An Analogue of Gårding's Inequality for Parabolic Operators and its Applications

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1. Introduction. We present a Hilbert space treatment of the Cauchy problem for the parabolic operator

$$P \equiv \frac{\partial}{\partial t} - L(t) \equiv \frac{\partial}{\partial t} - \sum_{|\alpha| \leq 2m} a_\alpha(x, t) D^\alpha,$$

where $x = \langle x_1, \dots, x_n \rangle$, $\alpha = \langle \alpha_1, \dots, \alpha_n \rangle$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, and

$$D^\alpha = \left(\frac{1}{i} \frac{\partial}{\partial x_1} \right)^{\alpha_1} \left(\frac{1}{i} \frac{\partial}{\partial x_2} \right)^{\alpha_2} \dots \left(\frac{1}{i} \frac{\partial}{\partial x_n} \right)^{\alpha_n},$$

following in rough outline the Hilbert space approach to the Dirichlet problem for strongly elliptic operators, as given, for example in [2], pp. 164–206. We start with a basic inequality (Theorem 1) which plays for us the role of Gårding's inequality in the elliptic theory. The spaces H^s in the elliptic case are here replaced by the doubly indexed family of Hilbert spaces $\mathcal{H}^{r,s}$ ($\equiv \mathcal{B}_{2,k}$ in the notation of [6], Chapter II, where $k(\xi, \tau) = k_{r,s}(\xi, \tau)$ is the temperate weight function defined for $\langle \xi, \tau \rangle = \langle \xi_1, \xi_2, \dots, \xi_n, \tau \rangle \in R^{n+1}$ by $k_{r,s}(\xi, \tau) = Q^r(\xi, \tau) q^s(\xi)$. Here $q(\xi) = \{1 + |\xi|^2\}^{1/2}$ with $|\xi|^2 = \sum_{i=1}^n \xi_i^2$, is the usual elliptic weight function in R^n , and $Q(\xi, \tau) = \{\tau^2 + q^{4m}(\xi)\}^{1/4m}$ is tailored to our operator P .)

We apply Theorem 1 along with a commutator estimate to show that $P + \lambda I$ is a topological isomorphism of $\mathcal{H}^{r,s}$ onto $\mathcal{H}^{r-2m,s}$, provided λ is large enough. To apply this result to the Cauchy problem, we introduce the space $\mathcal{H}^{r,s}(\Omega)$ of restrictions to Ω of elements of $\mathcal{H}^{r,s}$, where $\Omega = \Omega_{a,b} = \{ \langle x, t \rangle \in R^{n+1} : a < t < b \}$. Employing a generalization of the classical energy inequality (Theorem 3), we deduce the following:

(Theorem 5). *If $-\infty < a < b < +\infty$, if $r > m$, and if s is any real number, the mapping $u \rightarrow \langle Pu, u(a) \rangle$ is a topological isomorphism of $\mathcal{H}^{r,s}(\Omega)$ onto $\mathcal{H}^{r-2m,s}(\Omega) \oplus H^{r-m+s}$.*

We assume throughout that the functions $\{a_\alpha(x, t): |\alpha| \leq 2m\}$ belong to the class $C_B^\infty(R^{n+1})$ of complex-valued functions having bounded (and therefore continuous) partial derivatives of all orders for all $\langle x, t \rangle \in R^{n+1}$. Moreover, we assume that $L(t)$ is uniformly strongly elliptic, i.e., there exists $\mu > 0$ such that

$$\operatorname{Re} \left[- \sum_{|\alpha| = 2m} a_\alpha(x, t) \xi^\alpha \right] \geq \mu |\xi|^{2m}$$

for all $\langle x, t \rangle \in R^{n+1}$, $\xi \in R^n$, where we write ξ^α for $\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$.

Theorem 1 and its use are an outgrowth of an earlier work of the present author [7]; the key to its wider application here is our commutator estimate (Proposition 5 (ii) below). We note that results which overlap with our Theorem 5 have been obtained through a variety of methods different from ours, some requiring less stringent assumptions on the coefficients $a_\alpha(x, t)$; see, for example [5] and [4].

2. The Hilbert space $\mathcal{H}^{r,s}$.

Definition.

$$\mathcal{H}^{r,s} = \left\{ u \in \mathcal{S}'(R^{n+1}) : u \in L_{\text{loc}}^1(R^{n+1}) \quad \text{and} \right.$$

$$\left. \int_{R^{n+1}} Q^{2r}(\xi, \tau) q^{2s}(\xi) |\hat{u}(\xi, \tau)|^2 d\xi d\tau \equiv \|u\|_{r,s}^2 < \infty \right\}$$

with the scalar product $(u, v)_{r,s}$ defined in the obvious way; here, we write

$$\mathcal{F}u(\xi, \tau) = \hat{u}(\xi, \tau) = (2\pi)^{-(n+1)/2} \int_{R^{n+1}} e^{-i\langle(x, \xi) + t\tau\rangle} u(x, t) dx dt$$

when $u \in \mathcal{S}(R^{n+1})$, and extend to $\mathcal{S}'(R^{n+1})$ in the usual way. We use the notation

$$\tilde{u}(\xi) = (2\pi)^{-n/2} \int_{R^n} e^{-i\langle x, \xi \rangle} u(x) dx, \quad u \in \mathcal{S}(R^n),$$

for the n -dimensional Fourier transform; the Hilbert space H^s as usual is defined to be

$$\left\{ u \in \mathcal{S}'(R^n) : \tilde{u} \in L_{\text{loc}}^1(R^n) \quad \text{and} \quad \int_{R^n} q^{2s}(\xi) |\tilde{u}(\xi)|^2 d\xi \equiv \|u\|_s^2 < \infty \right\}$$

with the usual scalar product.

Proposition 1. *Let (r_i, s_i) , $i = 1, \dots, k$ be k pair of real numbers, k finite. Given any $u \in \bigcap_{i=1}^k \mathcal{H}^{r_i, s_i}$ and $\epsilon > 0$, there exists $\varphi \in C_0^\infty(R^{n+1})$ such that $\|u - \varphi\|_{r_i, s_i} < \epsilon$ for $i = 1, \dots, k$.*

Proof. See the proofs of Theorems 2.2.10 and 2.2.11 of [6].

Proposition 2. (i) $\mathcal{H}^{r_1, s_1} \subseteq \mathcal{H}^{r_2, s_2}$ if and only if $r_1 \geq r_2$ and $r_1 + s_1 \geq r_2 + s_2$; the inclusion, when it exists, is bounded, with norm 1.

(ii) If $s_1 > s_2 \geq s_3$, and r is any real number, then, given any $\epsilon > 0$, there exists $C = C(\epsilon, s_1, s_2, s_3, r)$ such that

$$\|u\|_{r,s_2}^2 \leq \epsilon \|u\|_{r,s_1}^2 + C \|u\|_{r,s_3}^2 \quad \text{for all } u \in \mathcal{H}^{r,s_1}.$$

Proof. (i) See Theorem 2.5.2 of [6]. (ii) Standard computation.

Notation. We write $[\varphi, \psi] = \int_{R^{n+1}} \varphi(x, t) \bar{\psi}(x, t) dx dt$ when φ and $\psi \in C_0^\infty(R^{n+1})$

Proposition 3. For $\varphi \in C_0^\infty(R^{n+1})$ and any r and s ,

$$\|\varphi\|_{r,s} = \sup \left\{ \frac{|[\varphi, \psi]|}{\|\psi\|_{-r,-s}} \mid 0 \neq \psi \in C_0^\infty(R^{n+1}) \right\}.$$

Thus $\mathcal{H}^{r,s}$ and $\mathcal{H}^{-r,-s}$ are dual Hilbert spaces, the duality being given by the sesquilinear form (which we again denote by $[\cdot, \cdot]$) obtained by extension.

Proof. [6], Theorem 2.2.9.

Remark. By Proposition 1, $[u, v]$ does not depend on the choice of r and s . Also, since the topology of $\mathcal{H}^{r,s}$ is stronger than the weak distribution topology, if u (respectively v) is in $C_0^\infty(R^{n+1})$, then $[u, v] = \bar{v}(u)$, i.e., the distribution $\bar{v}$ applied to the test function u (respectively, $u(\bar{v})$).

Proposition 4. (i) D^α is a bounded operator from $\mathcal{H}^{r,s}$ into $\mathcal{H}^{r,s-|\alpha|}$ for all r, s . (ii) $D_t \equiv (1/i) \partial/\partial t$ is a bounded operator from $\mathcal{H}^{r,s}$ into $\mathcal{H}^{r-2m,s}$ for all r, s .

Proof. Obvious.

Definition. By $\mathfrak{M}_{r,s}$ (r and s real numbers) we denote the unitary isomorphism of $\mathcal{H}^{\rho,\sigma}$ onto $\mathcal{H}^{\rho-r,\sigma-s}$ (for each real ρ and σ) given by $\mathfrak{F}(\mathfrak{M}_{r,s}\varphi)(\xi, \tau) = Q^r(\xi, \tau)q^s(\xi)\hat{\varphi}(\xi, \tau)$, when $\varphi \in C_0^\infty(R^{n+1})$, and extended.

