On the Inversion of Toeplitz Operators

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1. Introduction. Let m denote the normalized Haar measure on the circle group $\mathbf{T}$. For $1 \leq p \leq \infty$, L^p will denote the usual (complex) Lebesgue space of this measure space. The closed subspace of L^p consisting of all functions whose Fourier coefficients vanish on the negative integers will be denoted by H^p . For $g \in L^\infty$, the *Toeplitz operator* T_g is an operator on H^2 (bounded linear transformation of H^2 into itself) defined by

$$T_g f = P(gf), \quad f \in H^2$$

where P is the orthogonal projection of L^2 onto H^2 . Widom ([15], [16]) obtained necessary and sufficient conditions on g in order that T_g is an invertible operator *i.e.* a bijective mapping. Independently, Devinatz obtained the inversion theorem in [1]. Devinatz shows that the theorem is valid for H^2 -spaces of Dirichlet algebras. Their final result is the following:

T_g is invertible if and only if g can be written in the form $\gamma \exp [u + i(v + \bar{w})]$, where γ is a complex number of modulus 1, u, v, w are real functions in L^∞ , $\|v\|_\infty < \pi/2$ and $\bar{w}$ denotes the conjugate function of w .

In this paper we study the problem of invertibility of Toeplitz operators induced by essentially bounded operator valued functions on the H^2 space of functions taking values in a complex separable Hilbert space. We shall show in §4 that an appropriate reformulation of the above result holds in the vectorial case. This reformulation states that T_g is invertible if and only if g can be factored in the form uh where h and h^{-1} are in H^∞ , u is a function of modulus 1 almost everywhere and there exists an invertible element of H^∞ whose L^∞ -distance from u is less than 1. The equivalence of the two statements is a consequence of the formula for outer functions [8, p. 62] and a Lemma of Helson and Szegő [6, p. 126, Lemma 1].

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In §2, we define Toeplitz operators in the general setup mentioned above and state some basic results about them. §3 contains an extension of a variational principle of Helson and Szegö [6]. We shall need this result for determining conditions of invertibility on Toeplitz operators induced by operator valued functions whose values are unitary operators. In the proof of this variational principle we make use of some recent results of Sarason [13].

The variational principle mentioned above has close connections with a problem in prediction theory. This problem arises in connection with a strong mixing criterion of Rosenblatt [12] and consists of giving spectral conditions for strong mixing for a class of stationary processes. The earliest work on this problem is that of Kolmogorov and Rozanov [9]. Recently, Helson and Sarason [5] solved this problem in the case of complex valued processes. In §5 we present some partial results on this problem for vector valued processes.

2. Definition and basic properties of Generalized Toeplitz operators. Let $\mathcal{H}$ be a complex, separable (possibly finite dimensional) Hilbert space. Let $\mathcal{B}$ denote the Banach algebra of all operators on $\mathcal{H}$ equipped with the operator norm. We need to consider certain Banach spaces of functions defined on $\mathbf{T}$ and taking values in $\mathcal{H}$ or $\mathcal{B}$. These functions will all be measurable in the weak sense [3, §3.5]. Two functions which agree almost everywhere with respect to m will be identified. We shall use the same symbol for a function and its equivalence class under this identification. This will not cause any confusion.

$L^2(\mathcal{H})$ will denote the space of all (equivalence classes of) weakly measurable $\mathcal{H}$ -valued functions on $\mathbf{T}$ which satisfy the condition

$$\int \|F(z)\|^2 dm(z) < \infty.$$

$L^2(\mathcal{H})$ is a Hilbert space under pointwise linear operations and the inner product

$$(F, G)_2 = \int (F(z), G(z)) dm(z).$$

The norm of an element F in $L^2(\mathcal{H})$ will be denoted by $\|F\|_2$, to distinguish it from the norm $\|F(z)\|$ of the vector $F(z)$ in $\mathcal{H}$. An element F of $L^2(\mathcal{H})$ is said to be *analytic* if the scalar function $(F(\cdot), \phi)$ is in H^2 for each ϕ in $\mathcal{H}$. $H^2(\mathcal{H})$ will denote the closed subspace of all analytic functions in $L^2(\mathcal{H})$. In the classical case, the subspace of H^p consisting of all functions with mean value zero is usually denoted by H_0^p . In the vectorial case, we shall use the symbol $H_0^2(\mathcal{H})$ for the subspace of $H^2(\mathcal{H})$ consisting of all elements F for which $(F(\cdot), \phi)$ is in H_0^2 for each ϕ in $\mathcal{H}$. The orthogonal complement of $H_0^2(\mathcal{H})$ will be denoted by $K^2(\mathcal{H})$. The elements of $K^2(\mathcal{H})$ are characterized by the property that the scalar function $(\phi, F(\cdot))$ is in H^2 for each ϕ in $\mathcal{H}$. Because of this they will be called *conjugate analytic*.

