Representation of Semi-Groups of Nonlinear Nonexpansive Transformations in Banach Spaces

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Let S be a weakly complete Banach space with conjugate space S^* separable. A semi-group of nonlinear nonexpansive transformations on S is a function T from $[0, \infty)$ to the set of transformations on S such that

- (1) $T(x + y) = T(x)T(y)$ for all $x, y \geq 0$,
- (2) $T(x)$ is nonexpansive for each $x \geq 0$, i.e., if $p, q \in S$ then $\|T(x)p - T(x)q\| \leq \|p - q\|$,
- (3) if $p \in S$ and $g_p(x) = T(x)p$ for each $x \geq 0$, then g_p is continuous and $g_p(0) = p$.

If there is a point q in S such that $g'_q(0)$ exists, then the infinitesimal generator of T is defined to be the function A such that $Ap = g'_p(0)$ for all p in S for which $g'_p(0)$ exists.

If for each $x \geq 0$ it is required that $T(x)$ is a linear transformation on S then T is a semi-group of linear nonexpansive transformations on S and it can be shown (see [4] or [12]) that the infinitesimal generator A of T is densely defined, for each $\epsilon > 0$ $(I - \epsilon A)^{-1}$ exists and has domain S , and T may be represented as follows:

$$T(\epsilon)p = \lim_{n \rightarrow \infty} (I - (\epsilon/n)A)^{-n}p \quad \text{for } p \in S, \quad \epsilon \geq 0.$$

If T is a semi-group of nonlinear nonexpansive transformations on S , J. W. Neuberger [9] has shown the following theorem:

If $x > 0$ and $A_x = (1/x)[T(x) - I]$, then for each $\epsilon \geq 0$ $(I - \epsilon A_x)^{-1}$ exists, has domain S , and is nonexpansive. Further, if there is a dense subset D of S

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such that for each p in D g'_p exists and is continuous on $[0, \infty)$, then if $p \in S$ and $\epsilon \geq 0$

$$\lim_{n \rightarrow \infty} \limsup_{x \rightarrow 0+} \|(I - (\epsilon/n)A_x)^{-n}p - T(\epsilon)p\| = 0.$$

It may be noted that the hypothesis of this theorem is always satisfied if T is linear but not necessarily if T is nonlinear (see Example 1). With the same hypothesis as used by Neuberger in the result above J. R. Dorroh [2] has shown the following:

$$T(\epsilon)p = \lim_{x \rightarrow 0+} [\lim_{n \rightarrow \infty} (I - (\epsilon/n)A_x)^{-n}p] \quad \text{for } p \in S, \quad \epsilon \geq 0.$$

These two theorems give a representation of T in terms of the functions A_x , $x > 0$. The object of this paper is to obtain a representation for a semi-group of nonlinear nonexpansive transformations in terms of the infinitesimal generator itself. This representation is then used to show that no two semi-groups of nonlinear nonexpansive transformations can have the same infinitesimal generator. The main theorems to be proved are stated as follows:

Theorem 1. *Let T be a semi-group of nonlinear nonexpansive transformations on S with the property that there is a point p in the domain of its infinitesimal generator A . If $\epsilon > 0$ then $(I - \epsilon A)^{-1}$ exists, has domain the range of $I - \epsilon A$, and has an extension $\text{ext } (I - \epsilon A)^{-1}$ to all of S which is nonexpansive.*

Theorem 2. *Let T be a semi-group of nonlinear nonexpansive transformations on S such that the infinitesimal generator A of T is densely defined and if p is in the domain of A g'_p exists and is continuous almost everywhere on $[0, \infty)$. For each $\epsilon > 0$ let $\text{ext } (I - \epsilon A)^{-1}$ be a nonexpansive extension of $(I - \epsilon A)^{-1}$ to S . If $q \in S$, $x > 0$, and $c > 0$, there exists $d > 0$ such that if $\{t_i\}_{i=0}^n$ is a chain from 0 to x of mesh less than d then*

$$\left\| \prod_{i=1}^n \text{ext } (I - (t_i - t_{i-1})A)^{-1}q - T(x)q \right\| < c.$$

It may be noted that if S is the Hilbert space, Y. Kōmura [7] has shown that the infinitesimal generator must always be densely defined. Further, M. G. Crandall and A. Pazy [1] have shown that if p is in the domain of A then g'_p exists and is continuous almost everywhere on $[0, \infty)$. Thus, the hypothesis of Theorem 2 is always satisfied in the case that S is the Hilbert space.

Before proving Theorem 1 three lemmas will be proved, each under the hypothesis of Theorem 1.

