infty$, of the Riesz transforms, we have

$$c_1 \|g\|_p \leq \|S_{\alpha,k}(f)\|_p \leq c_2 \|g\|_p.$$

Finally,

$$\|f\|_p + \|S_{\alpha,k}(f)\|_p \leq c[\|\psi\|_p + \|g\|_p] \leq c\|\psi\|_p$$

and

$$\|\psi\|_p \leq c[\|f\|_p + \|g\|_p] \leq c[\|f\|_p + \|S_{\alpha,k}(f)\|_p].$$

This proves the first part of Theorem 3.

The proof of the second part of Theorem 3 is now fairly standard. Let $f \in L^p$, $1 < p < \infty$, and $S_{\alpha,k}(f) \in L^p$, $0 < \alpha < k$. Above we showed that $S_{\alpha,m}(f)(x)$ is equivalent to

$$(3.4) \quad \left(\iint_{|x-y| < \epsilon t} t^{2(m-\alpha-1)} |F^{(m)}(y, t)|^2 \frac{dy dt}{t^{n-1}} \right)^{1/2}$$

for $\alpha < m$. A simple application of Lemma 3 shows that (3.4) is in turn equivalent

to

$$\left(\iint_{|x-y| < \epsilon t} t^{2(k-\alpha-1)} |F^{(k)}(y, t)|^2 \frac{dy dt}{t^{n-1}} \right)^{1/2}.$$

Hence $S_{\alpha, k}(f)(x)$ and $S_{\alpha, m}(f)(x)$ are equivalent for $\alpha < k$, $\alpha < m$. In particular, we may assume $S_{\alpha, m}(f) \in L^p$ for $m \geq k$.

For $\delta > 0$, set $f_\delta(x) = f(x, \delta)$. Then $f_\delta \in L_\alpha^p$, $f_\delta = G_\alpha * \psi_\delta$. We claim that $\|S_{\alpha, m}(f_\delta)\|_p$ is uniformly bounded in δ for $m \geq k$. For since $f_\delta(x, t) = f(x, t + \delta)$,

$$S_{\alpha, m}(f_\delta)(x) = \left(\iint_{|x-y| < \epsilon t} t^{2(m-\alpha-1)} |f^{(m)}(y, t + \delta)|^2 \frac{dy dt}{t^{n-1}} \right)^{1/2}.$$

Choosing m so large that $2(m - \alpha - 1) + 1 - n \geq 0$, we have only to replace t by $t - \delta$ to obtain $S_{\alpha, m}(f_\delta)(x) \leq S_{\alpha, m}(f)(x)$, which proves our claim. Since $\|f_\delta\|_p \leq \|f\|_p$ and $f_\delta \in L_\alpha^p$, we have from the first part of Theorem 3 that

$$\|\psi_\delta\|_p = \|f_\delta\|_{p, \alpha} \leq C.$$

Choose a subsequence of δ 's, denoting them again δ , tending to zero so that ψ_δ converges in the sense of distributions to a function $\psi \in L^p$. Denoting by h the distribution satisfying $f = G_\alpha * h$, we must show $h \in L^p$. But, as distributions, $f_\delta \rightarrow f$, $\hat{f}_\delta \rightarrow \hat{f}$ or $(1 + |x|^2)^{-\alpha/2} \hat{\psi}_\delta \rightarrow \hat{f}$. Thus $\hat{\psi}_\delta \rightarrow (1 + |x|^2)^{\alpha/2} \hat{f} = \hat{h}$ as distributions. Therefore, since $\hat{\psi}_\delta \rightarrow \hat{\psi}$ as distributions, $\hat{h} = \hat{\psi}$ or $h = \psi \in L^p$. This completes the proof of Theorem 3.

For g^* we have the following analogue of Theorem 3.

Theorem 4. *Let k be a positive integer and $f \in L^p$, $1 < p < \infty$. Let $f(x, t)$, $t > 0$, denote the Poisson integral of f and for $0 < \alpha < k$ and $\lambda > 1$ set*

$$g_{\alpha, k}^*(f)(x; \lambda) = \left(\iint_{t>0} \frac{t^{\lambda n + 2(\alpha - k - 1)}}{(|x - y| + t)^{\lambda n}} |f^{(k)}(y, t)|^2 \frac{dy dt}{t^{n-1}} \right)^{1/2}.$$

Then if $f \in L_\alpha^p$, $p > 2/\lambda$,

$$c_1 \|f\|_{p, \alpha} \leq \|g_{\alpha, k}^*(f)\|_p + \|f\|_p \leq c_2 \|f\|_{p, \alpha}.$$

Moreover, if f is any L^p function, $1 < p < \infty$, $p > 2/\lambda$, and $g_{\alpha, k}^*(f) \in L^p$ then $f \in L_\alpha^p$.