The space $L^\infty(\mathcal{B})$ is the space of weakly measurable $\mathcal{B}$ -valued functions A on $\mathbf{T}$ for which the norm

$$\|A\|_{\infty} = \text{ess sup } \|A(z)\|$$

is finite. With this norm and pointwise algebraic operations and involution $L^{\infty}(\mathfrak{B})$ becomes a B^* -algebra with identity. The algebra L^{∞} of essentially bounded scalar functions can be regarded as a subalgebra of $L^{\infty}(\mathfrak{B})$ in an obvious manner.

An element A of $L^{\infty}(\mathfrak{B})$ induces an operator on $L^2(\mathfrak{H})$ which we shall denote by M_A . The action of M_A is given by

$$(M_A F)(z) = A(z)F(z)$$

Let χ be the function on $\mathbf{T}$ defined by $\chi(z) = z$. The unitary operator M_{χ} is of particular importance and it will be denoted by S . From the defining relation, it is clear that M_A commutes with S and that $\|M_A\| \leq \|A\|_{\infty}$. In fact, it can be shown that $\|M_A\| = \|A\|_{\infty}$ and that any operator on $L^2(\mathfrak{H})$ which commutes with S arises in this way [11, p. 499]. Moreover, the map $A \rightarrow M_A$ is a *star-homomorphism*.

An element A of $L^{\infty}(\mathfrak{B})$ is said to be *analytic* if the scalar function $(A(\cdot)\phi, \psi)$ is in H^{∞} for each pair of vectors ϕ, ψ in $\mathfrak{H}$. It is easy to see that A is analytic if and only if M_A maps $H^2(\mathfrak{H})$ into itself. The space of analytic elements of $L^{\infty}(\mathfrak{B})$ will be denoted by $H^{\infty}(\mathfrak{B})$. $H^{\infty}(\mathfrak{B})$ is clearly a closed subalgebra of $L^{\infty}(\mathfrak{B})$.

For A in $L^{\infty}(\mathfrak{B})$, the *Generalized Toeplitz operator* T_A is defined as the restriction of the operator $P M_A$ to $H^2(\mathfrak{H})$ where P is the orthogonal projection of $L^2(\mathfrak{H})$ onto $H^2(\mathfrak{H})$. We shall now state some of the basic properties of the map $A \rightarrow T_A$ from $L^{\infty}(\mathfrak{B})$ into the algebra of operators on $H^2(\mathfrak{H})$.

Proposition 2.1. *Let A and B denote elements of $L^{\infty}(\mathfrak{B})$.*

- (i) *The map $A \rightarrow T_A$ is linear.*
- (ii) *$T_A^* = T_{A^*}$.*
- (iii) *$T_{AB} = T_A T_B$ if either $B \in H^{\infty}(\mathfrak{B})$ or $A^* \in H^{\infty}(\mathfrak{B})$.*
- (iv) *$\|T_A\| = \|A\|_{\infty}$.*
- (v) *The spectrum of A is contained in the spectrum of T_A .*

Statements (i), (ii), and (iii) are easy to verify. As for (iv) and (v), the proofs given in [2] for the scalar case can be easily adapted to the present context.

3. A variational problem. For an element A of $L^{\infty}(\mathfrak{B})$, define

$$\begin{aligned} \rho(A) \\ (1) \quad &= \sup \left\{ \left| \int (AF, G) dm \right| : F \in H^2(\mathfrak{H}), G \in H^2(\mathfrak{H})^{\perp}, \|F\|_2 \leq 1, \|G\|_2 \leq 1 \right\}. \end{aligned}$$

It is easy to see that

$$(2) \quad \rho(A) = \sup \left| \int \chi(AF, G) dm \right|$$

where the supremum in (2) is taken over all F and G in the closed unit balls of

$H^2(\mathfrak{H})$ and $K^2(\mathfrak{H})$ respectively. The variational problem is to obtain a convenient formula for $\rho(A)$. When $\mathfrak{H}$ has dimension 1, Helson and Szegő [6, p. 125] showed that $\rho(A)$ is equal to the distance of A from H^∞ . We shall show that the same result extends to the present situation with H^∞ replaced by $H^\infty(\mathfrak{B})$. Helson and Szegő transform this problem into a linear variational problem in L^1 using a classical result of F. Riesz [8, p. 52] which states that every function in the unit ball of H^1 is expressible as the product of two functions in the unit ball of H^2 . They proceed to solve the problem in L^1 using the duality between L^1 and L^∞ .