Proposition 5. Suppose $a \in C_B^\infty(R^{n+1})$ is given. Denote by $a \cdot$ the operation of multiplication by a . Then, for every real r and s

- (i) $a \cdot$ is a continuous mapping from $\mathcal{H}^{r,s}$ into itself and
- (ii) for every real ρ and σ , and every $\theta < 1$, $\mathfrak{M}_{r,s}a \cdot - a \cdot \mathfrak{M}_{r,s}$ is a bounded mapping from $\mathcal{H}^{\rho,\sigma}$ into $\mathcal{H}^{\rho-r,\sigma-s+\theta}$.

Proof. See Section 6 below.

3. The basic inequality. Let H be the mapping of $\mathcal{H}^{r,s}$ into itself, defined, for $u \in C_0^\infty(R^{n+1})$, by

$$\mathfrak{F}(Hu)(\xi, \tau) = \frac{i\tau}{|\tau|} \hat{u}(\xi, \tau)$$

and let $S_\gamma = I - \gamma H$, for γ real.

Theorem 1. There exist positive numbers γ , λ_0 and λ_1 such that

$$(1) \quad \operatorname{Re} [S_\gamma P u, u] \geq \lambda_1 \|u\|_{m,0}^2 - \lambda_0 \|u\|_{0,0}^2 \quad \text{for all } u \in C_0^\infty(R^{n+1}).$$

Proof. Applying Plancherel's theorem,

$$\begin{aligned} \operatorname{Re} [S_\gamma Pu, u] &= \operatorname{Re} [Pu, u] - \gamma \operatorname{Re} [HPu, u] = -\operatorname{Re} [Lu, u] \\ &\quad + \gamma \int_{R^{n+1}} |\tau| |\hat{u}(\xi, \tau)|^2 d\xi d\tau \\ &\quad - \gamma \operatorname{Re} [HLu, u]. \end{aligned}$$

Gårding's inequality (see [1], p. 78) yields

$$(2) \quad \operatorname{Re} (-L(t)\varphi, \varphi)_0 \geq \mu_1 \|\varphi\|_m^2 - \mu_0 \|\varphi\|_0^2 \quad \text{for all } \varphi \in C_0^\infty(R^n),$$

and all $t \in R^1$, where μ_1 and μ_0 are positive numbers independent of φ and t . Integrating (2), and applying Propositions 3 and 4, we obtain

$$\begin{aligned} \operatorname{Re} [S_\gamma Pu, u] &\geq \mu_1 \|u\|_{0,m}^2 - \mu_0 \|u\|_{0,0}^2 \\ &\quad + \gamma \int_{R^{n+1}} |\tau| |\hat{u}(\xi, \tau)|^2 d\xi d\tau - C\gamma \|u\|_{0,m}^2. \end{aligned}$$

(C is used throughout to denote a positive constant depending only on P , not necessarily the same at each occurrence.) If we choose γ so that $C\gamma = \mu_1/2$, and if $c_1 > 0$ is such that $|\tau| + q^{2m}(\xi) \geq c_1 Q^{2m}(\xi, \tau)$ for all $(\xi, \tau) \in R^{n+1}$, then (1) follows immediately, with $\lambda_1 = c_1 \min(\mu_1/2, \gamma)$ and $\lambda_0 = \mu_0$.

Remarks. 1) If P^* is the formal adjoint of P , i.e.,

$$P^*u = -\frac{\partial u}{\partial t} - \sum_{|\alpha| \leq 2m} D^\alpha (\bar{a}_\alpha u),$$

and S_γ^* is the adjoint of S_γ , i.e., $S_\gamma^* = I + \gamma H$, then the same argument shows that

$$(1') \quad \operatorname{Re} [S_\gamma^* P^* u, u] \geq \lambda_1 \|u\|_{m,0}^2 - \lambda_0 \|u\|_{0,0}^2 \quad \text{for all } u \in C_0^\infty(R^{n+1}).$$

2) (1) and (1') are easily seen to hold for all $u \in \mathcal{H}^{m,0}$, by continuity.

Theorem 2. *Given any real r and s , for λ real and sufficiently large (depending on r, s , and P) $P + \lambda I$ is a topological isomorphism of $\mathcal{H}^{r,s}$ onto $\mathcal{H}^{r-2m,s}$.*

Proof. The continuity of $P + \lambda I$ for all λ follows from the results of Section 2; here I is of course the inclusion mapping from $\mathcal{H}^{r,s}$ into $\mathcal{H}^{r-2m,s}$. We next establish the following: there exists $C = C_{r,s} > 0$ such that

$$(3) \quad \|u\|_{r,s} \leq C \|(P + \lambda I)u\|_{r-2m,s}$$

for all $u \in C_0^\infty(R^{n+1})$, and for all sufficiently large λ .

To this end, note that

$$\begin{aligned} \operatorname{Re} (S_\gamma Pu, u)_{r,s} &= \operatorname{Re} [\mathfrak{M}_{r,s} S_\gamma Pu, \mathfrak{M}_{r,s} u] \\ &= \operatorname{Re} [S_\gamma P \mathfrak{M}_{r,s} u, \mathfrak{M}_{r,s} u] + \operatorname{Re} [(\mathfrak{M}_{r,s} P - P \mathfrak{M}_{r,s})u, S_\gamma^* \mathfrak{M}_{r,s} u] \\ &\geq \lambda_1 \|\mathfrak{M}_{r,s} u\|_{m,0}^2 - \lambda_0 \|\mathfrak{M}_{r,s} u\|_{0,0}^2 \\ &\quad - (1 + \gamma) \|(\mathfrak{M}_{r,s} P - P \mathfrak{M}_{r,s})u\|_{0,-m} \|\mathfrak{M}_{r,s} u\|_{0,m} \end{aligned}$$

by Theorem 1 and Proposition 3. Fixing a θ between 0 and 1, and applying Propositions 4 and 5, we obtain, with $C = C(\theta, r, s) > 0$,

$$|(\mathfrak{M}_{r,s}P - P\mathfrak{M}_{r,s})u|_{0,-m} = |(\mathfrak{M}_{r,s}L - L\mathfrak{M}_{r,s})u|_{0,-m} \leq C \|u\|_{r,s+m-\theta}.$$

Applying next Proposition 2 (ii), with $s_1 = s + m$, $s_2 = s + m - \theta$, and $s_3 = s$, we obtain, for arbitrary $\epsilon > 0$

$$\begin{aligned} (1 + \gamma) |(\mathfrak{M}_{r,s}P - P\mathfrak{M}_{r,s})u|_{0,-m} \|\mathfrak{M}_{r,s}u\|_{0,m} &\leq \epsilon \|u\|_{r,s+m}^2 + C(\epsilon) \|u\|_{r,s}^2 \\ &\leq \epsilon \|u\|_{r+m,s}^2 + C(\epsilon) \|u\|_{r,s}^2; \end{aligned}$$

thus,

$$\operatorname{Re} (S_\gamma Pu, u)_{r,s} \geq (\lambda_1 - \epsilon) \|u\|_{r+m,s}^2 - (\lambda_0 + C(\epsilon)) \|u\|_{r,s}^2.$$

Now choose $\epsilon = \lambda_1/2$; if $\lambda \geq \lambda_0 + C(\lambda_1/2)$, we have

$$\operatorname{Re} (S_\gamma(P + \lambda I)u, u)_{r,s} = \operatorname{Re} (S_\gamma Pu, u)_{r,s} + \lambda \|u\|_{r,s}^2 \geq \frac{\lambda_1}{2} \|u\|_{r+m,s}^2,$$

or,

$$\|u\|_{r+m,s}^2 \leq \frac{2}{\lambda_1} |[\mathfrak{M}_{r,s}S_\gamma(P + \lambda I)u, \mathfrak{M}_{r,s}u]| \leq C_{r,s} \|(P + \lambda I)u\|_{r-m,s} \|u\|_{r+m,s}$$

by Proposition 3. Replacing r by $r - m$, (3) follows immediately. As a consequence, we conclude that the range of $P + \lambda I$ is closed in $\mathcal{H}^{r-2m,s}$; thus, all we need show is that the range of $P + \lambda I$, for λ sufficiently large, has trivial orthogonal complement in $\mathcal{H}^{r-2m,s}$. To that end, first observe that by replacing (1) by (1') in the above argument, we obtain instead, for λ sufficiently large,

$$(3') \quad \|u\|_{r,s} \leq C \|(P^* + \lambda I)u\|_{r-2m,s} \quad \text{for all } u \in C_0^\infty(\mathbb{R}^{n+1}).$$

Now, suppose $v \in \mathcal{H}^{r-2m,s}$ is such that for all $u \in C_0^\infty(\mathbb{R}^{n+1})$

$$0 = ((P + \lambda I)u, v)_{r,s} = [(P + \lambda I)u, \mathfrak{M}_{2r,2s}v].$$

Then $w = \mathfrak{M}_{2r,2s}v \in \mathcal{H}^{-r-2m,-s}$ and $(P^* + \lambda I)w = 0$, which, by (3') applied to w , with $-r - 2m$ and $-s$ replacing r and s , implies $w = 0$, and thus $v = 0$, completing our proof.

