*Approximation-Extension Type Property of Continuous Functions of Bounded Variation**

FON-CHE LIU

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1. Let $f(x)$ be a real-valued continuous function of bounded variation on a closed interval $[a, b]$. It defines an additive function of an interval of bounded variation on $[a, b]$ [3] and therefore generates a totally finite signed measure $F(\cdot)$ on Borel subsets of $[a, b]$. Let $F(\cdot) = F_+(\cdot) - F_-(\cdot)$, where $F_+(\cdot)$ and $F_-(\cdot)$ are the positive and negative parts of $F(\cdot)$ respectively. Define a measure $|F|(\cdot)$ by $|F|(\cdot) = F_+(\cdot) + F_-(\cdot)$. $|F|(\cdot)$ is called the total variational measure generated by $f(x)$.

A theorem of Michael [2], in a form given by Goffman [1], shows that if $f(x)$ is absolutely continuous, then for any given $\epsilon > 0$ there is a continuously differentiable function $g(x)$ defined on $[a, b]$ such that $m(E) < \epsilon$, $|F|(E) < \epsilon$ and $|G|(E) < \epsilon$, where E is the subset of $[a, b]$ on which $f(x)$ and $g(x)$ are different, $|G|(\cdot)$ is the total variational measure generated by $g(x)$, and $m(\cdot)$ is the Lebesgue measure.

The purpose of this paper is to prove the same kind of approximation extension type property for $f(x)$ when $f(x)$ is continuous without being necessarily absolutely continuous. In order to state our main theorem we first make clear a terminology we shall use in the statement of the theorem. Let $[-\infty, +\infty]$ be the two-point compactification of the real line. An extended real-valued function defined on a closed interval is said to be weakly continuous if it is continuous when considered as a mapping from the interval into $[-\infty, +\infty]$. A continuous function defined on a closed interval is called a weak C' -function if its derivative (which may be $+\infty$ or $-\infty$) exists at every point of the interval and is weakly continuous.

We shall prove the following

Main Theorem. *Let $f(x)$ be a real-valued continuous function of bounded*

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variation on the closed interval $[a, b]$. Then, for any $\epsilon > 0$, there is a weak C' -function $g(x)$ defined on $[a, b]$ such that $m(E) < \epsilon$, $|F|(E) < \epsilon$ and $|G|(E) < \epsilon$, where $|F|(\cdot)$, $|G|(\cdot)$ are the total variational measures generated by $f(x)$, $g(x)$ respectively and E is the set on which $f(x)$ and $g(x)$ are different.

In intuitive terms, our main theorem asserts that continuous functions of bounded variation are essentially the functions whose graphs have continuously turning tangents. We prove this theorem in §4. §2 and §3 consist of some preliminary results which are of interest in themselves.

In §2 we get a decomposition $\{A_+, A_-, \mathfrak{N}\}$ of $[a, b]$ such that $A_+(A_-)$ is the union of a countable number of Borel sets on each of which $f(x)$ is monotonically nondecreasing (nonincreasing), and $|F|(\mathfrak{N}) = 0$. In order to do so we need an integral representation of the total variational measures generated by continuous functions of bounded variation. Let B be any Borel subset of $[a, b]$ and y any real number, we define $n(B, y) = N(f^{-1}(y) \cap B)$, where $N(\cdot)$ is the counting measure. The following integral representation of $|F|(\cdot)$ [5] is called the Banach indicatrix formula,

$$|F|(B) = \int_{-\infty}^{\infty} n(B, y) dy.$$

In §3 we deal with the extension problem of Whitney's type [4] with more stringent condition on both initial and extended functions.

Finally, we note that $F_+(\cdot)$, $F_-(\cdot)$ and $|F|(\cdot)$ are regular nonatomic measures.

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2. A decomposition theorem. In this section we always denote by I and J the closed interval $[a, b]$ and the half-open interval $[a, b)$ respectively. Let

$$A^+ = \{x \in J : \exists \delta_x > 0 \text{ such that } f(x) \leq f(y) \text{ if } x < y \leq x + \delta_x\},$$

$$A^- = \{x \in J : \exists \sigma_x > 0 \text{ such that } f(x) \geq f(y) \text{ if } x < y \leq x + \sigma_x\},$$

and

$$\mathfrak{N} = I \setminus (A^+ \cup A^-).$$

Lemma 1. A^+ , A^- and $\mathfrak{N}$ are Borel sets.

Proof. We only have to prove that A^+ and A^- are Borel. Let $B_k = \{x \in J : \exists y \text{ with } x < y < x + 1/k \text{ such that } f(x) > f(y)\}$ for each positive integer k . It is obvious that each B_k is relative open in J and therefore is Borel. So $\bigcap_{k=1}^{\infty} B_k$ is Borel. But $A^+ = J \setminus (\bigcap_{k=1}^{\infty} B_k)$, which shows that A^+ is Borel. Similarly, A^- is Borel. Q.E.D.

