Almost-Periodic Solutions of a Certain Abstract Differential Equation

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In this note we will present a generalization together with a new proof of a result contained in [8]. Let $\mathfrak{X}$ be a real or complex Banach-space, $\mathcal{L}(\mathfrak{X}, \mathfrak{X})$ denotes the set of all linear continuous operators $T: \mathfrak{X} \rightarrow \mathfrak{X}$ with the usual uniform topology. If $\mathcal{G}$ is a topological group, a function $G: \mathcal{G} \rightarrow \mathcal{L}(\mathfrak{X}, \mathfrak{X})$ will be called a representation of $\mathcal{G}$ in $\mathfrak{X}$ if $G(e) = I =$ the identity transformation in $\mathfrak{X}$, and $G(s \cdot t) = G(s)G(t)$ for $s, t \in \mathcal{G}$; $e =$ unit of G . We shall consider here only the special case when $\mathcal{G} = J =$ the additive group of the real numbers; a representation of J , $G(t)$, $-\infty < t < \infty \rightarrow \mathcal{L}(\mathfrak{X}, \mathfrak{X})$ will be then called a one-parameter group. We call a set $A \subset \mathfrak{X}$ almost-periodic under $G(t)$ if

$G(t)x$ is a.p. from J to $\mathfrak{X}$, for every $x \in A$.

If $\mathfrak{X}^*$ is the dual space of $\mathfrak{X}$, a set $A \subset \mathfrak{X}$ is weakly almost-periodic under $G(t)$ if $\langle x^*, G(t)x \rangle$ is scalar almost-periodic, for each $x \in A$ and $x^* \in \mathfrak{X}^*$. $G(t)$ is called strongly a.p. if $\mathfrak{X}$ is a.p. under $G(t)$. An almost-periodic function is by definition continuous; it is well known that for continuous $h: \mathcal{G} \rightarrow M =$ metric space, Bochner's and Maak's definition coincide; if $\mathcal{G} = J$ they coincide also with Bohr's definition (Maak [6], p. 94/95 and 151–153; the proofs given there work also for functions with values in metric spaces).

Denote by A the infinitesimal operator associated with a one parameter strongly continuous group $G(t)$, with domain of definition $D(A)$ ([5]). If $f: J \rightarrow \mathfrak{X}$ is given, a function $u: J \rightarrow \mathfrak{X}$ will be called a solution of

$$(1) \quad u' = Au + f \quad \text{on } J$$

if $u(J) \subset D_A$, $u \in C^1(J; X)$, and (1) holds everywhere in J . Our main result is given in the following

Theorem. *Let $\mathfrak{X}$ be a real or complex uniformly convex Banach space; let $G(t)$ be a one-parameter strongly continuous group, such that*

$$\sup_{t \in J} \|G(t)\|_{\mathcal{L}(\mathfrak{X}, \mathfrak{X})} < \infty.$$

Let $f(t) \in C^1(J; \mathfrak{X})$ be Stepanoff a.p. ($S^p(J; \mathfrak{X})$, $p \geq 1$) and the set $f(J)$ be almost-periodic under $G(t)$; let also $u_0 \in D(A)$, u_0 almost-periodic under $G(t)$, and $u(t)$ be the (unique) solution of (1) with $u(0) = u_0$. If $\sup_{t \in J} \|u(t)\| < \infty$ then $u(t)$ is almost-periodic.

Proof of the theorem. Under the assumptions made on G and f the function v

$$(2) \quad v(t) = G(t)u_0 + \int_0^t G(t-s)f(s) ds$$

is well defined on J , $v(J) \subset D_A$, v is a solution (the only one by Hille-Phillips [5], p. 622) of (2) with $v(0) = u_0$. Hence $v(t) = u(t)$ and $\sup_{t \in J} \|v(t)\| < \infty$. By our assumption on u_0 , $v_1 : t \rightarrow G(t)u_0$ is a.p. Then $v_2 = v - v_1$ is bounded; we have to prove that it is almost-periodic. Since

$$v_2(t) = G(t) \int_0^t G(-s)f(s) ds$$

and G is uniformly bounded

$$v_3(t) = \int_0^t G(-s)f(s) ds$$

is bounded too. As $f(J)$ is a.p. under $G(t)$, it is obviously a.p. under the uniformly bounded one-parameter group G_* ; $G_*(t) = G(-t)$. We shall prove now in the subsequent lemmas, that, under our hypothesis, $G(-s)f(s)$ is again Stepanoff a.p. Then, in an uniformly convex space, bounded integrals of almost-periodic functions in Stepanoff's sense are continuous Bohr a.p. functions and this will completely prove the theorem (see Amerio [1], Vasconi [7]).