4. The Space $\mathcal{H}^{r,s}(\Omega)$.

Definitions. For $-\infty \leq a < b \leq +\infty$, we set $\Omega_{a,b} = \{\langle x, t \rangle \in \mathbb{R}^{n+1} : a < t < b\}$. When there is no danger of confusion, we write simply Ω for $\Omega_{a,b}$. Then $\mathcal{H}^{r,s}(\Omega) = \{u \in \mathcal{D}'(\Omega) : \exists v \in \mathcal{H}^{r,s} \text{ such that } u = v|_\Omega\}$. We norm $\mathcal{H}^{r,s}(\Omega)$ with the usual quotient norm:

$$\|u\|_{r,s,\Omega} = \inf \{\|v\|_{r,s} : v \in \mathcal{H}^{r,s} \text{ and } u = v|_\Omega\}.$$

Observe that differential operators whose coefficients are restrictions to Ω of functions in $C_B^\infty(\mathbb{R}^{n+1})$ operate on the $\mathcal{H}^{r,s}(\Omega)$ spaces exactly as in Propositions

4 and 5; moreover, applying Proposition 6 below, it follows easily that $\mathfrak{M}_{0,s}$ takes $\mathcal{H}^{r,s}(\Omega)$ continuously into $\mathcal{H}^{r,s-\sigma}(\Omega)$ for all real r, s , and σ , and Proposition 5 (ii) is applicable in this case.

For any closed set $K \subseteq R^{n+1}$, we set $\mathcal{H}^{r,s}(K) = \{u \in \mathcal{H}^{r,s} : \text{supp } u \subseteq K\}$. For any open set $G \subseteq R^{n+1}$, we set $C_{(0)}^\infty(\bar{G}) = \{\varphi \mid G: \varphi \in C_0^\infty(R^{n+1})\}$. (The bar denotes closure; we write $\mathbf{C}A$ for the complement of A .)

Proposition 6. (i) $C_{(0)}^\infty(\bar{\Omega})$ is dense in $\mathcal{H}^{r,s}(\Omega)$ for all r and s .

(ii) $C_0^\infty(\Omega)$ is dense in $\mathcal{H}_0^{r,s}(\bar{\Omega})$ for all r and s .

(iii) $C_0^\infty(\mathbf{C}\bar{\Omega})$ is dense in $\mathcal{H}_0^{r,s}(\mathbf{C}\Omega)$ for all r and s .

Proof. (i) Obvious. (ii) For $b = +\infty$ or for $a = -\infty$, the proof is the same as for the corresponding statement in Theorem 2.5.1 of [6]. In the general case, choose ψ_1 and $\psi_2 \in C_B^\infty(R^{n+1})$ satisfying $\psi_1 + \psi_2 \equiv 1$, $\text{supp } \psi_1 \subseteq \Omega_{a,+\infty}$ and $\text{supp } \psi_2 \subseteq \Omega_{-\infty,b}$. Given $u \in \mathcal{H}_0^{r,s}(\bar{\Omega}_{a,b}) \subseteq \mathcal{H}_0^{r,s}(\bar{\Omega}_{a,+\infty}) \cap \mathcal{H}_0^{r,s}(\bar{\Omega}_{-\infty,b})$, choose sequences $\{\varphi_i^1\}$ and $\{\varphi_i^2\}$ of elements of $C_0^\infty(\Omega_{-\infty,b})$ and $C_0^\infty(\Omega_{a,+\infty})$ respectively, both of which converge to u in $\mathcal{H}^{r,s}$. Taking $\varphi_i = \psi_1 \varphi_i^1 + \psi_2 \varphi_i^2$, we have $\varphi_i \rightarrow u$ in $\mathcal{H}^{r,s}$ by Proposition 5 (i) and $\varphi_i \in C_0^\infty(\Omega_{a,b})$. (iii) Given $u \in \mathcal{H}_0^{r,s}(\mathbf{C}\Omega)$, write $u = \psi_1 u + \psi_2 u$ and apply part (ii) to each term.

Proposition 7. $\mathcal{H}^{r,s}(\Omega)$ and $\mathcal{H}_0^{-r,-s}(\bar{\Omega})$ are dual Hilbert spaces with respect to an extension of the sesquilinear form

$$[u, v] \equiv \int_{\Omega} u(x, t) \bar{v}(x, t) \, dx \, dt \quad u \in C_{(0)}^\infty(\bar{\Omega}), \quad v \in C_0^\infty(\Omega).$$

Proof. See Theorem 2.5.1 of [6].

Proposition 8. If $u \in \mathcal{H}^{r,s+2m}(\Omega)$, and $\partial u / \partial t \in \mathcal{H}^{r,s}(\Omega)$, then $u \in \mathcal{H}^{r+2m,s}(\Omega)$.

Lemma. There exist isometries $E_0 : \mathcal{H}^{r,s}(\bar{\Omega}_{a,+\infty}) \rightarrow \mathcal{H}_0^{r+2m,s-2m}(\bar{\Omega}_{a,+\infty})$ and $E_1 : \mathcal{H}_0^{r,s}(\bar{\Omega}_{a,+\infty}) \rightarrow \mathcal{H}_0^{r+2m,s}(\bar{\Omega}_{a,+\infty})$ such that $v = E_0 v + (\partial / \partial t)(E_1 v)$ for all $v \in \mathcal{H}_0^{r,s}(\bar{\Omega}_{a,+\infty})$. Here a, r , and s are any real numbers.

Proof of lemma. (Compare with Lemma 2.5.1 of [6].) For $v \in C_0^\infty(\Omega_{a,+\infty})$ define E_0 and E_1 by

$$\mathfrak{F}(E_0 v)(\xi, \tau) = \frac{q(\xi)^{2m}}{i\tau + q(\xi)^{2m}} \hat{v}(\xi, \tau)$$

and

$$\mathfrak{F}(E_1 v)(\xi, \tau) = \frac{\hat{v}(\xi, \tau)}{i\tau + q(\xi)^{2m}};$$

extend by continuity.

Proof of Proposition 8. If $a = -\infty$ and $b = +\infty$, our assertion is obvious. If $-\infty < a < +\infty$ and $b = +\infty$, we apply our lemma and Proposition 7: given $v \in \mathcal{H}_0^{r-2m,s}(\bar{\Omega}_{a,+\infty})$, we have

$$[u, v] = [u, E_0 v] + \left[u, \frac{\partial}{\partial t} (E_1 v) \right] = [u, E_0 v] - \left[\frac{\partial u}{\partial t}, E_1 v \right];$$

thus

$$\begin{aligned} |[u, v]| &\leq \|u\|_{r, s+2m, \Omega} \|E_0 v\|_{-r, -s-2m} + \left\| \frac{\partial u}{\partial t} \right\|_{r, s, \Omega} \|E_1 v\|_{-r, -s} \\ &= \left\{ \|u\|_{r, s+2m, \Omega} + \left\| \frac{\partial u}{\partial t} \right\|_{r, s, \Omega} \right\} \|v\|_{-r-2m, -s}. \end{aligned}$$

A similar argument takes care of the case $a = -\infty$, $b < +\infty$. Finally, if $-\infty < a < b < +\infty$, we apply the previous arguments to $\psi_1 u$ and $\psi_2 u$, where ψ_1 and ψ_2 are as in the proof of Proposition 6.

Definition. If X is a Banach space, and $-\infty \leq a < b \leq +\infty$, we denote by $C([a, b]; X)$ the Banach space of continuous functions u , defined on $[a, b]$, taking values in X , and, in addition, if $a = -\infty$ (or, if $b = +\infty$) vanishing at $-\infty$ (or, at $+\infty$); we define

$$\|u\|_{C([a, b]; X)} = \sup_{a < t < b} \|u(t)\|_X.$$

Proposition 9. (Compare with Theorems 2.2.8 and 2.5.6 of [6].) *The mapping which assigns to $\varphi \in C_{(0)}^\infty(\Omega)$ the function $t \rightarrow \varphi(\cdot, t)$ taking $[a, b]$ into $C_0^\infty(R^n)$, extends, in a unique way, to a continuous embedding of $\mathcal{H}^{r, s}(\Omega)$ into $C([a, b]; H^{r-m+s})$, provided $r > m$.*

Proof. If $r > m$, a simple calculation shows (see Theorem 2.2.8 of [6]) there exists $C = C_r > 0$ such that

$$(4) \quad \sup_{t \in \mathbb{R}^1} \|\psi(\cdot, t)\|_{r+s-m} \leq C \|\psi\|_{r, s} \quad \text{for all } \psi \in C_0^\infty(R^{n+1}).$$

Thus, $\mathcal{H}^{r, s}$ is mapped continuously into $C([a, b]; H^{r-m+s})$; applying Proposition 6, if $u \in \mathcal{H}_0^{r, s}(\mathbb{C}\Omega)$, u is mapped into zero in $C([a, b]; H^{r-m+s})$. Conversely if $u \in \mathcal{H}^{r, s}$ is mapped into zero in $C([a, b]; H^{r-m+s})$, taking a sequence u_i of elements of $C_0^\infty(R^{n+1})$ which converge to u in $\mathcal{H}^{r, s}$, and any $\varphi \in C_0^\infty(\Omega)$, we have

$$[u, \varphi] = \lim_{i \rightarrow \infty} [u_i, \varphi] = \lim_{i \rightarrow \infty} \int_a^b (u_i(\cdot, t), \varphi(\cdot, t))_0 dt.$$

Applying (4), and the theorem on bounded convergence, we obtain $[u, \varphi] = 0$; thus $\text{supp } u \subseteq \mathbb{C}\Omega$, and our assertion follows immediately.

Notation. If $u \in \mathcal{H}^{r, s}(\Omega)$, $r > m$, and $\theta \in [a, b]$, we write $u(\theta)$ for $u(\cdot, \theta) \in H^{r-m+s}$.