Lemma 1.1. *If p is in the domain of A then g_p is Lipschitz continuous on $[0, \infty)$ with Lipschitz constant $\leq \|Ap\|$ and if $x > 0$ such that $g'_p(x)$ exists then $g'_p(x) = Ag_p(x)$ and $\|g'_p(x)\| \leq \|Ap\|$.*

Proof. If $c > 0$ there exists $d > 0$ such that if $0 < x < d$ then $\|(1/x) \cdot [g_p(x) - g_p(0)] - Ap\| < c$. Let $0 \leq a < b$. Let $\{x_i\}_{i=0}^n$ be a chain from a to

b of mesh less than d . Then,

$$\begin{aligned} \|g_b(b) - g_a(a)\| &= \left\| \sum_{i=1}^n g_b(x_i) - g_b(x_{i-1}) \right\| \\ &\leq \sum_{i=1}^n \|T(x_i - x_{i-1})p - p\| \\ &< \sum_{i=1}^n (x_i - x_{i-1}) \cdot [c + \|Ap\|] \\ &= (b - a) \cdot [c + \|Ap\|]. \end{aligned}$$

Thus, g_b is Lipschitz continuous on $[0, \infty)$ with Lipschitz constant $\leq \|Ap\|$. Suppose now that $x > 0$ such that $g'_b(x)$ exists. If $c > 0$ there exists $d > 0$ such that if $0 < h < d$ then

$$\|(1/h)[g_b(x+h) - g_b(x)] - g'_b(x)\| < c.$$

Thus, if $0 < h < d$, then

$$\|(1/h)[g_{g_b(x)}(h) - g_{g_b(x)}(0)] - g'_b(x)\| < c$$

and so $g'_{g_b(x)}(0) = Ag_b(x) = g'_b(x)$. Further, if $h > 0$ then

$$\begin{aligned} \|(1/h)[g_b(x+h) - g_b(x)]\| &= (1/h) \|T(x)T(h)p - T(x)p\| \\ &\leq (1/h) \|T(h)p - p\| \\ &= \|(1/h)[g_b(h) - g_b(0)]\| \end{aligned}$$

and so $\|g'_b(x)\| \leq \|Ap\|$.

Lemma 1.2. Let $\epsilon > 0$. For each $x > 0$ $(I - \epsilon A_x)^{-1}$ exists, has domain S and is nonexpansive, where $A_x = (1/x)[T(x) - I]$, (for a proof see [9]). There exists $M > 0$ such that if $q \in S$ and $x > 0$ then $\|(I - \epsilon A_x)^{-1}q\| \leq \|q\| + M$.

Proof. Let p be a point in the domain of A . Then $\|(I - \epsilon A_x)^{-1}p - p\| \leq \|p - (I - \epsilon A_x)p\| = \epsilon \cdot \|A_x p\| = \epsilon \cdot \|(1/x)(g_b(x) - p)\| \leq \epsilon \cdot \|Ap\|$ (by Lemma 1.1). If $q \in S$ and $x > 0$ then $\|(I - \epsilon A_x)^{-1}q - (I - \epsilon A_x)^{-1}p\| \leq \|q - p\|$. Let $M = 2\|p\| + \epsilon\|Ap\|$ and the lemma is established.

Lemma 1.3. If $\epsilon > 0$ there is a sequence $\{x_i\}_{i=0}^{\infty}$ of positive numbers converging to 0 such that if $q \in S$ then $\{(I - \epsilon A_{x_i})^{-1}q\}_{i=0}^{\infty}$ is weakly convergent.

Proof. Since S^* is separable, S is separable (see [4], page 34). Let $\{q_m\}_{m=0}^{\infty}$ be a sequence with values dense in S . Let $\{h_n\}_{n=0}^{\infty}$ be a sequence with values dense in S^* . Let M be a positive number as in Lemma 1.2. If each of m and n is a nonnegative integer then

$$\{h_n((I - \epsilon A_{x_i})^{-1}q_m) \mid 0 < x\}$$

is a bounded set of numbers. Thus, there is a sequence $\{x_i\}_{i=0}^{\infty}$ of positive num-

bers converging to 0 such that if each of m and n are nonnegative integers, then $\{h_n((I - \epsilon A_{x_i})^{-1} q_m)\}_{i=0}^\infty$ converges. Let $q \in S$, let $h \in S^*$, and let $c > 0$. There is a positive integer m such that $\|q - q_m\| < c/(5 \cdot [1 + \|h\|])$ and there is a positive integer n such that $|h - h_n| < \min \{c/(5 \cdot [\|q\| + M]), 1\}$. There is a positive integer j such that if k is a nonnegative integer

$$|h_n((I - \epsilon A_{x_i})^{-1} q_m) - h_n((I - \epsilon A_{x_{i+k}})^{-1} q_m)| < c/5.$$