Lemma 1. Let $\mathfrak{X}$ be a real or complex normed space, $G(t)$ a one-parameter group of operators such that $\sup_{t \in J} \|G(t)\| < \infty$. Let also $f : J \rightarrow \mathfrak{X}$ be a (Bohr) a.p. function, such that $f(J)$ is almost-periodic under $G(t)$. Then, the function $G(t) \cdot f(t)$ is a.p. from J to $\mathfrak{X}$.

We shall use in the proof Maak's definition (equivalent with Bochner's) for almost-periodicity.

A continuous function $h(t)$, $t \in J$ to $\mathfrak{X}$ is a.p. if and only if for any $\epsilon > 0$ there is a partition $J = A_1 \cup \dots \cup A_m$ of sets in J such that $x, y \in A_i$ implies

$$\|h(x+t) - h(y+t)\| < \epsilon \quad \text{for } t \in J.$$

To prove the almost-periodicity of $G(t)f(t)$, let $\epsilon > 0$ be given. Since $f(t)$ is almost-periodic, its range $f(J)$ is relatively compact. There are henceforth finitely many $y_1 \dots y_k \in f(J)$, such that $f(J)$ is covered by the union of balls with center y_i and radius $\epsilon/4M$, $M = \sup_{t \in J} \|G(t)\| < \infty$. All functions $G(t)y_1, \dots, G(t)y_k$ are almost-periodic because we assumed $f(J)$ to be almost-periodic under $G(t)$. Using also the almost-periodicity of $f(t)$ we find, using Maak's definition of almost-periodicity, a partition $A_1 \dots A_n$ of J such that

the following is true: $\bigcup_{i=1}^n A_i = J$; moreover, if $\xi, \eta \in A_\mu$ and $t \in J$ arbitrary, then

$$||f(\xi + t) - f(\eta + t)|| < \epsilon/4M, \quad ||G(\xi + t)y_\nu - G(\eta + t)y_\nu|| < \epsilon/4, \\ \nu = 1, 2, \dots, k$$

(in fact from Maak's definition, such decomposition exists at first for each of the function $f(t)$, $G(t)y_1, \dots, G(t)y_k$, and using the common subpartition one can immediately find a partition which is good for all the functions involved simultaneously). Let now $\xi, \eta \in A_\mu$, μ fixed $= 1, 2, \dots, n$; t fixed $\in J$; then

$$\begin{aligned} G(\xi + t)f(\xi + t) - G(\eta + t)f(\eta + t) \\ = G(\xi + t)(f(\xi + t) - f(\eta + t)) + (G(\xi + t) - G(\eta + t))f(\eta + t) \\ = G(\xi + t)(f(\xi + t) - f(\eta + t)) + G(\xi + t)(f(\eta + t) - y_\rho) \\ - G(\eta + t)(f(\eta + t) - y_\rho) + (G(\xi + t) - G(\eta + t))y_\rho. \end{aligned}$$

Hence

$$\begin{aligned} ||G(\xi + t)f(\xi + t) - G(\eta + t)f(\eta + t)|| \\ \leq M(\epsilon/4M) + 2M ||f(\eta + t) - y_\rho|| + ||G(\xi + t)y_\rho - G(\eta + t)y_\rho||. \end{aligned}$$

As ξ, η, t are fixed, we find a ρ , $1 \leq \rho \leq n$, such that

$$||f(\eta + t) - y_\rho|| < \epsilon/4M.$$

Then we have

$$||G(\xi + t)f(\xi + t) - G(\eta + t)f(\eta + t)|| < \epsilon/4 + \epsilon/2 + \epsilon/4 = \epsilon$$

when $\xi, \eta \in A_\mu$, $t \in J$. The continuity of f following similarly, Lemma 1 is proved.