Proposition 10. (Compare Theorem 2.5.7 of [6].) *If $-\infty < a < b \leq +\infty$, $r > m$, and $u_0 \in H^{r-m+s}$, there exists $u \in \mathcal{H}^{r, s}(\Omega)$ such that $u(a) = u_0$.*

Proof. We may take $a = 0$. We construct a bounded mapping $E : H^{r-m+s} \rightarrow$

$\mathcal{H}^{r,s}$ such that $E(u_0)(0) = u_0$ for $u_0 \in H^{r-m+s}$; choose $\rho = \rho(t) \in C_0^\infty(R^1)$, with $\rho(0) = 1$ and define $u = E(u_0)$ for $u_0 \in C_0^\infty(R^n)$ by the formula

$$\hat{u}(\xi, \tau) = q(\xi)^{-2m} \hat{\rho}(\tau q(\xi)^{-2m}) \hat{u}_0(\xi),$$

where $\hat{\rho}(\tau) = (2\pi)^{-1/2} \int e^{-i\tau t} \rho(t) dt$. Clearly, $u \in \mathcal{S}(R^{n+1})$, and an easy computation shows that $\|u\|_{r,s}^2 = C_r \|u_0\|_{r-m+s}^2$, where

$$C_r = \int_{-\infty}^{\infty} (1 + \theta^2)^{r/2m} |\hat{\rho}(\theta)|^2 d\theta,$$

and $u(x, 0) = u_0(x)$ for all $x \in R^n$.

Proposition 11. Suppose r, s , and a are real numbers, with $|r| < m$. If $\varphi \in C_0^\infty(R^{n+1})$ define

$$H_a \varphi(x, t) = \begin{cases} \varphi(x, t) & \text{if } t \geq a \\ 0 & \text{if } t < a \end{cases}.$$

Then $H_a \varphi \in \mathcal{H}^{r,s}$; in fact, there exists $C = C_{r,s} > 0$ such that $\|H_a \varphi\|_{r,s} \leq C \|\varphi\|_{r,s}$. Thus, H_a defines a continuous projection of $\mathcal{H}^{r,s}$ onto $\mathcal{H}_0^{r,s}(\bar{\Omega}_{a,+\infty})$.

Proof. See Section 6 below.

Corollary. If $|r| < m$, and $-\infty < a < b \leq +\infty$, the mapping $u \rightarrow u|_{\Omega}$ gives a topological isomorphism of $\mathcal{H}_0^{r,s}(\bar{\Omega})$ onto $\mathcal{H}^{r,s}(\Omega)$.

Proposition 12. Given real r_1 and r_2 , satisfying $r_1 + r_2 \geq 0$ and $\min(r_1, r_2) > -m$, any real s_1 and s_2 , and $-\infty \leq a < b \leq +\infty$, the form $\int_a^b (\varphi(\cdot, t), \psi(\cdot, t))_\theta dt$, $\varphi, \psi \in C_{(0)}^\infty(\bar{\Omega})$, extends in a unique way to a continuous sesquilinear form on $\mathcal{H}^{r_1,s_1}(\Omega) \times \mathcal{H}^{r_2,s_2}(\Omega)$ (which we denote by $\int_a^b (u, v)_\theta dt$), provided that $\theta \leq \frac{1}{2}(r_1 + s_1 + r_2 + s_2)$.

Proof. Suppose first that φ and $\psi \in C_0^\infty(R^{n+1})$. Choose r real such that $|r| < m$ and $-r_2 \leq r \leq r_1$; then choose $s = \frac{1}{2}(s_1 + r_1 - s_2 - r_2) - r$, which implies that $\theta - (s_2 + r_2) \leq r + s \leq s_1 + r_1 - \theta$. Thus, applying Propositions 11, 3, and 2 (i), we obtain

$$\begin{aligned} \left| \int_a^b (\varphi(\cdot, t), \psi(\cdot, t))_\theta dt \right| &= |[(H_a - H_b)\mathfrak{M}_{0,\theta}\varphi, \mathfrak{M}_{0,\theta}\psi]| \\ &\leq C \|\varphi\|_{r,s+\theta} \|\psi\|_{-r,-s+\theta} \\ &\leq C \|\varphi\|_{r_1,s_1} \|\psi\|_{r_2,s_2}. \end{aligned}$$

Thus, $\int_a^b (u, v)_\theta dt$ extends to a continuous sesquilinear form on $\mathcal{H}^{r_1,s_1} \times \mathcal{H}^{r_2,s_2}$, which by Proposition 6 (iii) vanishes when $u \in \mathcal{H}_0^{r_1,s_1}(\mathbf{C}\Omega)$ or when $v \in \mathcal{H}_0^{r_2,s_2}(\mathbf{C}\Omega)$, and our proof is complete.

Note. We write $\int_a^b \|u\|_\theta^2 dt$ for $\int_a^b (u, u)_\theta dt$; this quantity is easily seen to equal $\|u\|_{0,\theta,\Omega}^2$.

5. The Cauchy problem.

Theorem 3. *Given real r and s , with $r > m$, there exist $\mu_1 > 0$ and μ_0 depending on P , r , and s such that*

$$(5) \quad \begin{aligned} \frac{1}{2} \|u(b)\|_{r-m+s}^2 - \frac{1}{2} \|u(a)\|_{r-m+s}^2 + \mu_1 \int_a^b \|u\|_{r+s}^2 dt - \mu_0 \int_a^b \|u\|_{r-m+s}^2 dt \\ \leq \operatorname{Re} \int_a^b (Pu, u)_{r+s-m} dt \end{aligned}$$

for all $u \in \mathcal{H}^{r,s}(\Omega)$.

Proof. First, suppose $r + s = m$. Then (5) holds for $u \in C_0^\infty(\mathbb{R}^{n+1})$, with μ_1 and μ_0 as in (2), as a simple calculation shows; it extends to the case where $u \in \mathcal{H}^{r,s}(\Omega)$ by application of Propositions 9 and 12. (For example, $\int_a^b (u, v)_{r+s-m} dt$ is continuous on $\mathcal{H}^{r-2m,s}(\Omega) \times \mathcal{H}^{r,s}(\Omega)$ by Proposition 12; since $P : \mathcal{H}^{r,s}(\Omega) \rightarrow \mathcal{H}^{r-2m,s}(\Omega)$ is continuous, it follows that $\int_a^b (Pu, v)_{r+s-m} dt$ is continuous on $\mathcal{H}^{r,s}(\Omega) \times \mathcal{H}^{r,s}(\Omega)$.) Now, let $r + s - m = \rho$. For $u \in \mathcal{H}^{r,s}(\Omega)$, apply what we have just shown to $\mathfrak{N}_{0,\rho} u \in \mathcal{H}^{r,s-\rho}(\Omega)$ and obtain

$$\begin{aligned} \frac{1}{2} \|u(b)\|_\rho^2 - \frac{1}{2} \|u(a)\|_\rho^2 + \mu_1 \int_a^b \|u\|_{\rho+m}^2 dt - \mu_0 \int_a^b \|u\|_\rho^2 dt \\ \leq \operatorname{Re} \int_a^b (P\mathfrak{N}_{0,\rho} u, \mathfrak{N}_{0,\rho} u)_0 dt \\ \leq \operatorname{Re} \int_a^b (Pu, u)_\rho dt + \left| \int_a^b ([P\mathfrak{N}_{0,\rho} - \mathfrak{N}_{0,\rho} P]u, \mathfrak{N}_{0,\rho} u)_0 dt \right|. \end{aligned}$$

Choosing θ between 0 and 1, and applying Propositions 12, 5(ii), and 2 (ii), we obtain, for arbitrary $\epsilon > 0$,

$$\begin{aligned} \left| \int_a^b ([P\mathfrak{N}_{0,\rho} - \mathfrak{N}_{0,\rho} P]u, \mathfrak{N}_{0,\rho} u)_0 dt \right| \\ \leq C \| [P\mathfrak{N}_{0,\rho} - \mathfrak{N}_{0,\rho} P]u \|_{0,-m,\Omega} \| \mathfrak{N}_{0,\rho} u \|_{0,m,\Omega} \\ \leq C \|u\|_{0,\rho+m-\theta,\Omega} \|u\|_{0,\rho+m,\Omega} \\ \leq \epsilon \|u\|_{0,\rho+m,\Omega}^2 + C(\epsilon) \|u\|_{0,\rho,\Omega}^2 \\ = \epsilon \int_a^b \|u\|_{\rho+m}^2 dt + C(\epsilon) \int_a^b \|u\|_\rho^2 dt. \end{aligned}$$

Choose now $\epsilon = \mu_1/2$ to complete our proof.