Then, if k is a nonnegative integer,

$$\begin{aligned} & |h((I - \epsilon A_{x_i})^{-1} q) - h((I - \epsilon A_{x_{i+k}})^{-1} q)| \\ & \leq |h((I - \epsilon A_{x_i})^{-1} q) - h_n((I - \epsilon A_{x_i})^{-1} q)| \\ & \quad + |h_n((I - \epsilon A_{x_i})^{-1} q) - h_n((I - \epsilon A_{x_i})^{-1} q_m)| \\ & \quad + |h_n((I - \epsilon A_{x_i})^{-1} q_m) - h_n((I - \epsilon A_{x_{i+k}})^{-1} q_m)| \\ & \quad + |h_n((I - \epsilon A_{x_{i+k}})^{-1} q_m) - h_n((I - \epsilon A_{x_{i+k}})^{-1} q)| \\ & \quad + |h_n((I - \epsilon A_{x_{i+k}})^{-1} q) - h((I - \epsilon A_{x_{i+k}})^{-1} q)| \\ & \leq c/5 + 2 \cdot \|h - h_n\| \cdot [\|q\| + M] + 2 \cdot \|h_n\| \cdot \|q - q_m\| \\ & < c. \end{aligned}$$

Thus, $\{h((I - \epsilon A_{x_i})^{-1} q)\}_{i=0}^\infty$ converges for each $q \in S$ and each $h \in S^*$ and so the lemma is proved.

Proof of Theorem 1. Let $\epsilon > 0$. Suppose that each of p and q is a point in the domain of A . If $x > 0$, then

$$\begin{aligned} & \|(I - \epsilon A_x)p - (I - \epsilon A_x)q\| \\ & = \|(I - (\epsilon/x)[T(x) - I])p - (I - (\epsilon/x)[T(x) - I])q\| \\ & \geq (x + \epsilon/x) \|p - q\| - (\epsilon/x) \|T(x)p - T(x)q\| \\ & \geq (x + \epsilon/x) \|p - q\| - (\epsilon/x) \|p - q\| \\ & = \|p - q\|. \end{aligned}$$

Then, $\|(I - \epsilon A)p - (I - \epsilon A)q\| \geq \|p - q\|$ and so $I - \epsilon A$ is reversible. Thus, $(I - \epsilon A)^{-1}$ exists and has domain the range of $I - \epsilon A$. By Lemma 1.3 there is a sequence $\{x_i\}_{i=0}^\infty$ of positive numbers converging to 0 such that if $q \in S$ then $\{(I - \epsilon A_{x_i})^{-1} q\}_{i=0}^\infty$ is weakly convergent. Since S is weakly complete there is member u of S such that if $h \in S^*$ then $\{h((I - \epsilon A_{x_i})^{-1} q)\}_{i=0}^\infty$ converges to $h(u)$. Define

$$\begin{aligned} \text{ext } (I - \epsilon A)^{-1} q &= u, \text{ i.e., } \text{ext } (I - \epsilon A)^{-1} q \\ &= w - \lim \{(I - \epsilon A_{x_i})^{-1} q\}_{i=0}^\infty \text{ for each } q \in S. \end{aligned}$$

Suppose now that w is in the domain of $(I - \epsilon A)^{-1}$, i.e., there exists z in S such

that $(I - \epsilon A)z = w$. If $x > 0$ then $\|(I - \epsilon A_x)^{-1}w - z\| \leq \|w - (I - \epsilon A_x)z\|$ and so $\lim_{x \rightarrow 0+} (I - \epsilon A_x)^{-1}w = z$. Thus, $w = \lim \{(I - \epsilon A_{x_i})^{-1}w\}_{i=0}^\infty = z$ and so $\text{ext } (I - \epsilon A)^{-1}$ is an extension of $(I - \epsilon A)^{-1}$ to all of S . Now let each of p and q be in S and let $c > 0$. By the Hahn-Banach theorem there is a member h of S^* such that $\|h\| = 1$ and $h(\text{ext } (I - \epsilon A)^{-1}p - \text{ext } (I - \epsilon A)^{-1}q) = \|\text{ext } (I - \epsilon A)^{-1}p - \text{ext } (I - \epsilon A)^{-1}q\|$. There is a positive integer i such that

$$\begin{aligned} |h(\text{ext } (I - \epsilon A)^{-1}p) - h(\text{ext } (I - \epsilon A)^{-1}q)| \\ < c + |h((I - \epsilon A_{x_i})^{-1}p) - h((I - \epsilon A_{x_i})^{-1}q)| \\ \leq c + \|(I - \epsilon A_{x_i})^{-1}p - (I - \epsilon A_{x_i})^{-1}q\| \\ \leq c + \|p - q\|. \end{aligned}$$

Thus, $\|\text{ext } (I - \epsilon A)^{-1}p - \text{ext } (I - \epsilon A)^{-1}q\| \leq \|p - q\|$ and so $\text{ext } (I - \epsilon A)^{-1}$ is nonexpansive. Hence, Theorem 1 is proved.