Obviously, the methods of these authors do not apply directly to the vectorial case. To overcome this difficulty we shall show that $\rho(A)$ is equal to an analogous expression which can be computed by the technique of Helson and Szegő. The generalization of the F. Riesz theorem is now provided by Sarason [13]. We have to introduce some more notations before we proceed.

Let $\mathfrak{S}$ be the space of all Hilbert-Schmidt operators from $\mathfrak{H}$ into itself. The space $\mathfrak{S}$ is a separable Hilbert space under the inner product

$$\langle X, Y \rangle = \text{tr} (XY^*)$$

where tr denotes the trace [14, pp. 29 ff.]. The norm of this Hilbert space will be denoted by $\sigma(\cdot)$. The Hilbert space $L^2(\mathfrak{S})$ and its subspace $H^2(\mathfrak{S})$ can be defined as in §2. We remark that an element X of $L^2(\mathfrak{S})$ is in $H^2(\mathfrak{S})$ if and only if the scalar function $(X(\cdot)\phi, \psi)$ is in H^2 for each pair of vectors ϕ, ψ in $\mathfrak{H}$.

For an element A in $L^\infty(\mathfrak{B})$, let

$$(3) \quad R(A) = \sup \left| \int \chi(z) \text{tr} (A(z)X(z)Y(z)) dm(z) \right|$$

where the supremum is taken over all X and Y in the closed unit ball of $H^2(\mathfrak{S})$.

Lemma 3.1. *Let $A \in L^\infty(\mathfrak{B})$. Then $\rho(A) = R(A)$.*

Proof. Let $\{e_i : i \in J\}$ be an orthonormal basis for $\mathfrak{H}$. For definiteness we shall assume that J is the set of positive integers if $\mathfrak{H}$ is infinite dimensional or the set $\{1, 2, \dots, N\}$ if the dimension of $\mathfrak{H}$ is $N < \infty$. Let F and G be in the closed unit balls of $H^2(\mathfrak{H})$ and $K^2(\mathfrak{H})$ respectively. Define an operator valued function X by

$$\begin{aligned} X(z)e_1 &= F(z) \quad \text{wherever } F \text{ is defined} \\ X(z)e_i &= 0 \quad j \neq 1. \end{aligned}$$

Then $X \in H^2(\mathfrak{S})$ and $\sigma(X)_2 = \|F\|_2 \leq 1$. Similarly, define an operator valued function Y_1 by

$$\begin{aligned} Y_1(z)e_1 &= G(z) \quad a \cdot e \cdot z \\ Y_1(z)e_i &= 0 \quad j \neq 1. \end{aligned}$$

Let $Y(z) = (Y_1(z))^*$. Then Y is in the closed unit ball of $H^2(\mathfrak{S})$. For almost

every z ,

$$(4) \quad \text{tr} (A(z)X(z)Y(z)) = (A(z)F(z), G(z))$$

From (2), (3) and (4) it follows that $\rho(A) \leq R(A)$.

To prove the reverse inequality choose X and Y in the closed unit ball of $H^2(\mathfrak{S})$ such that

$$\left| \int \chi(z) \text{tr} (A(z)X(z)Y(z)) \, dm(z) \right| > R(A) - \epsilon$$

where $\epsilon > 0$ is arbitrary. For each $j \in J$ consider the vector valued functions F_j and G_j defined by

$$F_j(z) = X(z)e_j \quad \text{wherever } X \text{ is defined}$$

$$G_j(z) = Y^*(z)e_j \quad \text{wherever } Y \text{ is defined.}$$

Then $F_j \in H^2(\mathfrak{H})$ and $G_j \in K^2(\mathfrak{H})$. For each j let $a_j = \|F_j\|_2$ and $b_j = \|G_j\|_2$. Then we have

$$\sum_j a_j^2 = \sigma(X)_2^2 \leq 1$$

and

$$\sum_j b_j^2 = \sigma(Y)_2^2 \leq 1.$$

For almost every z ,

$$(5) \quad \text{tr} (A(z)X(z)Y(z)) = \sum_{j \in J} (A(z)F_j(z), G_j(z))$$

The series on the right side of (5) converges absolutely for almost every z . In fact by Schwarz's inequality, for any positive integer k and almost every z ,

$$(6) \quad \sum_{j=1}^k |(A(z)F_j(z), G_j(z))| \leq \|A\|_\infty \left(\sum_{j \in J} \|F_j(z)\|^2 \right)^{1/2} \left(\sum_{j \in J} \|G_j(z)\|^2 \right)^{1/2} \\ \leq \|A\|_\infty \sigma(X(z))\sigma(Y(z)).$$