Theorem 4. *Given any real r , s , and c , $P + \lambda I$ is an isomorphism of $\mathcal{H}_0^{r,s}(\bar{\Omega}_{c,+,\infty})$ onto $\mathcal{H}_0^{r-2m,s}(\bar{\Omega}_{c,+,\infty})$ for all sufficiently large λ (depending on r , s , and P .)*

Proof. First choose $\rho_0 > m$, $\rho_0 \geq r$. Next, choose λ_0 so large that i) $\lambda_0 \geq \mu_0(\rho_0, s)$ (μ_0 the constant in Theorem 3, with ρ_0 replacing r) and ii)

$\lambda \geq \lambda_0$ implies that $P + \lambda I$ is an isomorphism of a) $\mathcal{H}^{r,s}$ onto $\mathcal{H}^{r-2m,s}$ and b) $\mathcal{H}^{\rho_0,s}$ onto $\mathcal{H}^{\rho_0-2m,s}$ (applying Theorem 2 in both cases.) We claim that for $\lambda \geq \lambda_0$, $P + \lambda I$ has the desired property; in fact, applying Theorem 2, all we need show is that if $u \in \mathcal{H}^{r,s}$ and $v = (P + \lambda I)u$ has its support in $\bar{\Omega}_{c,+\infty}$, then so does u . By Proposition 6 (ii), there exists a sequence $\{v_i\}$ of elements of $C_0^\infty(\Omega_{c,+\infty})$ converging to v in $\mathcal{H}^{r-2m,s}$. By Theorem 2, for each j there exists a unique $u_i \in \mathcal{H}^{r,s}$ satisfying $(P + \lambda I)u_i = v_i$; moreover, there exists also $u'_i \in \mathcal{H}^{\rho_0,s}$ such that $(P + \lambda I)u'_i = v_i$. Thus $u_i = u'_i \in \mathcal{H}^{\rho_0,s}$; applying Theorem 3 with $a = -\infty$ and any $b < c$, we see that $u_i(b) = 0$. Thus each u_i has support in $\bar{\Omega}_{c,+\infty}$, and the same remains true for u .

Theorem 5. *If s is any real number, $r > m$, and $-\infty < a < b < +\infty$, the mapping $u \rightarrow \langle Pu, u(a) \rangle$ is a topological isomorphism of $\mathcal{H}^{r,s}(\Omega)$ onto $\mathcal{H}^{r-2m,s}(\Omega) \oplus H^{r-m+s}$, where $\Omega = \Omega_{a,b}$.*

Proof. For convenience, we take $a = 0$. We write $\mathcal{O}u = \langle Pu, u(0) \rangle$ and $\mathcal{O}_\lambda u = \langle (P + \lambda I)u, u(0) \rangle$. It suffices to show that $\mathcal{O}_\lambda$ is an isomorphism, for sufficiently large λ ; in fact, $\mathcal{O}u = \langle f, u_0 \rangle$ if and only if $\mathcal{O}_\lambda(e^{-\lambda t}u) = \langle e^{-\lambda t}f, u_0 \rangle$. Since multiplication by $e^{\pm \lambda t}$ is an isomorphism of $\mathcal{H}^{r,s}(\Omega)$ onto itself for all real r and s , our reduction is accomplished. But, $\mathcal{O}_\lambda$ is one-to-one, if $\lambda \geq \mu_0(r, s)$, as is seen by setting $Pu = -\lambda u$ and $u(a) = 0$ in (5). We must still show that $\mathcal{O}_\lambda$ is onto; given $u_0 \in H^{r-m+s}$ and $f \in \mathcal{H}^{r-2m,s}(\Omega)$ we must find $u \in \mathcal{H}^{r,s}(\Omega)$ with $\mathcal{O}_\lambda u = \langle f, u_0 \rangle$. It suffices to treat the case $u_0 = 0$; in the general case, we apply Proposition 10, to find $U \in \mathcal{H}^{r,s}(\Omega)$ with $U(0) = u_0$. Then, since $f' = f - (P + \lambda I)U$ is in $\mathcal{H}^{r-2m,s}(\Omega)$, if $v \in \mathcal{H}^{r,s}(\Omega)$ can be found satisfying $\mathcal{O}_\lambda v = \langle f', 0 \rangle$ it follows that $u = v + U \in \mathcal{H}^{r,s}(\Omega)$ and satisfies $\mathcal{O}_\lambda u = \langle f, u_0 \rangle$.

Note now that $r = 2mk + \theta$, where $k = 1, 2, \dots$ and $-m < \theta \leq m$. We claim: for each $k = 1, 2, \dots$, for each $\theta, -m < \theta \leq m$, and for every real s , there exists $\lambda_0 = \lambda_0(k, \theta, s, P)$ such that whenever $\lambda \geq \lambda_0$ and $f \in \mathcal{H}^{2m(k-1)+\theta,s}(\Omega)$, there is a u in $\mathcal{H}^{2mk+\theta,s}(\Omega)$ such that $u(0) = 0$ and $(P + \lambda I)u = f$. We prove this by induction on k . For $k = 1$, we distinguish two cases: a) Suppose $-m < \theta < m$. Then $f \in \mathcal{H}^{\theta,s}(\Omega)$, and, by the corollary to Proposition 11, there is an f^* in $\mathcal{H}_0^{\theta,s}(\bar{\Omega})$ with $f = f^*|_\Omega$. By Theorem 4, if λ is large enough, there exists $u^* \in \mathcal{H}_0^{\theta+2m,s}(\bar{\Omega}_{a,+\infty})$ with $(P + \lambda I)u^* = f^*$. Taking $u = u^*|_\Omega$ and applying Proposition 6 (ii), we see that $u(0) = 0$, and $(P + \lambda I)u = f$.

b) If, on the other hand, $\theta = m$, we observe (by Proposition 2 (i)) that $f \in \mathcal{H}^{m,s}(\Omega) \subseteq \mathcal{H}^{0,s+m}(\Omega)$. By what we have just seen, there exists $u \in \mathcal{H}^{2m,s+m}(\Omega)$ such that $u(0) = 0$ and $(P + \lambda I)u = f$. But then, $\partial u / \partial t = f - \lambda u + Lu \in \mathcal{H}^{m,s}(\Omega)$, by Propositions 2(i), 4, and 5. Applying Propositions 2(i) and 8, we obtain $u \in \mathcal{H}^{3m,s}(\Omega)$, which was to be shown.

Suppose now our assertion is true for $k = 1, \dots, l-1$, where $l > 1$, and suppose $f \in \mathcal{H}^{2m(l-1)+\theta,s}(\Omega)$, with $-m < \theta \leq m$. Then, $f \in \mathcal{H}^{2m(l-2)+\theta,s+2m}(\Omega)$ and thus, if λ is large, there exists $u \in \mathcal{H}^{2m(l-1)+\theta,s+2m}(\Omega)$ such that $u(0) = 0$ and $(P + \lambda I)u = f$. But, as before, $\partial u / \partial t = f - \lambda u + Lu \in \mathcal{H}^{2m(l-1)+\theta,s}(\Omega)$ and thus $u \in \mathcal{H}^{2ml+\theta,s}(\Omega)$, which was to be shown. Finally, $\mathcal{O}^{-1}$ is seen to be continuous, by

the open mapping principle; alternatively, a bit more care in the preceding argument would yield this fact directly.

6. Technical Details.

Proof of Proposition 5. (i) It suffices to show: there exists $C = C_{r,s} > 0$ such that

$$(6) \quad \|\mathfrak{N}_{r,s} a \cdot \varphi\|_{0,0} \leq C \|\mathfrak{N}_{r,s} \varphi\|_{0,0} \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^{n+1}).$$

First, we assume $r = 4mk$ and $s = 2l$ where k and l are non-negative integers. Then, $\mathfrak{N}_{r,s}$ is a differential operator with constant coefficients; $\mathfrak{N}_{r,s} = R(D, D_t)$, where $R(\xi, \tau)$ is the polynomial $[\tau^2 + (1 + |\xi|^2)^{2m}]^k [1 + |\xi|^2]^l$. Applying Leibnitz' rule ([6], p. 10) we see that

$$\mathfrak{N}_{r,s}(a \cdot \varphi) = \sum_{\substack{\alpha \geq 0 \\ i \geq 0}} \frac{1}{\alpha! j!} D^\alpha D_t^i a \cdot R^{(\alpha,i)}(D, D_t) \varphi$$

where $R^{(\alpha,i)}(\xi, \tau)$ is the polynomial $D_\xi^\alpha D_\tau^i R(\xi, \tau)$ (and the sum is, of course, finite.) Thus,

$$\begin{aligned} \|\mathfrak{N}_{r,s} a \cdot \varphi\|_{0,0} &\leq C \sum_{\substack{\alpha \geq 0 \\ i \geq 0}} \|R^{(\alpha,i)}(D, D_t) \varphi\|_{0,0} \\ &= C \sum_{\substack{\alpha \geq 0 \\ i \geq 0}} \|R^{(\alpha,i)}(\cdot, \cdot) \hat{\varphi}(\cdot, \cdot)\|_{L^2(\mathbb{R}^{n+1})}. \end{aligned}$$

But, it is easily established that for each $\alpha, j \geq 0$ there is a constant $C_{\alpha,j} \geq 0$ such that

$$(7) \quad \left| \frac{R^{(\alpha,i)}(\xi, \tau)}{R(\xi, \tau)} \right| \leq C_{\alpha,i} q^{-|\alpha|}(\xi) Q^{-2mi}(\xi, \tau)$$

for all $(\xi, \tau) \in \mathbb{R}^{n+1}$; the right hand side of (7) being bounded, (6) follows at once, at least for our special case. For $r = -4mk, s = -2l$, we now appeal to Proposition 3.