It should be noted that in [6] Y. Kōmura used a similar argument to show that if S is the Hilbert space then for p in S and $\epsilon > 0$,

$$w = \lim (I - \epsilon A_h)^{-1}p, \quad h \in \Phi \in \Phi$$

exists where $\Phi = \{\phi\}$ is an ultra filter of sets contained in $(0, \infty)$ converging to 0 and this limit defines a nonexpansive extension of $(I - \epsilon A)^{-1}$ to all of S .

Before proving Theorem 2 two lemmas will be proved, each under the hypothesis of Theorem 2.

Lemma 2.1. *Let p be in the domain of A and let $\text{ext } (g'_p)$ be an extension of g'_p to $[0, \infty)$ bounded by $\|Ap\|$. If $[a, b] \subseteq [0, \infty)$ then $\text{ext } (g'_p)$ is Riemann integrable on $[a, b]$ and $g_p(b) - g_p(a) = \int_a^b \text{ext } (g'_p)$.*

Proof. Since g'_p is continuous almost everywhere on $[0, \infty)$ $\int_a^b \text{ext } (g'_p)$ exists. Further, for each $h \in S^*$ $h[g_p]$ is Lipschitz continuous on $[0, \infty)$. Thus, if $h \in S^*$ then $h[g_p](b) - h[g_p](a) = \int_a^b h[\text{ext } (g'_p)] = h(\int_a^b \text{ext } (g'_p))$ and so by the Hahn-Banach theorem $g_p(b) - g_p(a) = \int_a^b \text{ext } (g'_p)$.

Lemma 2.2. *Let p be in the domain of A , let $x > 0$, and let $c > 0$. There is a sequence of segments $\{c_i\}_{i=0}^m$ and a positive number d such that if $\{t_i\}_{i=0}^n$ is a chain from 0 to x of mesh less than d then*

- (1) *if $i \in [1, n]$ and $[t_{i-1}, t_i]$ intersects c_j for some $j \in [0, m]$, then if $u, v \in [t_{i-1}, t_i]$*
 $\|g'_p(u) - g'_p(v)\| < c/4x$

and

$$(2) \quad \sum_{i=1}^n t_i - t_{i-1} < c/(8 \cdot (\|Ap\| + 1)),$$

where $[t_{i-1}, t_i] \cap c_j = \emptyset, \quad j \in [0, m]$.

Proof. There is a subset E of $[0, x]$ such that E has measure 0 and if u is

in $[0, x] - E$ then g'_p exists and is continuous at u . There is a sequence of segments $\{d_i\}_{i=0}^{\infty}$ such that $E \subseteq \bigcup_{i=0}^{\infty} d_i$ and $\lim \sum_{i=0}^{\infty} l(d_i) < c/(8[|Ap| + 1])$. For each number u in $[0, x] - \bigcup_{i=0}^{\infty} d_i$ there is a positive number c_u such that if v is in $[0, x]$ and $|u - v| < 2c_u$ then

$$||g'_p(u) - g'_p(v)|| < c/8x.$$

There is a finite sequence $\{u_i\}_{i=0}^m$ of numbers in $[0, x] - \bigcup_{i=0}^{\infty} d_i$ such that

$$[0, x] - \bigcup_{i=0}^{\infty} d_i \subseteq \bigcup_{i=0}^m (u_i - c_{u_i}, u_i + c_{u_i}).$$

For each integer $i \in [0, m]$ let $c_i = (u_i - c_{u_i}, u_i + c_{u_i})$ and let $d = \min \{c_{u_0}, c_{u_1}, \dots, c_{u_m}\}$ and the lemma is proved.