We define two functions $\bar{\delta}(x)$ and $\bar{\sigma}(x)$ on A^+ and A^- respectively by

$$\bar{\delta}(x) = \text{largest } \delta_x > 0 \text{ such that } f(x) \leq f(y) \text{ if } x < y \leq x + \delta_x, x \in A^+;$$

$$\bar{\sigma}(x) = \text{largest } \sigma_x > 0 \text{ such that } f(x) \geq f(y) \text{ if } x < y \leq x + \sigma_x, x \in A^-.$$

The existences of $\bar{\delta}(x)$ and $\bar{\sigma}(x)$ on respective defining domains follow from the continuity of $f(x)$. Obviously, both $\bar{\delta}(x)$ and $\bar{\sigma}(x)$ are upper semi-continuous on their defining domains and therefore are Borel measurable. For every positive integer n define two Borel sets B_n^+ and B_n^- as follows

$$B_n^+ = \left\{ x \in A^+ : \frac{b-a}{n+1} \leq \bar{\delta}(x) < \frac{b-a}{n} \right\},$$

$$B_n^- = \left\{ x \in A^- : \frac{b-a}{n+1} \leq \bar{\sigma}(x) < \frac{b-a}{n} \right\}.$$

Then both families $\{B_n^+\}_{n=1}^\infty$, $\{B_n^-\}_{n=1}^\infty$ are pairwise disjoint and $A^+ = \bigcup_{n=1}^\infty B_n^+$, $A^- = \bigcup_{n=1}^\infty B_n^-$. For positive integers n and k , $1 \leq k \leq n+1$, let J_{nk} be the half-open interval

$$\left(a + \frac{(k-1)(b-a)}{n+1}, a + \frac{k(b-a)}{n+1} \right), \quad C_{nk}^+ = B_n^+ \cap J_{nk}$$

and $C_{nk}^- = B_n^- \cap J_{nk}$. From the defining properties of $C_{nk}^+(C_{nk}^-)$, $n = 1, 2, \dots$, $k = 1, \dots, n+1$, it follows at once that $f(x)$ is monotone increasing (decreasing) when restricted to $C_{nk}^+(C_{nk}^-)$. We rearrange the sequences $\{C_{nk}^+\}$, $\{C_{nk}^-\}$, $n = 1, 2, \dots$, $k = 1, \dots, n+1$ and denote them by $\{A_n^+\}_{n=1}^\infty$, $\{A_n^-\}_{n=1}^\infty$ respectively. Then $\{A_n^+\}_{n=1}^\infty$ is a decomposition of A_+ into Borel sets such that on each of them $f(x)$ is monotone increasing, $\{A_n^-\}_{n=1}^\infty$ is a decomposition of A^- into Borel sets such that on each of them $f(x)$ is monotone decreasing.

We shall prove in the rest of this section that A^+ is a support of $F_+(\cdot)$ and A^- is a support of $F_-(\cdot)$ i.e. $F_+(B) = F(B \cap A^+)$ and $F_-(B) = -F(B \cap A^-)$ for any Borel set $B \subset I$.

Lemma 2. *If $E \subset I$ is Borel and $f(x)$ is monotone increasing on E , then $F(E) \geq 0$.*

Proof. It is sufficient to prove the case that E is compact. So suppose E is compact. Let $\epsilon > 0$ be given. Let J_E be the smallest closed interval containing E , $\theta = J_E \setminus E$ and $\theta = \bigcup_{n=1}^\infty J_n$ be the decomposition of θ into disjoint open intervals. If only finitely many of J_n 's are nondegenerate, the lemma is trivial. Otherwise, since $F_-(\theta) = \sum_{n=1}^\infty F_-(J_n) < \infty$, there is N such that

$$\alpha) \quad \sum_{n=N+1}^\infty F_-(J_n) < \epsilon.$$

We consider the set $J_E \setminus \bigcup_{n=1}^N J_n$. It is the union of finitely many closed intervals, say K_n , $n = 1, 2, \dots, l$. Let $\{J_{i_n}\}$ be the subfamily of $\{J_n\}_{n=N+1}^\infty$ which lies in K_n , $n = 1, 2, \dots, l$. Consider any arbitrary finite system $I_i = [a_i, b_i]$, $i = 1, \dots, k$, of nonoverlapping intervals in K_n . By considering separately those points of a_i, b_i , $i = 1, \dots, k$ which belong to some of $\{J_{i_n}\}$ and those in $K_n \cap E$, it is clear from the definition of $F_-(\cdot)$ for intervals, the fact that $f(x)$

is monotone increasing on E and the continuity of $f(x)$ that

$$\beta) \quad \sum_{i=1}^k (f(b_i) - f(a_i)) \geq - \sum_{i_n} F_-(J_{i_n}).$$

Since $-F_-(K_n) = \inf \sum_{i=1}^k (f(b_i) - f(a_i))$, where the infimum is taken over all possible finite systems of nonoverlapping intervals in K_n , it follows from $\beta)$ that $-F_-(K_n) \geq - \sum_{i_n} F_-(J_{i_n})$ or

$$\gamma) \quad F_-(K_n) \leq \sum_{i_n} F_-(J_{i_n}), \quad n = 1, \dots, l.$$

Consequently, from $\alpha)$ and $\gamma)$

$$\delta) \quad F_- \left(J_E \setminus \bigcup_{n=1}^N J_n \right) = \sum_{n=1}^l F_-(K_n) \leq \sum_{n=N+1}^{\infty} F_-(J_n) < \epsilon.$$

Finally, by $\delta)$

$$F_-(E) \leq F_- \left(J_E \setminus \bigcup_{n=1}^N J_n \right) < \epsilon$$

and therefore

$$F(E) = F_+(E) - F_-(E) \geq -E(E) > -\epsilon.$$

But $\epsilon > 0$ is arbitrary, so $F(E) \geq 0$.