Since X and Y are in $H^2(\mathfrak{S})$, the right hand side of (6) is summable. By the dominated convergence theorem, we can integrate term by term in (5). This yields the inequality

$$\left| \int \chi \text{tr} (AXY) \, dm \right| \leq \sum_{j \in J} \left| \int \chi (AF_j, G_j) \, dm \right|.$$

Since $F_j \in H^2(\mathfrak{H})$ and $G_j \in K^2(\mathfrak{H})$, by (2) we have

$$(7) \quad \left| \int \chi \text{tr} (AXY) \, dm \right| \leq \rho(A) \sum_{j \in J} a_j b_j.$$

By applying Schwarz's inequality to the series in (7), it follows that

$|\int \chi \operatorname{tr} (AXY) dm| \leq \rho(A)$. Thus $R(A) - \epsilon \leq \rho(A)$. ϵ being arbitrary, $R(A) \leq \rho(A)$. Hence $\rho(A) = R(A)$.

We shall show that $R(A)$ equals the distance of A from $H^\infty(\mathfrak{B})$. To do this we have to consider still another Banach space of operator valued functions. Let $\mathfrak{J}$ denote the trace class of operators on $\mathfrak{H}$ [14, pp. 36 ff.]. The trace norm on $\mathfrak{J}$ will be denoted by $\tau(\cdot)$. It is known that $\mathfrak{J}$ is separable Banach space with this norm. For an operator $A \in \mathfrak{B}$, let

$$\Phi_A(D) = \operatorname{tr} (AD)$$

The correspondence $A \rightarrow \Phi_A$ is an isometric isomorphism of $\mathfrak{B}$ onto the dual space of $\mathfrak{J}$. Let $L^1(\mathfrak{J})$ be the space $\mathfrak{J}$ -valued functions D on $\mathbf{T}$ such that

$$\tau(D)_1 = \int \tau(D(z)) dm(z) < \infty.$$

Under this norm and pointwise linear operations, $L^1(\mathfrak{J})$ is a Banach space. The subspace $H^1(\mathfrak{J})$ is defined as the set of all D in $L^1(\mathfrak{J})$ for which the function $(D(\cdot)\phi, \psi)$ is in H^1 for every pair of vectors ϕ and ψ in $\mathfrak{H}$. The subspace $H_0^1(\mathfrak{J})$ is defined analogously. The dual space of $L^1(\mathfrak{J})$ was identified by Sarason [13, p. 197]. For convenience we shall state some of his results here.

Proposition 3.1. (i) For $A \in L^\infty(\mathfrak{B})$ define

$$\Phi_A(D) = \int \operatorname{tr} (A(z)D(z)) dm(z), \quad D \in L^1(\mathfrak{J}).$$

The map $A \rightarrow \Phi_A$ is an isometric isomorphism of $L^\infty(\mathfrak{B})$ onto the dual space $L^1(\mathfrak{J})$.

(ii) The annihilator of $H_0^1(\mathfrak{J})$ in $L^\infty(\mathfrak{B})$ is $H^\infty(\mathfrak{B})$.

It is easy to see that if X and Y are in the closed unit ball of $H^2(\mathfrak{S})$ then the product XY is in the closed unit ball of $H^1(\mathfrak{J})$. The converse of this is true and forms a special case of Sarason's generalization of the F. Riesz factorization theorem [13, p. 198, Theorem 4].

Proposition 3.2. Each function D in $H^1(\mathfrak{J})$ can be factored in the form XY where X and Y are in $H^2(\mathfrak{S})$ and

$$Y^*Y = (D^*D)^{1/2}, \quad X^*X = YY^*.$$

We are now ready to evaluate $R(A)$.