Proof of Theorem 2. Let p be in the domain of A , let $x > 0$, and let $c > 0$. Let $\{c_i\}_{i=0}^m$ be a sequence of segments and d a positive number as in Lemma 2.2. Let $\{t_i\}_{i=0}^n$ be a chain from 0 to x of mesh less than d . Let $\{t'_i\}_{i=0}^n$ be a sequence of numbers such that if $i \in [1, n]$ then $t_{i-1} < t'_i < t_i$, $g'_p(t'_i)$ exists, and $||g_p(t_i) - g_p(t'_i)|| < c/4n$. Then,

$$\begin{aligned} & \left\| \prod_{i=1}^n \text{ext} (I - (t_i - t_{i-1})A)^{-1} p - T(x)p \right\| \\ &= \left\| \sum_{i=1}^{n-1} \left[\prod_{j=i}^n \text{ext} (I - (t_j - t_{j-1})A)^{-1} T(t_{i-1})p \right. \right. \\ & \quad \left. \left. - \prod_{j=i+1}^n \text{ext} (I - (t_j - t_{j-1})A)^{-1} T(t_i)p \right] \right. \\ & \quad \left. + \text{ext} (I - (t_n - t_{n-1})A)^{-1} T(t_{n-1})p - T(t_n)p \right\| \\ &\leq \sum_{i=1}^n ||\text{ext} (I - (t_i - t_{i-1})A)^{-1} T(t_{i-1})p - T(t_i)p|| \\ &< c/4 + \sum_{i=1}^n ||\text{ext} (I - (t_i - t_{i-1})A)^{-1} T(t_{i-1})p - T(t'_i)p|| \\ &= c/4 + \sum_{i=1}^n ||\text{ext} (I - (t_i - t_{i-1})A)^{-1} T(t_{i-1})p \\ & \quad - (I - (t_i - t_{i-1})A)^{-1} (I - (t_i - t_{i-1})A) T(t'_i)p|| \\ &\leq c/4 + \sum_{i=1}^n ||T(t_{i-1})p - (I - (t_i - t_{i-1})A) T(t'_i)p|| \\ &\leq c/4 + \sum_{i=1}^n \left[\left\| \int_{t_{i-1}}^{t'_i} [\text{ext} (g'_p) - \text{ext} (g'_p)(t'_i)] \right\| + ||T(t'_i)p - T(t_i)p|| \right] \\ &< c/2 + \sum_{i=1}^n \int_{t_{i-1}}^{t'_i} ||\text{ext} (g'_p) - \text{ext} (g'_p)(t'_i)|| \end{aligned}$$

$$\begin{aligned}
 &< c/2 + \sum_{i=1}^n (c/4x) \cdot [t_i - t_{i-1}], \quad [t_{i-1}, t_i] \cap c_j \text{ for some } j \text{ in } [0, m] \\
 &+ \sum_{i=1}^n 2 \cdot \|Ap\| \cdot [t_i - t_{i-1}], \quad [t_{i-1}, t_i]c_i = \emptyset \text{ for any } j \text{ in } [0, m] \\
 &< c.
 \end{aligned}$$

Now let $q \in S$, let $x > 0$, and let $c > 0$. There is a point p in the domain of A such that $\|p - q\| < c/3$. There is a positive number d such that if $\{t_i\}_{i=0}^n$ is a chain from 0 to x of mesh less than d then

$$\left\| \prod_{i=1}^n \text{ext} (I - (t_i - t_{i-1})A)^{-1} p - T(x)p \right\| < c/3.$$

Then,

$$\begin{aligned}
 &\left\| \prod_{i=1}^n \text{ext} (I - (t_i - t_{i-1})A)^{-1} q - T(x)q \right\| \\
 &\leq \left\| \prod_{i=1}^n \text{ext} (I - (t_i - t_{i-1})A)^{-1} q - \prod_{i=1}^n \text{ext} (I - (t_i - t_{i-1})A)^{-1} p \right\| \\
 &+ \left\| \prod_{i=1}^n \text{ext} (I - (t_i - t_{i-1})A)^{-1} p - T(x)p \right\| + \|T(x)p - T(x)q\| \\
 &< \|q - p\| + c/3 + \|p - q\| \\
 &< c.
 \end{aligned}$$

Hence, the theorem is proved.

Corollary. Let each of T_1 and T_2 be a semi-group of nonlinear nonexpansive transformations on S with the property that the infinitesimal generator is densely defined and if p is in the domain of the infinitesimal generator then g'_p exists and is continuous almost everywhere on $[0, \infty)$. If T_1 and T_2 have the same infinitesimal generator A , then T_1 is T_2 .

Proof. For each $\epsilon > 0$ let $\text{ext} (I - \epsilon A)^{-1}$ be a nonexpansive extension of $(I - \epsilon A)^{-1}$ to all of S (at least one such extension exists by Theorem 1). Let $p \in S$, $x > 0$, and $c > 0$. There is a positive number d such that if $\{t_i\}_{i=0}^n$ is a chain from 0 to x of mesh less than d , then

$$\left\| \prod_{i=1}^n \text{ext} (I - (t_i - t_{i-1})A)^{-1} p - T_1(x)p \right\| < c/2$$

and

$$\left\| \prod_{i=1}^n \text{ext} (I - (t_i - t_{i-1})A)^{-1} p - T_2(x)p \right\| < c/2$$

Thus, $T_1(x)p = T_2(x)p$ and so the corollary is established.