Q.E.D.

Similarly, we prove the following

Lemma 3. *If $E \subset I$ is Borel and $f(x)$ is monotone decreasing on E , then $F(E) \leq 0$.*

We remark that Lemma 2 and Lemma 3 hold only when $f(x)$ is continuous. It is very easy to find examples for which Lemma 2 and Lemma 3 do not hold when $f(x)$ is not continuous.

A Borel set $E \subset [a, b]$ is called a positive (or negative) set of $F(\cdot)$ if $F(s) \geq 0$ (or ≤ 0) for every Borel subset S of E .

Lemma 4. *A^+ is a positive set of $F(\cdot)$ and A^- is a negative set of $F(\cdot)$.*

Proof. Let S be any Borel subset of A^+ , then $S = \bigcup_{n=1}^{\infty} (S \cap A_n^+)$ and consequently $F(S) = \sum_{n=1}^{\infty} F(S \cap A_n^+)$. But $f(x)$ is monotone increasing on $S \cap A_n^+$, $n = 1, 2, \dots$. So by Lemma 2 $F(s) \geq 0$, which shows that A^+ is a positive set of $F(\cdot)$. Similarly, A^- is a negative set of $F(\cdot)$ Q.E.D.

Lemma 5. $|F|(\mathfrak{N}) = 0$.

Proof. By Banach indicatrix formula

$$|F|(\mathfrak{N}) = \int_{-\infty}^{\infty} n(y, \mathfrak{N}) dy,$$

so the lemma follows if we show that $n(y, \mathfrak{N}) = 0$ for almost all y . In order to

see this fact we note first that $n(y, \mathfrak{N}) > 0$ implies that $n(y, I) = \infty$, if $y \neq f(b)$, by the defining properties of $\mathfrak{N}$. But, since $|F|(I) = \int_{-\infty}^{\infty} n(y, I) dy < \infty$, $n(y, I) < \infty$ for almost all y . Q.E.D.

It follows from Lemma 4 and Lemma 5 that

Lemma 6. A^+ is a support of $F_+(\cdot)$ and A^- is a support of $F_-(\cdot)$.

We summarize our results in this section into a theorem,

Theorem 1. There is a decomposition $\{A^+, A^-, \mathfrak{N}\}$ of I such that

- 1) $A^+ = \bigcup_{n=1}^{\infty} A_n^+$, where $A_1^+, A_2^+, \dots$ are pairwise disjoint Borel sets on each of which $f(x)$ is monotone increasing,
- 2) $A^- = \bigcup_{n=1}^{\infty} A_n^-$, where $A_1^-, A_2^-, \dots$ are pairwise disjoint Borel sets on each of which $f(x)$ is monotone decreasing, and
- 3) $|F|(\mathfrak{N}) = 0$ and $F(B) = F_+(B \cap A^+) - F_-(B \cap A^-)$, if B is a Borel subset of I .

3. Extension theorems of Whitney's type. For any compact set K in R^1 we shall denote by J_K the smallest closed interval containing K . Let $I = [c, d]$ be a given nondegenerate closed interval. Suppose $v_1 < v_2$, $d_1 \geq 0$, $d_2 \geq 0$ and $\epsilon > 0$ are also given. We denote by $\mathfrak{F}_+(I; v_1, v_2; d_1, d_2; \epsilon)$ the family of all continuously differentiable and strictly increasing functions $f(x)$ on I such that

$$P1) \quad f(c) = v_1, \quad f(d) = v_2; \quad f'(c) = d_1, \quad f'(d) = d_2,$$

$$P2) \quad \min. \left\{ d_1, d_2, \frac{v_2 - v_1}{d - c} - \epsilon \right\}$$

$$\leq f'(x) \leq \max \left\{ d_1, d_2, \frac{v_2 - v_1}{d - c} + \epsilon \right\}, \quad x \in I.$$

We first show that $\mathfrak{F}_+(I; v_1, v_2; d_1, d_2; \epsilon)$ is nonempty. We actually construct such a function. Let $l_1(x)$ and $l_2(x)$ be the linear functions such that $l_1(c) = v_1$, $l_1'(x) \equiv d_1$; $l_2(d) = v_2$, $l_2'(x) \equiv d_2$. It is obvious that we can choose x_1, x_2 such that $a < x_1 < x_2 < b$ and

P3) if $l(x)$ is the linear function with $l(x_1) = l_1(x_1)$, $l(x_2) = l_2(x_2)$, then $l'(x) \neq 0$, $l'(x) \neq d_i$, $i = 1, 2$, and $|l'(x) - (v_2 - v_1)/(d - c)| < \epsilon$,

P4) there are x_3, x_4 with $x_1 < x_3 < x_4 < x_2$ such that $(x_1 - c)^2 + [l_1(x_1) - l_1(c)]^2 = (x_3 - x_1)^2 + [l(x_3) - l(x_1)]^2$ and $(d - x_2)^2 + [l_2(d) - l_2(x_2)]^2 = (x_2 - x_4)^2 + [l(x_2) - l(x_4)]^2$.