For general r and s , we employ the complex interpolation theorem ([3]). Observe that, in the notation of [3], $[\mathcal{H}^{r_1, s_1}, \mathcal{H}^{r_2, s_2}]^\theta = \mathcal{H}^{r, s}$ where

$$r = (1 - \theta)r_1 + \theta r_2, \quad s = (1 - \theta)s_1 + \theta s_2 \quad \text{and} \quad 0 \leq \theta \leq 1.$$

Given any real r and s , we may write $r/2m = k + \theta_1$ and $s = l + \theta_2$, where k and l are integers, and $0 \leq \theta_1, \theta_2 < 1$. Then, since $\mathcal{H}^{r, s} = [\mathcal{H}^{2r, 0}, \mathcal{H}^{0, 2s}]^{1/2}$, $\mathcal{H}^{2r, 0} = [\mathcal{H}^{4mk, 0}, \mathcal{H}^{4m(k+1), 0}]^{\theta_1}$, and $\mathcal{H}^{0, 2s} = [\mathcal{H}^{0, 2l}, \mathcal{H}^{0, 2(l+1)}]^{\theta_2}$, our proof is complete.

(ii) Again, we first assume $r = 4mk$ and $s = 2l$, where k and l are non-negative integers. Then, as in the proof of (i).

$$(\mathfrak{N}_{r,s} a \cdot - a \cdot \mathfrak{N}_{r,s}) \varphi = \sum_{|\alpha| + i > 0} \frac{1}{\alpha! j!} D^\alpha D_t^i a \cdot R^{(\alpha,i)}(D, D_t) \varphi.$$

Applying (6) and (7), we obtain

$$\begin{aligned}
\|(\mathfrak{M}_{r,s}a \cdot - a \cdot \mathfrak{M}_{r,s})\varphi\|_{\rho-r, \sigma-s+1} &\leq C \sum_{|\alpha|+i>0} \|R^{(\alpha,i)}(D, D_t)\varphi\|_{\rho-r, \sigma-s+1} \\
&= C \sum_{|\alpha|+i>0} \|Q^{\rho-r} q^{\sigma-s+1} R^{(\alpha,i)}\hat{\varphi}\|_{L^2(\mathbb{R}^{n+1})} \\
&\leq C \sum_{|\alpha|+i>0} \|Q^{\rho-2mi} q^{\sigma+1-|\alpha|}\hat{\varphi}\|_{L^2(\mathbb{R}^{n+1})} \\
&\leq C \|\varphi\|_{\rho, \sigma}.
\end{aligned}$$

Since $\mathfrak{M}_{-r,-s}a \cdot - a \cdot \mathfrak{M}_{-r,-s} = \mathfrak{M}_{-r,-s}(a \cdot \mathfrak{M}_{r,s} - \mathfrak{M}_{r,s}a \cdot)\mathfrak{M}_{-r,-s}$ our assertion follows for k and l negative integers as well, even with $\theta = 1$. For general r and s , we need the following

Lemma. *Given $\gamma = \langle \gamma_1, \dots, \gamma_k \rangle$ where $\gamma_1, \dots, \gamma_k \geq 1$, and ρ , a positive C^∞ function on $\mathbb{R}^k \sim \{0\}$ satisfying*

$$\rho(\lambda^{\gamma_1} z_1, \lambda^{\gamma_2} z_2, \dots, \lambda^{\gamma_k} z_k) = \lambda \rho(z_1, \dots, z_k)$$

for all $\lambda > 0$, and all $z = \langle z_1, \dots, z_k \rangle \neq 0$ in $\mathbb{R}^k$

denote by $\wedge^\theta$ the operator taking $C_0^\infty(\mathbb{R}^k)$ into $L^2(\mathbb{R}^k)$ given by $\mathfrak{F}(\wedge^\theta \varphi)(z) = \rho^\theta(z) \hat{\varphi}(z)$, $z \in \mathbb{R}^k$, $\varphi \in C_0^\infty(\mathbb{R}^k)$, where $\mathfrak{F}(\varphi) = \hat{\varphi}$ is the k -dimensional Fourier transform of φ , and $\theta \geq 0$. Let $a \in C_B^\infty(\mathbb{R}^k)$. Then for every $\theta < 1$, there exists $C > 0$ (depending only on ρ, θ) such that

$$\|a \cdot \wedge^\theta \varphi - \wedge^\theta(a \cdot \varphi)\|_{L^2(\mathbb{R}^k)} \leq C \left\{ \|a\|_{L^\infty(\mathbb{R}^k)} + \sum_{i=1}^k \|D_i a\|_{L^\infty(\mathbb{R}^k)} \right\} \|\varphi\|_{L^2(\mathbb{R}^k)}.$$

Proof. We rely on the theory of quasi-homogeneous distributions (see [8], for example.) Define the transformation M_λ on functions defined on $\mathbb{R}^k$ by

$$M_\lambda \varphi(x_1, \dots, x_k) = \varphi(\lambda^{\gamma_1} x_1, \dots, \lambda^{\gamma_k} x_k),$$

and extend it to a mapping: $\mathfrak{D}'(\mathbb{R}^k) \rightarrow \mathfrak{D}'(\mathbb{R}^k)$ by means of

$$M_\lambda T(\varphi) = \lambda^{-|\gamma|} T(M_{1/\lambda} \varphi) \quad \varphi \in C_0^\infty(\mathbb{R}^k), \quad |\gamma| = \gamma_1 + \dots + \gamma_k.$$

Then a function (distribution) f is quasi-homogeneous of degree s if $M_\lambda f = \lambda^s f$ for all $\lambda > 0$.

Note that the function ρ^s is locally integrable at zero (respectively, at infinity) if and only if $s > -|\gamma|$ (respectively, $s < -|\gamma|$). ρ^θ , considered as a tempered distribution on $\mathbb{R}^k$, is quasi-homogeneous of degree θ ; its inverse Fourier transform $\mathfrak{F}^{-1}(\rho^\theta)$ is thus quasi-homogeneous of degree $-\theta - |\gamma|$. Since $D_z^\alpha(z^\beta \rho^\theta(z))$ is, for $z \neq 0$, quasi-homogeneous of degree $\theta + \langle \beta, \gamma \rangle - \langle \alpha, \gamma \rangle$ and C^∞ , it follows easily that $\mathfrak{F}^{-1}(\rho^\theta)$ is given by a C^∞ function g in $\mathbb{R}^k \sim \{0\}$; moreover, g is quasi-homogeneous of degree $-\theta - |\gamma|$. Thus, for all $\varphi \in C_0^\infty(\mathbb{R}^k)$

$$(8) \quad \mathfrak{F}^{-1}(\rho^\theta)(\varphi) = \int_{\mathbb{R}^k} (\varphi(y) - \varphi(0))g(y) dy.$$

In fact, (8) is clearly true for $\varphi \in C_0^\infty(R^k \sim \{0\})$; hence, the difference of the two distributions displayed in (8) is quasi-homogeneous of degree $-\theta - |\gamma|$, and has support at $\{0\}$, and is therefore zero. Note that the right hand side of (8) does indeed define a distribution, since, for $|y| < 1$,

$$|\varphi(y) - \varphi(0)| \leq C_\varphi |y| \leq C_\varphi \sum_{i=1}^k |y_i| \leq C_\varphi \sum_{i=1}^k |y_i|^{1/\gamma_i} \leq C'_\varphi \rho(y).$$

Now, writing $g(y) = h(y)\rho^{-\theta-\gamma}(y)$, where $h \in C^\infty(R^k \sim \{0\})$ is quasihomogeneous of degree 0 (and thus, in particular, bounded), we obtain from (8),

$$\wedge^\theta \varphi(x) = (2\pi)^{-k/2} \int_{R^k} \frac{h(y)}{\rho(y)^{|\gamma|+\theta}} (\varphi(x-y) - \varphi(x)) dy$$

and therefore

$$(a \cdot \wedge^\theta - \wedge^\theta a \cdot) \varphi(x) = (2\pi)^{-k/2} \int_{R^k} \frac{h(y)}{\rho(y)^{|\gamma|+\theta}} [a(x) - a(x-y)] \varphi(x-y) dy$$

from which our lemma follows immediately, using Young's inequality.

Proof of Proposition 5 concluded. For $0 < \theta < 1$, define $\mathfrak{M}_{\theta,0}$ and $\mathfrak{M}_{0,\theta}$ by $\mathfrak{F}(\mathfrak{M}_{\theta,0}\varphi)(\xi, \tau) = (|\xi|^{4m} + \tau^2)^{\theta/4m} \hat{\varphi}(\xi, \tau)$ and $\mathfrak{F}(\mathfrak{M}_{0,\theta}\varphi)(\xi, \tau) = |\xi|^\theta \hat{\varphi}(\xi, \tau)$. Applying the inequality

$$(9) \quad 0 \leq \{1+t\}^\theta - t^\theta \leq \max(\theta, 1) \cdot \{1+t\}^{\theta-1} \quad \text{for } t, \theta \geq 0$$

and the mean-value theorem, we obtain

$$0 \leq Q^\theta(\xi, \tau) - (|\xi|^{4m} + \tau^2)^{\theta/4m} \leq C_\theta Q^{\theta-2}(\xi, \tau)$$