With the hypothesis of Theorem 2 a product integral representation for

T is obtained. If one requires only that the infinitesimal generator of T be densely defined then the following representation theorem may be shown:

Theorem 3. *Let T be a semi-group of nonlinear nonexpansive transformations on S such that the infinitesimal generator A of T is densely defined. For each $\epsilon > 0$ let $\text{ext } (I - \epsilon A)^{-1}$ be a nonexpansive extension of $(I - \epsilon A)^{-1}$ to S . If $p \in S$, $x > 0$, $c > 0$, and $b > 0$, then there is a chain $\{t_i\}_{i=0}^n$ from 0 to x of mesh less than b such that*

$$\left\| \prod_{i=1}^n \text{ext } (I - (t_i - t_{i-1})A)^{-1} p - T(x)p \right\| < c.$$

Before proving Theorem 3 the following two lemmas will be proved under the hypothesis of Theorem 3.

Lemma 3.1. *If p is in the domain of A then g_p is differentiable almost everywhere on $[0, \infty)$ and if $[a, b] \subseteq [0, \infty)$ then $g_p(b) - g_p(a) = (B) \int_a^b g'_p$ (where (B) denotes the Bochner integral).*

Proof. Let W be a countable dense subset of S^* and let h belong to W . In the proof of Lemma 1.1 it was shown that if p is in the domain of A then g_p is Lipschitz on $[0, \infty)$. Then $h[g_p]$ is Lipschitz on $[0, \infty)$ and so by the Lebesgue theorem $h[g_p]$ is differentiable almost everywhere on $[0, \infty)$. There is a subset E of $[0, \infty)$ such that E has measure 0 and if $a \in [0, \infty) - E$ and $h \in W$ then $h[g_p]$ is differentiable at a . Let $a \in [0, \infty) - E$, $k \in S^*$, and $c > 0$. There is a member h of W^* such that $|k - h| < c/4 \cdot [|Ap| + 1]$. There is a positive number $x < a$ such that if $0 < |u| < x$ and $0 < |v| < x$ then

$$|(1/u)(h[g_p](a+u) - h[g_p](a)) - (1/v)(h[g_p](a+v) - h[g_p](a))| < c/2.$$

Then, if $0 < |u| < x$ and $0 < |v| < x$,

$$\begin{aligned} & |(1/u)(k[g_p](a+u) - k[g_p](a)) - (1/v)(k[g_p](a+v) - k[g_p](a))| \\ & \leq |(k - h)((1/u)(g_p(a+u) - g_p(a)) - (1/v)(g_p(a+v) - g_p(a)))| \\ & \quad + |h((1/u)(g_p(a+u) - g_p(a)) - (1/v)(g_p(a+v) - g_p(a)))| \\ & \leq |k - h| \cdot 2 \cdot [|Ap| + 1] + c/2 \\ & < c. \end{aligned}$$

Thus, $\lim_{x \rightarrow 0} ((1/x)(k[g_p](a+x) - k[g_p](a)))$ exists for each $k \in S^*$. Since S is weakly complete there is a member z of S such that if $k \in S^*$ $\lim_{x \rightarrow 0} ((1/x) \cdot (k[g_p](a+x) - k[g_p](a))) = k(z)$. Thus, g_p has a weak derivative at a for each $a \in [0, \infty) - E$. Further, g_p is Lipschitz on $[0, \infty)$ by Lemma 1.1 and so by the Bochner theorem (see [4], Theorem 3.8.6, page 88) g_p is differentiable almost everywhere on $[0, \infty)$ and g_p can be represented as the indefinite Bochner integral of g'_p .

Lemma 3.2. *Let p be in the domain of A , let $x > 0$ such that $g'_p(x)$ exists,*

let $c > 0$, and let $b > 0$. There is a chain $\{t_i\}_{i=0}^{2n+1}$ from 0 to x of mesh less than b such that

(1) if $1 \leq i \leq 2n + 1$ then $g'_p(t_i)$ exists;

(2) if $1 \leq i \leq n$ then $\frac{1}{t_{2i} - t_{2i-1}} \int_{t_{2i-1}}^{t_{2i}} \|g'_p - g'_p(t_{2i})\| < c/2x$;

and

(3) $\sum_{i=1}^{n+1} t_{2i-1} - t_{2i-2} < c/(4 \cdot [\|Ap\| + 1])$.