Let L_1, L_2 and L be the lines determined by $l_1(x)$, $l_2(x)$ and $l(x)$ in R^2 . Let $f_1(x)$, $f_2(x)$ be the functions on $[c, x_3]$ and $[x_4, d]$ respectively whose graphs are the arcs of the circles C_1 and C_2 with L_1, L tangent to C_1 at $(c, l_1(c))$, $(x_3, l(x_3))$ and L_2, L tangent to C_2 at $(d, l_2(d))$, $(x_4, l(x_4))$. $f_1(x)$ and $f_2(x)$ are so chosen

that $f(x)$ defined below is continuous,

$$\begin{aligned} f(x) &= l(x) & \text{if } x_3 < x < x_4; \\ &= f_1(x) & \text{if } c \leq x \leq x_3; \text{ and} \\ &= f_2(x) & \text{if } x_4 \leq x \leq d. \end{aligned}$$

It follows from $P3$) and the fact $\min. \{d_1, d_2, l'(x)\} \leq f'(x) \leq \max. \{d_1, d_2, l'(x)\}$, $x \in I$ that $f(x) \in \mathfrak{I}_+(I; v_1, v_2; d_1, d_2; \epsilon)$. So $\mathfrak{I}_+(I; v_1, v_2; d_1, d_2; \epsilon)$ is nonempty.

Lemma 7 below is the Egoroff's theorem with continuous parameter. It follows from an obvious modification of the proof for the case of sequences of functions. (See Saks [3], p. 18.)

Lemma 7. *Let (μ, Σ, M) be a measure space, where M is a topological space, Σ is the σ -algebra of Borel sets of M , and μ is a totally finite regular measure on Σ . If f_h is a family of measurable functions depending on a real parameter h which lies in a neighborhood of 0 but different from 0 and $\lim_{h \rightarrow 0} f_h(x)$ exists for almost all $x \in M$. Then for any $\epsilon > 0$ there is a closed subset K of M such that $\mu(M \setminus K) < \epsilon$ and $f_h(x)$ converges uniformly for $x \in K$ as $h \rightarrow 0$.*

Theorem 3. *Let $K \subset R'$ be compact and $g(x)$ be a finite real-valued and strictly increasing function on K . Then, for any $\epsilon > 0$, there is a continuously differentiable and strictly increasing function $g_1(x)$ on J_K and a compact $\hat{K} \subset K$ such that*

- 1) $m(K \setminus \hat{K}) < \epsilon$,
- 2) $g_1(x) = g(x)$ if $x \in \hat{K}$.

Proof. We divide the proof into four steps.

Step 1. We extend $g(x)$ to J_K in the following way. Decompose the complement of K in J_K into at most countably many disjoint open intervals and define the extended function to be linear in each of these intervals and coincide with $g(x)$ at the end points of the interval. We denote this function by $\hat{g}(x)$. $\hat{g}(x)$ is strictly monotone on J_K .

Step 2. $\hat{g}'(x)$ exists and is finite for almost all $x \in J_K$, so for the given $\epsilon > 0$, by Lusin's Theorem, the regularity of $m(\cdot)$ and Lemma 7, we can find a compact set $F \subset J_K$ such that

$$\begin{aligned} \alpha_1) \quad & m(J_K \setminus F) < \epsilon, \\ \beta_1) \quad & \hat{g}'(x) \text{ is continuous relative to } F, \text{ and} \\ \gamma_1) \quad & \frac{\hat{g}(y) - \hat{g}(x)}{y - x} \rightarrow \hat{g}'(x) \text{ uniformly for } x \in F \text{ as } y \rightarrow x. \end{aligned}$$

Take $\hat{K} = F \cap K$; then $\hat{K}$ is compact and $m(K \setminus \hat{K}) < \epsilon$.

Step 3. Decompose the complement of $\hat{K}$ in J_K into at most countably many disjoint intervals, say $J_K \setminus \hat{K} = \bigcup_{n=1}^{\infty} I_n$. We suppose there are infinitely many of I_n 's, otherwise the proof is trivial. For each $n = 1, 2, \dots$, let $a_n < b_n$

be the end points of $\bar{I}_n$ and choose a function $f_n(x)$ from $\mathfrak{I}_+(\bar{I}_n; \mathcal{G}(a_n), \mathcal{G}(b_n); \mathcal{G}'(a_n), \mathcal{G}'(b_n); 1/n)$. Now let a function $g_1(x)$ on J_K be defined by

$$\begin{aligned} g_1(x) &= \mathcal{G}(x) \quad \text{if } x \in \hat{K}; \\ &= f_n(x) \quad \text{if } x \in I_n \text{ for some } n = 1, 2, \dots. \end{aligned}$$

Obviously, $g_1(x)$ is strictly increasing and $g_1(x) = g(x)$ if $x \in \hat{K}$.