Lemma 3.2. Let $A \in L^\infty(\mathfrak{B})$. Then

$$R(A) = \inf \{ \|A - B\|_\infty : B \in H^\infty(\mathfrak{B}) \}$$

Proof. The first step is to show that $R(A)$ is the supremum over $H_0^1(\mathfrak{J})$ of the linear functional induced by A on $L^1(\mathfrak{J})$. By definition

$$R(A) = \sup \left| \int \chi \operatorname{tr} (AXY) dm \right|$$

where X and Y vary over the closed unit ball of $H^2(\mathfrak{S})$. The product XY is in the closed unit ball of $H^1(\mathfrak{J})$. Therefore $R(A) \leq \sup |\int \chi \operatorname{tr} (AD) dm|$, where the supremum is taken over all D in the closed unit ball of $H^1(\mathfrak{J})$. If D is in the closed unit ball of $H^1(\mathfrak{J})$, by Proposition 3.2, it can be factored in the form XY where X and Y are in the closed unit ball of $H^2(\mathfrak{S})$. Thus $R(A) = \sup |\int \chi \operatorname{tr} (AD) dm|$, the supremum being taken over all D in the closed unit ball of $H^1(\mathfrak{J})$. It is easy to see that $R(A) = \sup |\int \operatorname{tr} (AD) dm|$, where the supremum is now taken over all D in the closed unit ball of $H_0^1(\mathfrak{J})$. We shall now use the duality between $L^1(\mathfrak{J})$ and $L^\infty(\mathfrak{B})$ and the fact that the annihilator of $H_0^1(\mathfrak{J})$ is $H^\infty(\mathfrak{B})$ (Proposition 3.1). By the Hahn-Banach theorem, it follows that $R(A) = \inf \|A - B\|_\infty$ where B varies over $H^\infty(\mathfrak{B})$.

Lemmas 3.1 and 3.2 combine to give the main result of this section.

Theorem 1. *Let $A \in L^\infty(\mathfrak{B})$ and let $\rho(A)$ be defined by (1). Then*

$$\rho(A) = \inf \{ \|A - B\|_\infty : B \in H^\infty(\mathfrak{B}) \}.$$

4. The invertibility of Generalized Toeplitz operators. In this section we shall prove our main theorem on Generalized Toeplitz operators. When $\mathfrak{H}$ is finite dimensional, this theorem has been obtained independently by H. R. Pousson [17].

Theorem 2. *Let $A \in L^\infty(\mathfrak{B})$. The operator T_A on $H^2(\mathfrak{H})$ is invertible if and only if A can be factored in the form*

$$(8) \quad A = B_1^* U B_2$$

where

- (i) B_1 and B_2 are in $H^\infty(\mathfrak{B})$ and have inverses in $H^\infty(\mathfrak{B})$
- (ii) U is an element of $L^\infty(\mathfrak{B})$ taking unitary values and there exists an element B_3 of $H^\infty(\mathfrak{B})$ having an inverse there and satisfying the condition

$$(9) \quad \|U - B_3\|_\infty < 1.$$

We need three lemmas to complete the proof of this theorem. Lemma 3.1 forms part of a proof in [4, p. 500]. This lemma shows that every invertible element of $L^\infty(\mathfrak{B})$ can be expressed as a product of a unitary valued element and an element of $H^\infty(\mathfrak{B})$ having an analytic inverse. (For convenience we shall call an element of $L^\infty(\mathfrak{B})$ which takes unitary values a *unitary function*.) This lemma shows that our problem reduces to the study of invertibility of two special kinds of Toeplitz operators. Lemma 4.2 takes care of Toeplitz operators induced by elements of $H^\infty(\mathfrak{B})$. The result in this case is not too surprising. Lemma 4.3 gives conditions for the invertibility of Toeplitz operators induced by unitary functions. In the proof of this lemma we shall make use of the result obtained in §3 on the Helson-Szegö variational problem.

Lemma 4.1. *Let A be an invertible element of $L^\infty(\mathfrak{B})$. Then there exists a unitary function U such that $M_A(H^2(\mathfrak{H})) = M_U(H^2(\mathfrak{H}))$. In particular, A can*

be factored in the form UB where U is a unitary function and B is in $H^\infty(\mathfrak{B})$ and is invertible there.

Proof. The proof depends on a well-known theorem on the structure of subspaces of $L^2(\mathfrak{H})$ that are invariant under S [3, p. 61]. Let $\mathfrak{M} = M_A(H^2(\mathfrak{H}))$. $\mathfrak{M}$ is a closed subspace of $L^2(\mathfrak{H})$ invariant under S . $\mathfrak{M}$ has the additional property that

$$\bigcap_{n \geq 1} S^n(\mathfrak{M}) = \{0\}.$$

Moreover, the set of $F(z)$ with F in $\mathfrak{M}$ spans $\mathfrak{H}$ for almost every z . Then there exists [3, p. 64] a unitary function U such that $M_A(H^2(\mathfrak{H})) = M_U(H^2(\mathfrak{H}))$. To obtain the factoring for A , simply set $B = U^{-1}A$. It is clear that B and B^{-1} are in $H^\infty(\mathfrak{B})$.