Proof. There is a subset E of $[0, x]$ such that E has measure 0 and if $a \notin [0, x] - E$ then $g'_p(a)$ exists and

$$\lim_{h \rightarrow 0} (1/h) \int_a^{a+h} \|g'_p - g'_p(a)\| = 0$$

(see [4], Theorem 3.8.5, page 87). For each number u in $[0, x] - E$ let c_u be a positive number less than b such that if $0 < |h| < c_u$ then

$$(1/h) \int_u^{u+h} \|g'_p - g'_p(u)\| < c/2x.$$

There is a sequence $\{u_i\}_{i=0}^\infty$ of numbers in $[0, x] - E$ such that $\bigcup_{i=0}^\infty (u_i - c_{u_i}, u_i]$ covers $[0, x] - E$. There is a finite increasing sequence $\{t_i\}_{i=0}^{2n+1}$ such that

(i) $t_0 = 0, t_{2n+1} = x$, and if $1 \leq i \leq 2n, t_i \notin E$;

(ii) if $1 \leq i \leq n$ then $t_{2i} - t_{2i-1} < c_{u_i}$;

and

(iii) $m\left([0, x] - \bigcup_{i=1}^n [t_{2i-1}, t_{2i}]\right) < \min \{b, c/(4 \cdot [\|Ap\| + 1])\}$.

Then $\{t_i\}_{i=0}^{2n+1}$ is a chain from 0 to x of mesh less than b having properties (1), (2), and (3) and so the lemma is established.

Proof of Theorem 3. Let p be in the domain of A , let $x > 0$ such that $g'_p(x)$ exists, let $c > 0$, and let $b > 0$. Let $\{t_i\}_{i=0}^{2n+1}$ be a chain from 0 to x of mesh less than b as in Lemma 3.2. Then,

$$\begin{aligned} & \left\| \prod_{i=1}^{2n+1} \text{ext} (I - (t_i - t_{i-1})A)^{-1} p - T(x)p \right\| \\ &= \left\| \sum_{i=1}^{2n} \left[\prod_{j=i}^{2n+1} \text{ext} (I - (t_j - t_{j-1})A)^{-1} T(t_{i-1})p \right. \right. \\ & \quad \left. \left. - \prod_{j=i+1}^{2n+1} \text{ext} (I - (t_j - t_{j-1})A)^{-1} T(t_i)p \right] \right. \\ & \quad \left. + \text{ext} (I - (t_{2n+1} - t_{2n})A)^{-1} T(t_{2n})p - T(t_{2n+1})p \right\| \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=1}^{2n+1} \|\text{ext } (I - (t_i - t_{i-1})A)^{-1}T(t_{i-1})p - T(t_i)p\| \\
&\leq \sum_{i=1}^{2n+1} \|T(t_{i-1})p - (I - (t_i - t_{i-1})A)T(t_i)p\| \\
&\leq \sum_{i=1}^{2n+1} \left\| (B) \int_{t_{i-1}}^{t_i} [g'_p - g'_p(t_i)] \right\| \\
&\leq \sum_{i=1}^{2n+1} \int_{t_{i-1}}^{t_i} \|g'_p - g'_p(t_i)\| \\
&= \sum_{i=1}^{n+1} \int_{t_{2i-2}}^{t_{2i-1}} \|g'_p - g'_p(t_{2i-1})\| \\
&\quad + \sum_{i=1}^n (t_{2i} - t_{2i-1}) \left[\frac{1}{t_{2i} - t_{2i-1}} \int_{t_{2i-1}}^{t_{2i}} \|g'_p - g'_p(t_{2i})\| \right] \\
&< \sum_{i=1}^{n+1} (t_{2i-1} - t_{2i-2}) \cdot 2 \cdot \|Ap\| + \sum_{i=1}^n (t_{2i} - t_{2i-1}) \cdot (c/2x) \\
&< c.
\end{aligned}$$

Using the nonexpansive property of the extensions $\text{ext } (I - \epsilon A)^{-1}$, $\epsilon > 0$, and the denseness of the domain of A , the above argument may be readily extended to all $p \in S$ and $x > 0$ and thus the theorem is established.