Step 4. We prove that $g_1(x)$ is continuously differentiable. First, we show that $g'_1(x)$ exists for every $x \in J_K$. We prove this for those x every neighborhood of which contains infinitely many I_n 's, for at other points the existence of $g'_1(x)$ is obvious. Let x_0 be such a point and for definiteness suppose every neighborhood of x_0 contains infinitely many I_n 's on the left side of x_0 and prove that left side derivative of $g_1(x)$ exists at x_0 and equals $\mathcal{G}'(x_0)$. For any $\delta > 0$, by β_1) and γ_1) we can choose $h_0 > 0$ so that

$$\alpha_2) \quad \left| \frac{\mathcal{G}(y) - \mathcal{G}(x)}{y - x} - \mathcal{G}'(x) \right| < \delta/4, \quad \text{if } x \in \hat{K}, \quad |y - x| < h_0,$$

$$\beta_2) \quad |\mathcal{G}'(y) - \mathcal{G}'(x)| < \delta/4, \quad \text{if } x, y \in \hat{K} \text{ and } |y - x| < h_0.$$

Let N be a positive integer such that $1/N < \delta/4$ and $0 < h < h_0$ be such that $I_n \cap (x_0 - h, x_0) \neq \emptyset$ only when $n \geq N$ and $b_n - a_n < h_0$. We show that $|[g_1(y) - g_1(x_0)]/(y - x_0) - \mathcal{G}'(x_0)| < \delta$ when $y < x_0$ and $|y - x_0| < h$. Suppose $|y - x_0| < h, y < x_0$. Either $y \in \hat{K}$ or $y \in I_n$ for some $n \geq N$ and $b_n - a_n < h_0$.

i) If $y \in \hat{K}$, then

$$\left| \frac{g_1(y) - g_1(x_0)}{y - x_0} - \mathcal{G}'(x_0) \right| = \left| \frac{\mathcal{G}(y) - \mathcal{G}(x_0)}{y - x_0} - \mathcal{G}'(x_0) \right| < \delta/4 < \delta$$

as follows from α_2).

ii) If $y \in I_n$ for some $n \geq N$ and $b_n - a_n < h_0$, then

$$\begin{aligned} \alpha_3) \quad \left| \frac{g_1(y) - g_1(x_0)}{y - x_0} - \mathcal{G}'(x_0) \right| &= \left| \frac{g_1(y) - g_1(b_n) + g_1(b_n) - g_1(x_0)}{y - x_0} - \mathcal{G}'(x_0) \right| \\ &= \left| \frac{f_n(y) - f_n(b_n)}{y - x_0} + \frac{f_n(b_n) - \mathcal{G}(x_0)}{y - x_0} - \mathcal{G}'(x_0) \right| \\ &= \left| \frac{y - b_n}{y - x_0} (f'_n(\tilde{y}) - \mathcal{G}'(x_0)) + \frac{b_n - x_0}{y - x_0} \left(\frac{\mathcal{G}(b_n) - \mathcal{G}(x_0)}{b_n - x_0} - \mathcal{G}'(x_0) \right) \right| \\ &\leq |f'_n(\tilde{y}) - \mathcal{G}'(x_0)| + \left| \frac{\mathcal{G}(b_n) - \mathcal{G}(x_0)}{b_n - x_0} - \mathcal{G}'(x_0) \right| \\ &< |f'_n(\tilde{y}) - \mathcal{G}'(x_0)| + \delta/4 \end{aligned}$$

by α_2), where $\tilde{y}$ lies between y and b_n . From α_2),

$$\alpha_4) \quad \frac{\mathcal{G}(b_n) - \mathcal{G}(a_n)}{b_n - a_n} - \delta/4 < \mathcal{G}'(a_n), \quad \mathcal{G}'(b_n) < \frac{\mathcal{G}(b_n) - \mathcal{G}(a_n)}{b_n - a_n} + \delta/4.$$

It follows from α_4), $P2$) and the fact $1/n \leq 1/N < \delta/4$ that

$$\alpha_5) \quad \left| f'_n(\tilde{y}) - \frac{\hat{g}(b_n) - \hat{g}(a_n)}{b_n - a_n} \right| < \delta/4.$$

Now, from α_3), α_2), β_2) and α_5), we have

$$\begin{aligned} \left| \frac{g_1(y) - g_1(x_0)}{y - x_0} - \hat{g}'(x_0) \right| &< |f'_n(\tilde{y}) - \hat{g}'(x_0)| + \delta/4 \\ &\leq \left| f'_n(\tilde{y}) - \frac{\hat{g}(b_n) - \hat{g}(a_n)}{b_n - a_n} \right| + \left| \frac{\hat{g}(b_n) - \hat{g}(a_n)}{b_n - a_n} - \hat{g}'(b_n) \right| \\ &\quad + |\hat{g}'(b_n) - \hat{g}'(x_0)| + \delta/4 < \delta. \end{aligned}$$

We have proved that $g'_1(x)$ exists for every $x \in J_K$ and

$$\alpha_6) \quad g'_1(x) = \hat{g}'(x) \quad \text{if } x \in \hat{K}.$$

It is clear from β_1), γ_1), $P2$) and α_6) that $g'_1(x)$ is continuous on J_K . Q.E.D.