Lemma 4.2. *Let $A \in H^\infty(\mathfrak{B})$. Then T_A is invertible if and only if A has an inverse in $H^\infty(\mathfrak{B})$.*

Proof. If A is invertible and A^{-1} is in $H^\infty(\mathfrak{B})$, $T_{A^{-1}}$ is the inverse of T_A , which proves the sufficiency of the condition.

Conversely, suppose T_A is invertible. By the spectral inclusion theorem (Proposition 2.1 (v)), A is invertible in $L^\infty(\mathfrak{B})$. It remains to show that this inverse is in $H^\infty(\mathfrak{B})$. The invertibility of T_A implies that for any vector ϕ in $\mathfrak{H}$ there exists a unique $F \in H^2(\mathfrak{H})$ such that $T_A F = \phi$. A being analytic, this means that $A(z)F(z) = \phi$ almost everywhere. Hence $(A^{-1}(\cdot)\phi, \psi)$ is analytic for every pair of vectors ϕ, ψ in $\mathfrak{H}$.

Lemma 4.3. *Let U be a unitary function. Then T_U is invertible if and only if there exists a B in $H^\infty(\mathfrak{B})$ such that B has an inverse in $H^\infty(\mathfrak{B})$ and $\|U - B\|_\infty < 1$.*

Proof. Suppose U satisfies the conditions of the lemma. Then $\|I - U^*B\|_\infty < 1$ where I is the identity of $L^\infty(\mathfrak{B})$. By Proposition 2.1 (i) and (iv), $\|T_I - T_{U^*B}\| < 1$. T_I is nothing but the identity operator of $H^2(\mathfrak{H})$. Hence by a well-known theorem in Banach algebras, T_{U^*B} is invertible. But B being analytic, by Proposition 2.1 (ii) and (iii), $T_{U^*B} = T_U^* T_B$. Since T_B is invertible, it follows that T_U^* is invertible and hence T_U itself.

Conversely, assume that T_U is invertible. We shall first show that there exists a B in $H^\infty(\mathfrak{B})$ such that $\|U - B\|_\infty < 1$. Suppose this is not the case. Then $\inf \|U - B\|_\infty = 1$ where B varies over $H^\infty(\mathfrak{B})$. By Theorem 1, we have $\sup |(M_U F, G)_2| = 1$, where the supremum is taken over all F and G in the closed unit balls of $H^2(\mathfrak{H})$ and $H^2(\mathfrak{H})^\perp$ respectively. Given $\epsilon > 0$ we can find unit vectors F and G in $L^2(\mathfrak{H})$ such that $F \in H^2(\mathfrak{H})$, $G \in H^2(\mathfrak{H})^\perp$ and $|(M_U F, G)_2| > 1 - \epsilon$. This inequality can be rewritten as $|((I - P)M_U F, G)_2| > 1 - \epsilon$, where I is the identity operator of $L^2(\mathfrak{H})$ and P is the orthogonal projection onto $H^2(\mathfrak{H})$. From this and the Schwarz inequality it follows that $\|(I - P)M_U F\|_2 > 1 - \epsilon$. Since U is a unitary function,

$$1 = \|F\|_2^2 = \|M_U F\|_2^2 = \|PM_U F\|_2^2 + \|(I - P)M_U F\|_2^2$$

Thus $\|PM_U F\|_2^2 < 2\epsilon$, which shows that 0 is an approximate eigenvalue of T_U . This contradicts the invertibility of T_U . Thus there exists a B in $H^\infty(\mathfrak{B})$ such that $\|U - B\|_\infty < 1$. It remains to show that B is invertible in $H^\infty(\mathfrak{B})$. In view of Lemma 4.2, it suffices to prove that T_B is invertible. To do this we can repeat the arguments used to prove the first part of the present lemma, remembering that T_U is invertible now.

Proof of Theorem 2. We shall first prove the sufficiency of the conditions. If A admits a factoring (8), by Proposition 2.1 (ii) and (iii), $T_A = T_{B_1}^* T_U T_{B_2}$. T_{B_1} and T_{B_2} are invertible by Lemma 4.2. Lemma 4.3 implies the invertibility of T_U . Thus T_A is invertible.