Corollary. *Let each of T_1 and T_2 be a semi-group of nonlinear nonexpansive transformations on S with the property that the infinitesimal generator is densely defined. If T_1 and T_2 have the same infinitesimal generator A , then T_1 is T_2 .*

Proof. For each $\epsilon > 0$ let $\text{ext } (I - \epsilon A)^{-1}$ be a nonexpansive extension of $(I - \epsilon A)^{-1}$ to all of S . Let p be in the domain of A and let $g_p(x) = T_1(x)p$ for $x \geq 0$ and $h_p(x) = T_2(x)p$ for $x \geq 0$. Let $x > 0$ such that both $g'_p(x)$ and $h'_p(x)$ exist and let $c > 0$. There is a chain $\{t_i\}_{i=0}^{2n+1}$ from 0 to x such that

(1) if $1 \leq i \leq 2n+1$ then both $g'_p(t_i)$ and $h'_p(t_i)$ exist;

(2) if $1 \leq i \leq n$ then $\frac{1}{t_{2i} - t_{2i-1}} \int_{t_{2i-1}}^{t_{2i}} \|g'_p - g'_p(t_{2i})\| < c/4x$

and

$$\frac{1}{t_{2i} - t_{2i-1}} \int_{t_{2i-1}}^{t_{2i}} \|h'_p - h'_p(t_{2i})\| < c/4x;$$

and

$$(3) \sum_{i=1}^{n+1} t_{2i-1} - t_{2i-2} < c/8 \cdot [\|Ap\| + 1].$$

Then,

$$\left\| \prod_{i=1}^{2n+1} \text{ext} (I - (t_i - t_{i-1})A)^{-1}p - T_1(x)p \right\| < c/2$$

and

$$\left\| \prod_{i=1}^{2n+1} \text{ext} (I - (t_i - t_{i-1})A)^{-1}p - T_2(x)p \right\| < c/2.$$

Thus, $T_1(x)p = T_2(x)p$. Then $g_p(x) = h_p(x)$ for almost all $x \in [0, \infty)$ and so $g_p = h_p$. Further, using the denseness of the domain of A it can be easily shown that $T_1(x)p = T_2(x)p$ for all $p \in S$ and $x \geq 0$ and so the corollary is established.

In conclusion two examples of semi-groups of nonlinear nonexpansive transformations will be given. The first example is due to J. W. Neuberger [9].

Example 1. Let $S = E_1$ and let T be defined as follows:

$$T(x)p = \begin{cases} \max \{p - x, 1\} & \text{if } p \geq 1, \quad x \geq 0, \\ p & \text{if } p < 1, \quad x \geq 0. \end{cases}$$

It is easily seen that T satisfies the hypothesis of Theorem 2. The nonexpansive extensions $\text{ext} (I - \epsilon A)^{-1}$, $\epsilon > 0$, obtained in Theorem 1 are given by $\text{ext} (I - \epsilon A)^{-1}p = \lim_{x \rightarrow 0+} (I - \epsilon A_x)^{-1}p$, where

$$\lim_{x \rightarrow 0+} (I - \epsilon A_x)^{-1}p = \begin{cases} \max \{p - \epsilon, 1\} & \text{if } p \geq 1, \\ p & \text{if } p < 1. \end{cases}$$

Further, Example 1 shows that T cannot be represented as a product integral in terms of $(I - \epsilon A)^{-1}$, $\epsilon > 0$, and the nonexpansive extensions $\text{ext} (I - \epsilon A)^{-1}$, $\epsilon > 0$, must be used.

The second example will show that if the Banach space S is not weakly complete then the infinitesimal generator may not be densely defined.

Example 2. Let $S = C_{[0,1]}$, i.e., S is the space of continuous real-valued functions on $[0, 1]$ with supremum norm. Let $F(x) = x$ if $x \geq 0$ and $F(x) = 2x$ if $x < 0$. Let T be defined as follows:

$$T(x)f = F[x + F^{-1}[f]] \quad \text{for } f \in C_{[0,1]} \quad \text{and } x \geq 0.$$

One easily sees that the domain of the infinitesimal generator of T is the set to which the member f of $C_{[0,1]}$ belongs only if $f \geq 0$ or $f < 0$. Thus, the infinitesimal generator is not densely defined. For Example 2 it may be shown that if $f \in C_{[0,1]}$ and $\epsilon > 0$ then $\lim_{x \rightarrow 0+} (I - \epsilon A_x)^{-1}f$ exists. Further, if for each $\epsilon > 0$ and $f \in C_{[0,1]}$ $L(\epsilon)f$ is defined to be $\lim_{x \rightarrow 0+} (I - \epsilon A_x)^{-1}f$, then $L(\epsilon)$ is a nonexpansive extension of $(I - \epsilon A)^{-1}$ and for each $f \in C_{[0,1]}$ and $\epsilon > 0$ $\lim_{n \rightarrow \infty} L^n(\epsilon/n)f = T(\epsilon)f$.

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