Similarly, we can prove the following

Theorem 4. *Let $K \subset R'$ be compact and $g(x)$ be a finite real-valued and strictly decreasing function on K . Then, for any $\epsilon > 0$, there is a continuously differentiable and strictly decreasing function $g_1(x)$ on J_K and a compact $\hat{K} \subset K$ such that*

- 1) $m(K \setminus \hat{K}) < \epsilon$,
- 2) $g_1(x) = g(x)$ if $x \in \hat{K}$.

4. Proof of the main theorem. We divide the proof into five steps.

Step 1. As in §2, we denote by $F(\cdot)$ the signed measure generated by $f(x)$ in the sense of Lebesgue-Stieltjes, by $F_+(\cdot)$, $F_-(\cdot)$ the positive and negative parts of $F(\cdot)$, and by $|F|(\cdot) = F_+(\cdot) + F_-(\cdot)$ the total variational measure generated by $f(x)$. By Theorem 2 of §2, there is a pair of disjoint Borel sets A_+ and A_- with the following properties:

- α_1) $A_+ = \bigcup_{n=1}^{\infty} A_n^+$, where A_1^+ , A_2^+ , $\dots$ are disjoint Borel sets on each of which $f(x)$ is monotone increasing,
- β_1) $A_- = \bigcup_{n=1}^{\infty} A_n^-$, where A_1^- , A_2^- , $\dots$ are disjoint Borel sets on each of which $f(x)$ is monotone decreasing,
- γ_1) $F(B) = F_+(B \cap A_+) - F_-(B \cap A_-)$ for every Borel subset B of $[a, b]$.

Step 2. From now on $\epsilon > 0$ is given and fixed. By a theorem of Whitney [4], there is a compact set $K \subset [a, b]$ and a continuously differentiable function $h(x)$ on $[a, b]$ such that

$$\alpha_2) \quad m([a, b] \setminus K) < \epsilon,$$

$$\beta_2) \quad h(x) = f(x) \quad \text{if } x \in K.$$

Decompose the complement of K in $[a, b]$ into at most countably many disjoint intervals, say $[a, b] \setminus K = \bigcup_{n=1}^{\infty} I_n$. Since $|h'(x)|$ is integrable and $|F|([a, b] \setminus K) = \sum_{n=1}^{\infty} |F|(I_n) < \infty$ we can find N_0 such that

$$\alpha_3) \quad \int_{\bigcup_{n=N_0+1}^{\infty} I_n} |h'(x)| \, dx < \epsilon/2,$$

$$\beta_3) \quad |F| \left(\bigcup_{n=N_0+1}^{\infty} I_n \right) = \sum_{n=N_0+1}^{\infty} |F|(I_n) < \epsilon/2.$$

Choose any interval from $I_1, I_2, \dots, I_{N_0}$ and denote it by I . This interval will be fixed in the following discussion. Since $|F|(I \cap A_+) = F_+(I \cap A_+) = F_+(\bigcup_{n=1}^{\infty} \{I \cap A_n^+\}) = \sum_{n=1}^{\infty} F_+(I \cap A_n^+) = F_+(I) < \infty$, there is N_1 such that

$$\alpha_4) \quad \sum_{n=1}^{N_1} F_+(I \cap A_n^+) > F_+(I) - \epsilon/32N_0.$$

For each $n = 1, 2, \dots, N_1$, $f(x)$ is monotone increasing on $I \cap A_n^+$, it follows from Banach indicatrix formula that

$$\alpha_5) \quad F_+(I \cap A_n^+) = m(F(I \cap A_n^+)), \quad n = 1, 2, \dots, N_1.$$

As it is obvious that there are at most countably many points in $f(I \cap A_n^+)$ each of which is the image of more than one point in $I \cap A_n^+$ under the mapping f , it is possible to find a compact set K_n^+ in $f(I \cap A_n^+)$ such that

$$\alpha_6) \quad m(f(I \cap A_n^+) \setminus K_n^+) < \frac{\epsilon}{32N_0N_1},$$