Now, suppose that T_A is invertible. By Proposition 2.1 (v), A is invertible in $L^\infty(\mathfrak{B})$. By Lemma 4.1, A can be factored in the form UB_2 where U is a unitary function and B_2 is an invertible element of $H^\infty(\mathfrak{B})$. Now $T_A = T_U T_{B_2}$ and so T_U is invertible. Lemma 4.2 implies the existence of an invertible element B_3 of $H^\infty(\mathfrak{B})$ satisfying (9).

5. A problem in prediction theory. A *weight function* W is a weakly measurable function on $\mathbf{T}$ whose values are non-negative operators on $\mathfrak{H}$ and which satisfies the condition

$$\int \|W(z)\| dm(z) < \infty.$$

(We remark that this definition of weight function is quite restrictive in the infinite dimensional case. It is more meaningful to consider infinite matrices of summable functions satisfying a positive definiteness condition [3, p. 112].)

For a weight function W , define

$$(10) \quad \alpha_n(W) = \sup \left| \int \chi^n(WF, G) dm \right|.$$

The supremum in (10) is taken over all $\mathfrak{H}$ -valued trigonometric polynomials satisfying the following conditions:

$$\int \langle WF, F \rangle dm \leq 1$$

$$\int \langle WG, G \rangle dm \leq 1$$

$$F = \sum_{k=0}^N \phi_k \chi^k$$

$$G = \sum_{k=0}^N \psi_k \chi^{-k}$$

where ϕ_k and ψ_k are in $\mathfrak{H}$ for each k and N is a non-negative integer. Let $\mathfrak{W}$

be the class of all weight functions W for which $\alpha_n(W) \rightarrow 0$ as $n \rightarrow \infty$. The problem is to give a characterization of this class $\mathfrak{W}$.

As we have already mentioned in §1, this problem arises in connection with a strong mixing condition of Rosenblatt [12]. It was Kolmogorov and Rozanov [9] who pointed out the application of the present problem to the problem of Rosenblatt. They also gave some sufficient conditions for a weight function in dimension one to be in the class $\mathfrak{W}$. The recent paper of Helson and Sarason [5] settles the problem in the one dimensional case by giving necessary and sufficient conditions on a weight function to be in $\mathfrak{W}$. Their paper also contains a complete bibliography on this problem.

First, we shall state a fact which helps us to construct new members of $\mathfrak{W}$ given an element there. The proof of this proposition is easy and is omitted.

Proposition 5.1. *Let W be in the class $\mathfrak{W}$ and let Q be a $\mathfrak{B}$ -valued trigonometric polynomial; i.e. a finite linear combination of the form $\sum_{k=-N}^N A_k \chi^k$, where A_k are in $\mathfrak{B}$. Then the weight function Q^*WQ is in $\mathfrak{W}$.*

Hereafter we shall restrict our attention to weight functions which are bounded above and below; i.e. there exist two constants c and d , $0 < c \leq d < \infty$, such that for every ϕ in $\mathfrak{H}$ and almost every point z ,

$$(11) \quad c\|\phi\|^2 \leq (W(z)\phi, \phi) \leq d\|\phi\|^2.$$

The following theorem gives necessary and sufficient conditions for such a weight function to belong to $\mathfrak{W}$.

Theorem 3. *Let W be a weight function satisfying (11). Then W is in $\mathfrak{W}$ and only if*

$$(12) \quad \lim_{n \rightarrow \infty} \inf_{B \in H^\infty(\mathfrak{B})} \|\chi^n W - B\|_\infty = 0.$$

Proof. For each z , let $\Theta(z)$ denote the positive square root of $W(z)$. By a standard argument it follows that Θ is weakly measurable. We can rewrite the definition of $\alpha_n(W)$ in terms of Θ :

$$\alpha_n(W) = \sup \left| \int \chi^n(\Theta F, \Theta G) dm \right|$$

where F and G are $\mathfrak{H}$ -valued trigonometric polynomials, $F \in H^2(\mathfrak{H})$, $G \in K^2(\mathfrak{H})$ and satisfy the conditions

$$(13) \quad \begin{aligned} \int \|\Theta F\|^2 dm &\leq 1 \\ \int \|\Theta G\|^2 dm &\leq 1. \end{aligned}$$