Lemma 2. Let $\mathfrak{X}$ be a B -space, $G(t)$ a strongly a.p. one parameter group in $\mathfrak{X}$ and $f: J \rightarrow \mathfrak{X}$ a Stepanoff S^p a.p. function, $p \geq 1$. Then $G(t)f(t)$ is again S^p a.p.

Let us remember that " f a.p. in $S^p(J; \mathfrak{X})$ sense" means that $f(t) \in L^p_{\text{loc}}(J; \mathfrak{X})$ and that the function $f_0: J \rightarrow L^p([0, 1]; \mathfrak{X})$ defined by

$$(f_0(t))(s) = f(t + s), \quad 0 \leq s \leq 1,$$

is Bohr almost-periodic in t (see for example Bochner [3]). This means precisely that to each $\epsilon > 0$ there is a relatively dense set of numbers τ such that

$$\sup_{-\infty < t < \infty} ||f_0(t + \tau) - f_0(t)||_{L^p([0, 1]; \mathfrak{X})} < \epsilon$$

or

$$\sup_{-\infty < t < \infty} \int_0^1 ||f(t + s + \tau) - f(t + s)||_{\mathfrak{X}}^p ds < \epsilon^p$$

or

$$\sup_{-\infty < t < \infty} \int_t^{t+1} ||f(\sigma + \tau) - f(\sigma)||_{\mathfrak{X}}^p d\sigma < \epsilon^p.$$

Let us consider now, for any $h > 0$,

$$f_h(t) = \frac{1}{h} \int_0^h f(t + \sigma) d\sigma \quad (\text{Steklov's function}).$$

As f is $S^p(\mathfrak{X})$ a.p., f is also $S^1(J; \mathfrak{X})$ a.p.; hence it follows easily that $f_h(t)$ is a.p. Bohr for each fixed $h > 0$ and it can be proved as for scalar-valued functions (see Besicovitch [2], p. 80–81) that $f_h \rightarrow f$ as $h \rightarrow 0$ in the S^p sense, that is

$$\sup_{t \in J} \left(\int_t^{t+1} \|f(\sigma) - f_h(\sigma)\|^p d\sigma \right)^{1/p} \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

Then we have $G(t)f(t) - G(t)f_h(t) = G(t)(f(t) - f_h(t))$, $G(t)f(t) \in L_{10}^p(J; \mathfrak{X})$, and

$$\sup_{t \in J} \left(\int_t^{t+1} \|G(\sigma)f(\sigma) - G(\sigma)f_h(\sigma)\|_{\mathfrak{X}}^p d\sigma \right)^{1/p} \\ \leq M \sup_{t \in J} \left(\int_t^{t+1} \|f(\sigma) - f_h(\sigma)\|_{\mathfrak{X}}^p d\sigma \right)^{1/p} \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

Since by the previous Lemma the $G(t)f_h(t)$ are a.p. Bohr, it follows that $G(t)f(t)$ is a.p. $S^p(J; \mathfrak{X})$.

The case when $G(t)$ is not necessarily strongly a.p. but $f(J)$ is a.p. under $G(t)$ can be reduced to the above situation by introducing a suitable subspace $Y \subset \mathfrak{X}$.

Finally, bounded integrals of S^p a.p. functions with values in a uniformly convex space are a.p. Bohr by the Love-Amerio theorem (cfr. Vasconi [7]), Amerio [1]).

Let us finally mention some possible generalizations: In the theorem it is enough if u_0 and $f(J)$ are only weakly a.p.; instead of uniformly convex one can consider also for example substitution spaces $l^p(I, \mathfrak{X}_i)$ with $l \leq p < \infty$, $\mathfrak{X}_i$ uniformly convex; Lemma 1 holds for arbitrary semigroups of operators $\subset \mathcal{L}(\mathfrak{X}, \mathfrak{X})$, the boundness assumption on u can be relaxed a little; if $G(t)$ is a.p., then $\sup_{t \in J} \|G(t)\| < \infty$ automatically by the uniform boundness theorem.

All this and another proof of Lemma 2 can be found in [4].

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