$\beta_6)$ f is 1-1 when considered as a mapping from $f^{-1}(K_n^+) \cap (I \cap A_n^+)$ onto K_n^+ . We denote this mapping by f_n . By $\beta_6)$, the inverse mapping f_n^{-1} of f_n exists and is strictly increasing on K_n^+ . By Theorem 3 of §3, there is a compact set $\hat{K}_n \subset K_n^+$ and a continuously differentiable and strictly increasing function h_n^{-1} on $J_{K_n^+}$ such that

$$\alpha_7) \quad m(K_n^+ \setminus \hat{K}_n) < \frac{\epsilon}{32N_0N_1},$$

$$\beta_7) \quad h_n^{-1}(z) = f_n^{-1}(z) \quad \text{if } z \in \hat{K}_n.$$

Because h_n^{-1} is strictly increasing and continuously differentiable, its inverse h_n is defined and is a c' -function on $h_n^{-1}(J_{K_n^+})$. By $\beta_7)$, we have

$$\alpha_8) \quad h_n(x) = f_n(x) = f(x) \quad \text{if } x \in h_n^{-1}(\hat{K}_n).$$

Combining $\alpha_6)$, $\alpha_7)$ and Banach indicatrix formula, we have

$$\alpha_9) \quad F_+(\{I \cap A_n^+\} \setminus h_n^{-1}(\hat{K}_n)) < \frac{\epsilon}{16N_0N_1}.$$

As $h_1^{-1}(\hat{K}_1), \dots, h_{N_1}^{-1}(\hat{K}_{N_1})$ are disjoint and compact,

$$d = \min_{\substack{i=j \\ i, j=1, \dots, N_1}} d(h_i^{-1}(\hat{K}_i), h_i^{-1}(\hat{K}_i)) > 0,$$

where $d(A, B)$ is the distance of two sets A, B in R^1 . Now, for each $n = 1, 2, \dots, N_1$, decompose the complement of $h_n^{-1}(\hat{K}_n)$ in $h_n^{-1}(J_{\hat{K}_n})$ into at most countably many open intervals, say $h_n^{-1}(J_{\hat{K}_n}) \setminus h_n^{-1}(\hat{K}_n) = \bigcup_{k=1}^{\infty} I_{nk}$, $n = 1, 2, \dots, N_1$. Let $H_n(\cdot)$ be the Lebesgue-Stieltjes measure on Borel sets in $h_n^{-1}(J_{\hat{K}_n})$ generated by h_n . Then since $H_n(\bigcup_{k=1}^{\infty} I_{nk}) = \sum_{k=1}^{\infty} H_n(I_{nk}) < \infty$ and $|F|(\bigcup_{k=1}^{\infty} I_{nk}) = \sum_{k=1}^{\infty} |F|(I_{nk}) < \infty$, we can find M_n such that

$$\alpha_{10}) \quad m(I_{nk}) < \frac{d}{2} \quad \text{for } k = M_n + 1, M_n + 2, \dots,$$

$$\beta_{10}) \quad H_n\left(\bigcup_{k=M_n+1}^{\infty} I_{nk}\right) < \frac{\epsilon}{16N_0N_1},$$

$$\gamma_{10}) \quad |F|\left(\bigcup_{k=M_n+1}^{\infty} I_{nk}\right) < \frac{\epsilon}{32N_0N_1}.$$

Let $C_n = h_n^{-1}(J_{\hat{K}_n}) \setminus \bigcup_{k=1}^{M_n} I_{nk}$, $C = \bigcup_{n=1}^{N_1} C_n$. Then $\alpha_{10})$ and the definition of $d > 0$ guarantee that

$\alpha_{11})$ $C_1, C_2, \dots, C_{N_1}$ are disjoint.

Define a function h_+ on C by

$$h_+(x) = h_n(x) \quad \text{if } x \in C_n, \quad n = 1, 2, \dots, N_1.$$

Because of $\alpha_{11})$, h_+ is well defined. C is the union of a finite number of disjoint closed intervals on each of which h_+ is a weak C' -function. We denote by $H_+(\cdot)$ the total variational measure generated by h_+ on C . Thus we have constructed a compact set C and a continuous function h_+ on C with the following properties:

$\alpha_{12})$ C is the union of a finite number of disjoint closed intervals on each of which h_+ is a weak C' -function,

$$\beta_{12}) \quad F_-(C) < \frac{\epsilon}{32N_0},$$

$$\gamma_{12}) \quad F_+(I \setminus C) < \frac{\epsilon}{8N_0},$$

$$\delta_{12}) \quad |F|(E_+) < \frac{\epsilon}{16N_0}, \quad \text{where } E_+ = \{x \in C : h_+(x) \neq f(x)\},$$

$$\epsilon_{12}) \quad H_+(E_+) < \frac{\epsilon}{16N_0}.$$

$\beta_{12})$ follows from $\gamma_{10})$, because

$$\begin{aligned} F_-(C) &= F_-\left(\bigcup_{n=1}^{N_1} h_n^{-1}(\hat{K}_n)\right) + F_-\left(\bigcup_{n=1}^{N_1} \bigcup_{k=M_n+1}^{\infty} I_{nk}\right) \\ &= F_-\left(\bigcup_{n=1}^{N_1} \bigcup_{k=M_n+1}^{\infty} I_{nk}\right) \leq \sum_{n=1}^{N_1} |F|\left(\bigcup_{k=M_n+1}^{\infty} I_{nk}\right) < \frac{\epsilon}{32N_0}, \end{aligned}$$

by γ_{10}). γ_{12}) is a consequence of α_4), α_9) and γ_{10}), while δ_{12}) and ϵ_{12}) follow from γ_{10}) and β_{10}) respectively.