The supremum does not change even if we allow F and G to vary over $H^2(\mathfrak{H})$ and $K^2(\mathfrak{H})$ respectively, subject to the restrictions imposed by (13). This

follows easily from the norm convergence of the partial sums of the generalized Fourier series of functions in $L^2(\mathcal{H})$ [3, p. 55]. Thus, denoting the subspaces $M_\theta(H^2(\mathcal{H}))$ and $M_\theta(K^2(\mathcal{H}))$ respectively by $\mathfrak{M}$ and $\mathfrak{N}$, we have

$$\alpha_n(W) = \sup \left| \int \chi^n(F_1, G_1) dm \right|$$

the supremum being taken over all F_1 and G_1 in the closed unit balls of $\mathfrak{M}$ and $\mathfrak{N}$ respectively. By (12), θ is invertible in $L^\infty(\mathcal{B})$. Lemma 4.1 implies the existence of a unitary function U such that $\mathfrak{M} = M_U(H^2(\mathcal{H}))$. This also gives a factoring for θ in the form UA_1 where A_1 is in $H^\infty(\mathcal{B})$ and is invertible in $H^\infty(\mathcal{B})$. For the subspace $\mathfrak{N} = M_\theta(K^2(\mathcal{H}))$ we have an analogous representation theorem: there exists a unitary function V such that $\mathfrak{N} = M_V(K^2(\mathcal{H}))$. This can be obtained by applying Lemma 4.1 to the subspace $\mathfrak{N}^\perp = M_A(H^2(\mathcal{H}))$, where $A = \chi(\theta^*)^{-1}$. This representation for $\mathfrak{N}$ leads to the factoring, $\theta = VA_2^*$ where A_2 and A_2^{-1} are in $H^\infty(\mathcal{B})$.

We have now a formula for $\alpha_n(W)$ in terms of U and V :

$$\alpha_n(W) = \sup \left| \int \chi^n(UF, VG) dm \right|$$

where F and G vary in the closed unit balls of $H^2(\mathcal{H})$ and $K^2(\mathcal{H})$ respectively. By Theorem 1,

$$\alpha_n(W) = \inf_{B \in H^\infty(\mathcal{B})} \|\chi^{n-1}V^*U - B\|_\infty.$$

Now, $V^*U = A_2^{-1}WA_1^{-1}$ and hence

$$\alpha_n(W) = \inf_{B \in H^\infty(\mathcal{B})} \|\chi^{n-1}A_2^{-1}WA_1^{-1} - B\|_\infty.$$

Thus $\alpha_n(W) \rightarrow 0$ if and only if the distance of $\chi^{n-1}A_2^{-1}WA_1^{-1}$ from $H^\infty(\mathcal{B})$ tends to zero. Since the A_j for $j = 1, 2$ are in $H^\infty(\mathcal{B})$ and have analytic inverses, $\alpha_n(W) \rightarrow 0$ if and only if the distance of $\chi^{n-1}W$ from $H^\infty(\mathcal{B})$ tends to zero.

As a consequence of this theorem we have the following simple criterion. This criterion is due to Kolmogorov and Rozanov [9] in dimension 1. After our work was completed Professor M. G. Nadkarni informed us that he had obtained this result in the finite dimensional case [10].

Corollary 1. *Let W be a weight function satisfying (11). Also assume that W is a continuous function of $\mathbf{T}$ into $\mathcal{B}$. Then W is in $\mathfrak{W}$.*

Proof. Let $\epsilon > 0$ be given. Using the continuity of W , we can find a $\mathcal{B}$ -valued trigonometric polynomial $Q = \sum_{k=-N}^N A_k \chi^k$ such that $\|W - Q\|_\infty < \epsilon$. Choose $n > N$. Then for any $B \in H^\infty(\mathcal{B})$,

$$\|\chi^n W - B\|_\infty \leq \|\chi^n W - \chi^n Q\|_\infty + \|\chi^n Q - B\|_\infty.$$

The first term on the right is at most ϵ and $\chi^n B \in H^\infty(\mathcal{B})$. Therefore,

$$\inf_{B \in H^\infty(\mathfrak{B})} \|\chi^n W - B\|_\infty < \epsilon.$$

The corollary now follows from Theorem 3.

Let C denote the algebra of all complex valued continuous functions on $\mathbf{T}$. It is known that $C + H^\infty$ is a closed subalgebra of L^∞ [13, p. 191]. An element f of L^∞ has the property

$$\lim_{n \rightarrow \infty} \inf_{g \in H^\infty} \|\chi^n f - g\|_\infty = 0$$

if and only if f is in $C + H^\infty$. From these observations we have the following corollary.

Corollary 2. *Let $\mathfrak{H}$ be finite dimensional and let W be a weight function satisfying (11). Let $(w_{ij}(z))$ be the matrix of $W(z)$ in an orthogonal basis for $\mathfrak{H}$. Then W is in $\mathfrak{W}$ if and only if each w_{ij} is in the algebra $C + H^\infty$.*

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