and

$$0 \leq q^\theta(\xi) - |\xi|^\theta \leq C_\theta q^{\theta-2}(\xi)$$

with $C_\theta > 0$ a constant depending only on θ . Thus, for all $\theta \geq 0$, and all ρ and σ , $\mathfrak{M}_{\theta,0} - \mathfrak{M}_{0,\theta}$ is a bounded operator from $\mathcal{H}^{\rho,\sigma}$ into $\mathcal{H}^{\rho-\theta+2,\sigma}$ and $\mathfrak{M}_{0,\theta} - \mathfrak{M}_{\theta,0}$ is a bounded operator from $\mathcal{H}^{\rho,\sigma}$ into $\mathcal{H}^{\rho,\sigma-\theta+2}$. But $a \cdot \mathfrak{M}_{\theta,0} - \mathfrak{M}_{\theta,0} a \cdot = a \cdot \mathfrak{M}_{\theta,0} - \mathfrak{M}_{\theta,0} a \cdot + a \cdot (\mathfrak{M}_{\theta,0} - \mathfrak{M}_{0,\theta}) - (\mathfrak{M}_{\theta,0} - \mathfrak{M}_{0,\theta}) a \cdot$ and, applying our lemma with $k = n+1$, $\gamma_1 = \gamma_2 = \dots = \gamma_n = 1$, $\gamma_{n+1} = 2m$, and $\rho(\xi, \tau) = (|\xi|^{4m} + \tau^2)^{1/4m}$, we conclude that $a \cdot \mathfrak{M}_{\theta,0} - \mathfrak{M}_{\theta,0} a \cdot$ is bounded, $\mathcal{H}^{0,0} \rightarrow \mathcal{H}^{0,0}$, provided $\theta < 1$. Thus, for $0 \leq \theta < 1$, $a \cdot \mathfrak{M}_{\theta,0} - \mathfrak{M}_{\theta,0} a \cdot$ is bounded: $\mathcal{H}^{0,0} \rightarrow \mathcal{H}^{0,0}$. Moreover, for $\rho = 4mk$, $\sigma = 2l$, with k and l non-negative integers, we have, for $\varphi \in C_0^\infty(R^{n+1})$,

$$\begin{aligned} \mathfrak{M}_{\rho,\sigma}[a \cdot \mathfrak{M}_{\theta,0} - \mathfrak{M}_{\theta,0} a \cdot] \varphi &= R(D, D_t)[a \cdot \mathfrak{M}_{\theta,0} - \mathfrak{M}_{\theta,0} a \cdot] \varphi \\ &= \sum_{\substack{\alpha \geq 0 \\ j \geq 0}} \frac{1}{\alpha!} \frac{1}{j!} [D^\alpha D^j a \cdot \mathfrak{M}_{\theta,0} - \mathfrak{M}_{\theta,0} D^\alpha D^j a \cdot] R^{(\alpha,j)}(D, D_t) \varphi \end{aligned}$$

and thus

$$||[a \cdot \mathfrak{M}_{\theta,0} - \mathfrak{M}_{\theta,0} a \cdot] \varphi||_{\rho,\sigma} \leq C \sum_{\substack{\alpha \geq 0 \\ j \geq 0}} ||R^{(\alpha,j)}(D, D_t) \varphi||_{0,0} \leq C ||\varphi||_{\rho,\sigma}.$$

Arguing now by duality and interpolation, as in (i), we conclude that $a \cdot \mathfrak{M}_{\theta,0} - \mathfrak{M}_{\theta,0} a \cdot$ is a bounded operator from $\mathcal{H}^{\rho,\sigma}$ into itself for all real ρ and σ , provided $0 \leq \theta < 1$. Similarly, if we can show that $a \cdot \mathfrak{M}_{0,\theta} - \mathfrak{M}_{0,\theta} a \cdot$ is bounded: $\mathcal{H}^{0,0} \rightarrow \mathcal{H}^{0,0}$, for $0 \leq \theta < 1$, it will follow that $a \cdot \mathfrak{M}_{0,\theta} - \mathfrak{M}_{0,\theta} a \cdot$ is a bounded mapping of $\mathcal{H}^{\rho,\sigma}$ into itself for all real ρ and σ . Again we apply our Lemma, this time with $k = n$, $\gamma_1 = \gamma_2 = \dots = \gamma_n = 1$, and $\rho(\xi) = |\xi|$, to conclude: there exists $C > 0$ such that for each real t ,

$$\begin{aligned} & ||a \cdot \mathfrak{M}_{0,\theta} \varphi(\cdot, t) - \mathfrak{M}_{0,\theta} a \cdot \varphi(\cdot, t)||_{L^s(\mathbb{R}^n)} \\ & \leq C \left\{ ||a(\cdot, t)||_{L^\infty(\mathbb{R}^n)} + \sum_{i=1}^n ||D_i a(\cdot, t)||_{L^\infty(\mathbb{R}^n)} \right\} ||\varphi(\cdot, t)||_{L^s(\mathbb{R}^n)}, \end{aligned}$$

and, integrating in t , we obtain

$$||a \cdot \mathfrak{M}_{0,\theta} \varphi - \mathfrak{M}_{0,\theta} a \cdot \varphi||_{0,0} \leq C \left\{ ||a||_{L^\infty(\mathbb{R}^{n+1})} + \sum_{i=1}^n ||D_i a||_{L^\infty(\mathbb{R}^{n+1})} \right\} ||\varphi||_{0,0}.$$

Finally, given any real r and s , and any $\theta < 1$, we may write $r = 4km(1 - \theta_1)$ and $s = 2l(1 - \theta_2)$ where k and l are integers, and $\theta \leq \theta_1, \theta_2 < 1$. From what we have shown, it follows that $a \cdot \mathfrak{M}_{-\theta_1,0} - \mathfrak{M}_{-\theta_1,0} a \cdot$ is bounded: $\mathcal{H}^{\rho,\sigma} \rightarrow \mathcal{H}^{\rho+2\theta_1,\sigma}$ and $a \cdot \mathfrak{M}_{0,-\theta_2} - \mathfrak{M}_{0,-\theta_2} a \cdot$ is bounded: $\mathcal{H}^{\rho,\sigma} \rightarrow \mathcal{H}^{\rho,\sigma+2\theta_2}$ for all real ρ, σ . This fact, what we proved in the special case $r = 4km, s = 2l$, and the operator identity

$$ABC - CAB = A[BC - CB] + [AC - CA]B$$

applied repeatedly to $\mathfrak{M}_{r,s} = \mathfrak{M}_{4km,0} \mathfrak{M}_{0,2l} (\mathfrak{M}_{-\theta_1,0})^{4km} (\mathfrak{M}_{0,-\theta_2})^{2l}$, complete our proof.

Proof of Proposition 11. Recall first the definition of the partial Fourier transform which takes $\mathcal{S}(\mathbb{R}^{n+1})$ isomorphically onto itself, and is given by

$$\tilde{\varphi}(\xi, t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} \varphi(x, t) dx \quad \text{for } \varphi \in \mathcal{S}(\mathbb{R}^{n+1}).$$

This mapping extends to an isomorphism of $\mathcal{S}'(\mathbb{R}^{n+1})$ onto itself, given by

$$\tilde{u}(\varphi) = u(\tilde{\varphi}) \quad \text{for } u \in \mathcal{S}'(\mathbb{R}^{n+1}), \quad \varphi \in \mathcal{S}(\mathbb{R}^{n+1}).$$

Lemma. (i) For every real s , $u \in \mathcal{H}^{0,s}$ if and only if $\tilde{u} \in L^1_{loc}(\mathbb{R}^{n+1})$ and

$$\int_{\mathbb{R}^{n+1}} q^{2s}(\xi) |\tilde{u}(\xi, t)|^2 d\xi dt < \infty;$$

moreover, for $u \in \mathcal{H}^{0,s}$ the integral just given equals $||u||_{0,s}^2$.

(ii) If $0 < r < 2m$, and s is any real number, $u \in \mathcal{H}^{r,s}$ if and only if $u \in \mathcal{H}^{0,s+r}$ and

$$I_{r,s}(u) = \int_{R^n} q^{2s}(\xi) \int_{-\infty}^{\infty} d\theta \int_{-\infty}^{\infty} d\sigma \left\{ \frac{|\tilde{u}(\xi, \theta) - \tilde{u}(\xi, \sigma)|^2}{|\theta - \sigma|^{1+r/m}} \right\} < \infty;$$

moreover, $\{\|u\|_{0,s+r}^2 + I_{r,s}(u)\}^{1/2}$ is an equivalent norm for $\mathcal{H}^{r,s}$.

Proof. (i) The asserted equality is clear for $u \in \mathcal{S}$. Let

$$\mathcal{G}^{0,s} = \left\{ u \in \mathcal{S}'(R^{n+1}) : \tilde{u} \in L_{\text{loc}}^1(R^{n+1}) \text{ and } \int_{R^n} q^{2s}(\xi) |\tilde{u}(\xi, t)|^2 d\xi dt \equiv (\|u\|_{0,s}^2) < \infty \right\}.$$

We need only show that $\mathcal{G}^{0,s}$ is complete in the $\|\cdot\|_{0,s}$ norm, and contains $\mathcal{S}$ as a dense subset. This follows as in the proof of Theorem 2.2.1 of [6].