We obtain the proof of the first part of Theorem 4 by the same argument we used for Theorem 3 provided we replace Lemma 4 by Lemma 6 and Lemma 7 by

Lemma 9. *Let $G(x, t)$, $t > 0$, belong to H^p for some $p > (n - 1)/n$ and let $G(x) = \lim_{t \rightarrow 0} G(x, t)$. There exist two positive constants c_1 and c_2 independent of G such that for $p > 2/\lambda$ and $\lambda > 1$*

$$c_1 \|G\|_p \leq \|g_\gamma^*(G)(x; \lambda)\|_p \leq c_2 \|G\|_p.$$

From Theorem 2 we see Lemma 9 holds if and only if it holds for $\gamma = 1$.

For $\gamma = 1$, however, Lemma 9 is a corollary of Lemma 6 and Proposition 2 of [6].

We can prove the second part of Theorem 4 by the same argument we used for Theorem 3 if we observe that $t^{\lambda n}(|x| + t)^{-\lambda n}$ is an increasing function of t .

Remark. The characterization of L_α^p in terms of the g -function is now very simple and uses a method from [7]. We shall be brief.

Theorem 5. *Let $f \in L^p$, $1 < p < \infty$, and let $f(x, t)$ be its Poisson kernel. For $0 < \alpha < k$, set*

$$g_{\alpha, k}(f)(x) = \left(\int_0^\infty t^{1+2(k-\alpha-1)} |f^{(k)}(x, t)|^2 dt \right)^{1/2}.$$

If $f \in L_\alpha^p$,

$$c_1 \|f\|_{p, \alpha} \leq \|g_{\alpha, k}(f)\|_p + \|f\|_p \leq c_2 \|f\|_{p, \alpha}.$$

Moreover, if $f \in L^p$, $1 < p < \infty$, and $g_{\alpha, k}(f) \in L^p$ then $f \in L_\alpha^p$.

For since $g_{\alpha, k}(f)(x) \leq c S_{\alpha, k}(f)(x)$ we have by Theorem 3 that $\|g_{\alpha, k}(f)\|_p \leq \|f\|_{p, \alpha}$. If $f \in L_\alpha^p$, $f = G_\alpha * \psi$, $g = \psi * d\mu$, $\psi = f * d\tau + g * d\sigma$ then, essentially, $\tilde{f}(x) = |x|^{-\alpha} \hat{g}(x)$. A simple approximation argument and Plancherel's formula show

$$\begin{aligned} \int_{\mathbb{R}^n} g(x)h(x) dx &= c \iint_{t>0} t^{k-\alpha} f^{(k)}(x, t) h'(x, t) dx dt \\ &= c \iint_{t>0} t^{k-\alpha-1/2} f^{(k)}(x, t) t^{1/2} h'(x, t) dx dt \\ &\leq c \int_{\mathbb{R}^n} g_{\alpha, k}(f)(x) g(h)(x) dx \end{aligned}$$

where $g(h)$ is the ordinary Littlewood-Paley function of h . Thus with $1/p + 1/p' = 1$,

$$\begin{aligned} \left| \int_{\mathbb{R}^n} g(x)h(x) dx \right| &\leq c \|g_{\alpha, k}(f)\|_p \|g(h)\|_{p'} \\ &\leq c \|g_{\alpha, k}(f)\|_p \|h\|_{p'}. \end{aligned}$$

Hence $\|g\|_p \leq c \|g_{\alpha, k}(f)\|_p$ and, as usual, $\|f\|_{p, \alpha} = \|\psi\|_p \leq c[\|g_{\alpha, k}(f)\|_p + \|f\|_p]$.

To prove the second part of Theorem 5, let $\phi \geq 0$ be smooth, $\int_{\mathbb{R}^n} \phi(x) dx = 1$ and set $\phi_\delta(x) = \delta^{-n} \phi(x/\delta)$. Then it is enough to show that $\|g_{\alpha, k}(f_\delta)\|_p \leq \|g_{\alpha, k}(f)\|_p$. But since

$$f_\delta^{(k)}(x, t) = \int_{\mathbb{R}^n} f^{(k)}(y, t) \phi_\delta(x - y) dy,$$

we obtain by multiplying by $t^{(1+2(k-\alpha-1))/2}$ and taking L^2 norms in t that

$$g_{\alpha,k}(f_\delta)(x) \leq \int_{E_n} g_{\alpha,k}(f)(y) \phi_\delta(x-y) dy.$$

The result now follows from Young's convolution theorem.

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Princeton University

Rutgers, The State University

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