Step 3. Similarly, if we start with A_- and use Theorem 4 of §3 we can construct a compact set $D \subset I$ and a continuous function h_- on D with the following properties:

α_{13}) D is the union of a finite number of disjoint closed intervals on each of which h_- is a weak C' -function,

$$\beta_{13}) \quad F_+(D) < \frac{\epsilon}{32N_0},$$

$$\gamma_{13}) \quad F_-(I \setminus D) < \frac{\epsilon}{8N_0},$$

$$\delta_{13}) \quad |F|(E_-) < \frac{\epsilon}{16N_0}, \quad \text{where } E_- = \{x \in D : h_-(x) \neq f(x)\},$$

ϵ_{13}) $H_-(E_-) < \epsilon/16N_0$, where $H_-(\cdot)$ is the total variational measure generated by h_- on D .

Step 4. From β_{12}) and β_{13}), it follows that

$$\alpha_{14}) \quad |F|(C \cap D) \leq F_+(D) + F_-(C) < \frac{\epsilon}{16N_0}.$$

Let $\hat{C} = C \setminus \{C \cap D\}^0$, $\hat{D} = D \setminus \{C \cap D\}^0$, then

$$\begin{aligned} \alpha_{15}) \quad |F|(I \setminus (\hat{C} \cup \hat{D})) &= F_+(I \setminus (\hat{C} \cup \hat{D})) + F_-(I \setminus (\hat{C} \cup \hat{D})) \\ &\leq F_+(I \setminus C) + F_+(C \cap D) + F_-(I \setminus D) + F_-(C \cap D) \\ &= F_+(I \setminus C) + F_-(I \setminus D) + |F|(C \cap D) < \frac{5\epsilon}{16N_0} \end{aligned}$$

by γ_{12}), γ_{13}) and α_{14}). Both $\hat{C}$ and $\hat{D}$ are unions of a finite number of closed intervals some of which may meet at end points. If necessary, we shrink some of the closed intervals of $\hat{C}$ and $\hat{D}$ to get new closed sets C_+ and D_- such that $C_+ \cap D_- = \emptyset$, $(C_+ \cup D_-) \cap \{c, d\} = \emptyset$, where $c < d$ are the end points of I , and

$$\alpha_{16}) \quad |F|(I \setminus (C_+ \cup D_-)) < \frac{5\epsilon}{16N_0},$$

β_{16}) if $I \setminus (C_+ \cup D_-) = J_1 \cup J_2 \cup \dots \cup J_l$, where J_i 's are disjoint intervals with end points $c_i < d_i$, and if we define a function h_0 on $C_+ \cup D_- \cup \{c, d\}$ by

$$\begin{aligned} h_0(x) &= h_+(x) \quad \text{if } x \in C_+, \\ &= h_-(x) \quad \text{if } x \in D_-, \\ h_0(c) &= h(c) = f(c), \quad h_0(d) = h(d) = f(d), \end{aligned}$$

then

$$\sum_{i=1}^l |h_0(d_i) - h_0(c_i)| < \frac{5\epsilon}{16N_0}.$$

This can be done, because $|F|(\cdot)$ is nonatomic and regular, the values taken by h_+ and h_- at the end points of the closed intervals of $\hat{C}$ and $\hat{D}$ coincide with those of f , h_+ and h_- are continuous on $\hat{C}$ and $\hat{D}$ respectively and α_{15}) holds. From β_{16}), it is possible to extend h_0 to a weak C' -function h_I on I whose derivative coincides with that of h at the end points of I such that if $|H_I|(\cdot)$ is the total variational measure generated by h_I on I , then

$$\alpha_{17}) \quad |H_I|(I \setminus (C_+ \cup D_-)) < \frac{5\epsilon}{16N_0}.$$

Now, it follows from ϵ_{12}), ϵ_{13}) and α_{17}) that

$$\alpha_{18}) \quad |H_I|(E_I) < \frac{\epsilon}{2N_0}, \quad \text{where } E_I = \{x \in I : h_I(x) \neq f(x)\},$$

and from δ_{12}), δ_{13}) and α_{16}) that

$$\alpha_{19}) \quad |F|(E_I) < \frac{\epsilon}{2N_0}.$$

Step 5. For each $i = 1, 2, \dots, N_0$, we construct h_{I_i} on I_i which satisfies α_{18}) and α_{19}) and define $g(x)$ by

$$\begin{aligned} g(x) &= h(x) \quad \text{if } x \in [a, b] \setminus \bigcup_{i=1}^{N_0} I_i, \\ &= h_{I_i}(x) \quad \text{if } x \in I_i. \end{aligned}$$

Then $g(x)$ is a weak C' -function on $[a, b]$. That $g(x)$ satisfies the conclusions of the theorem follows from $\alpha_2)$, $\beta_2)$, $\alpha_3)$, $\beta_3)$, α_{18}) and α_{19}). Q.E.D.

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Wayne State University

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