(ii) Again, for $u \in \mathcal{S}(R^{n+1})$, the two norms given are seen to be equivalent by an easy calculation. Setting $\mathcal{G}^{r,s} = \{u \in \mathcal{S}'(R^{n+1}) : \tilde{u} \in L_{\text{loc}}^1(R^{n+1}) \text{ and } \|u\|_{0,s+r}^2 + I_{r,s}(u) \equiv (\|u\|_{r,s}^2) < \infty\}$ it again suffices to show that $\mathcal{G}^{r,s}$ is complete in the $\|\cdot\|_{r,s}$ norm, and contains $\mathcal{S}$ as a dense subset. Given $\{u_i\}$ Cauchy in $\|\cdot\|_{r,s}$, there exists $u \in \mathcal{H}^{0,s+r}$ such that $u_i \rightarrow u$ in $\mathcal{H}^{0,s+r}$; passing to a subsequence if necessary, we may assume that $\tilde{u}_i \rightarrow \tilde{u}$ pointwise a.e. in R^{n+1} ; applying Fatou's lemma,

$$I_{r,s}(u_i - u) \leq \liminf_{k \rightarrow \infty} I_{r,s}(u_i - u_k) \rightarrow 0 \text{ as } j \rightarrow \infty.$$

To show that $\mathcal{S}$ is dense in $\mathcal{G}^{r,s}$, first choose $\psi \in C_0^\infty(R^n)$ with $\psi(0) = 1$, and define, for $\epsilon > 0$, $\psi_\epsilon(x) = \psi(\epsilon x)$. Applying part (i), if $u \in \mathcal{G}^{r,s}$, in particular, $\int_{R^n} q^{2(r+s)}(\xi) |\tilde{u}(\xi, t)|^2 d\xi < \infty$ for a.e. $t \in R^1$; by the statement and proof of Theorem 2.2.11 of [6], it follows that $u_\epsilon = \psi_\epsilon u$ satisfies

$$\int_{R^n} q^{2(r+s)}(\xi) |\tilde{u}_\epsilon(\xi, t) - u(\xi, t)|^2 d\xi \rightarrow 0$$

for a.e. t , and

$$\int_{R^n} q^{2(r+s)}(\xi) |\tilde{u}_\epsilon(\xi, t) - u(\xi, t)|^2 d\xi \leq C \int_{R^n} q^{2(r+s)}(\xi) |u(\xi, t)|^2 d\xi$$

from which we conclude that $u_\epsilon \rightarrow u$ in $\mathcal{H}^{0,r+s}$ as $\epsilon \rightarrow 0$. Passing to a subsequence if necessary, we may assume that $\tilde{u}_\epsilon \rightarrow \tilde{u}$ a.e. in R^{n+1} ; but, since by Theorem 2.2.11 of [6] again,

$$\int_{R^n} q^{2s}(\xi) |\tilde{u}_\epsilon(\xi, \theta) - \tilde{u}_\epsilon(\xi, \sigma)|^2 d\xi \leq C \int_{R^n} q^{2s}(\xi) |\tilde{u}(\xi, \theta) - \tilde{u}(\xi, \sigma)|^2 d\xi$$

for a.e. $\langle \theta, \sigma \rangle \in R^1 \times R^1$, we conclude that $I_{r,s}(u_\epsilon - u) \rightarrow 0$ as $\epsilon \rightarrow 0$. Similarly, we choose $\zeta \in C_0^\infty(R^1)$, with $\zeta(0) = 1$, $\zeta_\epsilon(t) = \zeta(\epsilon t)$, $\epsilon > 0$. Clearly, $u_\epsilon = \zeta_\epsilon \cdot u \rightarrow u$ in $\mathcal{H}^{0,s+r}$ as $\epsilon \rightarrow 0$. Moreover,

$$I_{r,s}(u_\epsilon - u) \leq 2 \int_{\mathbb{R}^n} q^{2s}(\xi) d\xi \int_{-\infty}^{\infty} d\theta \int_{-\infty}^{\infty} d\sigma (\zeta_\epsilon(\theta) - 1)^2 \frac{|\tilde{u}(\xi, \theta) - \tilde{u}(\xi, \sigma)|^2}{|\theta - \sigma|^{1+r/m}} \\ + 2 \int_{\mathbb{R}^n} q^{2s}(\xi) d\xi \int_{-\infty}^{\infty} d\theta \int_{-\infty}^{\infty} d\sigma |\tilde{u}(\xi, \sigma)|^2 \frac{|\zeta_\epsilon(\theta) - \zeta_\epsilon(\sigma)|^2}{|\theta - \sigma|^{1+r/m}}.$$

The first integral on the right tends to zero as $\epsilon \rightarrow 0$ by Lebesgue's theorem; the second integral is dominated by $C\epsilon^{r/m} \|u\|_{0,s}^2$, which goes to zero as $\epsilon \rightarrow 0$. Thus $\mathcal{E}' \cap \mathcal{G}'^s$ is dense in $\mathcal{G}'^s$; to complete our proof, a standard mollifier argument, first in x , then in t , works with no difficulty.

Proof of Proposition 11 concluded. As the case $r = 0$ is trivial, by part (i) of the lemma, we treat only the case $0 < r < m$; the case $-m < r < 0$ then follows from Proposition 3. We may assume $a = 0$, without loss of generality. By part (ii) of the lemma, we need only show there exists $C = C_{r,s} > 0$ such that

$$I_{r,s}(H_0\varphi) \leq C \|\varphi\|_{r,s}^2 \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^{n+1}).$$

But,

$$I_{r,s}(H_0\varphi) = \int_{\mathbb{R}^n} q^{2s}(\xi) \int_0^\infty d\theta \int_0^\infty d\sigma \frac{|\tilde{\varphi}(\xi, \theta) - \tilde{\varphi}(\xi, \sigma)|^2}{|\theta - \sigma|^{1+r/m}} \\ + \frac{m}{r} \int_{\mathbb{R}^n} q^{2s}(\xi) \int_0^\infty \frac{|\tilde{\varphi}(\xi, \theta)|^2}{\theta^{r/m}} d\theta.$$

The first integral on the right is, by our lemma, dominated by $C \|\varphi\|_{r,s}^2$; for the second term, we need the inequality

$$(10) \quad \int_0^\infty \frac{|\psi(\theta)|^2}{|\theta|^{2\alpha}} d\theta \leq C_\alpha \int_{-\infty}^\infty |t|^{2\alpha} |\hat{\psi}(t)|^2 dt$$

valid for $0 < \alpha < \frac{1}{2}$, $\psi \in \mathcal{S}(\mathbb{R}^1)$. Applying (10) with $\alpha = r/2m$, and $\psi(\theta) = \tilde{\varphi}(\xi, \theta)$, and observing that $|t|^{r/m} \leq Q^{2r}(\xi, t)$, would complete our proof. Although (10) is well-known, we are unable to find a proof in the literature; for completeness we include a short one here. In fact, we prove

$$(10') \quad \int_{-\infty}^\infty \frac{|\psi(\theta)|^2}{|\theta|^{2\alpha}} d\theta \leq C_\alpha \int_{-\infty}^\infty |t|^{2\alpha} |\hat{\psi}(t)|^2 dt.$$

Let $f(\theta) = |\theta|^{-\alpha} \psi(\theta)$, $g(\tau) = |\tau|^\alpha \hat{\psi}(\tau)$, and $h(\tau) = \hat{f}(\tau)$. Then, as is easily seen,

$$h(\tau) = C_\alpha \int_{-\infty}^\infty |\tau - \sigma|^{\alpha-1} \hat{\psi}(\sigma) d\sigma = C_\alpha \int_{-\infty}^\infty |\tau - \sigma|^{\alpha-1} |\sigma|^{-\alpha} g(\sigma) d\sigma.$$

For $\tau > 0$, we have

$$h(\tau) = C_\alpha \int_{-\infty}^\infty |1 - \rho|^{\alpha-1} |\rho|^{-\alpha} g(\tau\rho) d\rho$$

and thus

$$\begin{aligned}
||h||_{L^2(0,\infty)} &\leq C_\alpha \int_{-\infty}^{\infty} |1 - \rho|^{\alpha-1} |\rho|^{-\alpha} \left[\int_0^{\infty} |g(\tau\rho)|^2 d\tau \right]^{1/2} d\rho \\
&\leq C_\alpha ||g||_{L^2(-\infty,\infty)} \int_{-\infty}^{\infty} |1 - \rho|^{\alpha-1} |\rho|^{-\alpha-1/2} d\rho.
\end{aligned}$$

A similar argument yields

$$||h||_{L^2(-\infty,0)} \leq C_\alpha ||g||_{L^2(-\infty,\infty)} \int_{-\infty}^{\infty} |1 + \rho|^{\alpha-1} |\rho|^{-\alpha-1/2} d\rho$$

and completes our proof.

REFERENCES

- [1] S. AGMON, *Lectures on Elliptic Boundary Value Problems*, Van Nostrand, Princeton, 1965.
- [2] L. BERS, F. JOHN & M. SCHECHTER, *Partial Differential Equations*, Interscience, New York, 1964.
- [3] A. CALDERÓN, Intermediate spaces and interpolation, the complex method, *Studia Math.*, **24**(1964) 119-190.
- [4] E. FABES & N. RIVIERE, Systems of parabolic equations with uniformly continuous coefficients, *J. Analyse Math.*, **17** (1966) 305-336.
- [5] A. FRIEDMAN, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, N. J., 1964.
- [6] L. HÖRMANDER, *Linear Partial Differential Operators*, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1964.
- [7] S. KAPLAN, Abstract boundary value problems for parabolic equations, *Ann. Scuola Norm. Sup. Pisa*, **20** (1966) 395-419.
- [8] P. KRÉE, Distributions quasihomogenes. Généralizations des intégrales singulieres et du calcul symbolique de Calderon-Zygmund., *C. R. Acad. Sci.*, **261** (1965) 2560-2563.

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