*On the Topological Entropy of Solenoids*¹

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In this note we prove a theorem which relates the entropy of a shift on an inverse limit space to the entropy of the map used in the construction of the inverse limit which is analogous to a theorem for measure theoretic entropy [3]. Using this theorem we then compute the topological entropy of the shift on a p -adic solenoid.

I. The main theorem. In the following an open cover $\mathcal{U}$ of a topological space X will be called minimal if no sub cover of $\mathcal{U}$ covers X and a sub cover $\mathcal{U}'$ of an open cover $\mathcal{U}$ of X will be called a minimal sub cover if $\mathcal{U}'$ is a minimal cover of X and no sub cover of $\mathcal{U}$ contains fewer sets than $\mathcal{U}'$.

Throughout, X will denote a compact topological space and φ a continuous map of X onto X . If $\mathcal{U}$ is an open cover of X , $N(\mathcal{U})$ will denote the number of sets in a minimal sub cover of $\mathcal{U}$. We will denote the topological entropy of φ with respect to $\mathcal{U}$ by $h(\varphi, \mathcal{U})$, and $h(\varphi)$ will denote the topological entropy of φ . For the definitions of these objects and their properties one may consult [1].

By X_∞ we shall denote the subspace of the product space of countably many copies of X with the product topology which consists of all sequences $(x_1, x_2, \dots)$ such that $\varphi(x_n) = x_{n-1}$ for $n \geq 2$. The projection maps of X_∞ onto X will be denoted by π_k for $k \geq 1$ and φ_* will denote the shift on X_∞ , i.e. $\pi_k \circ \varphi_* = \pi_{k+1}$. It is known [2, Chapter VIII] that X_∞ is a compact topological space, φ_* is a homeomorphism of X_∞ and π_k are continuous surjections. Further, note that

$$(A) \quad \varphi \circ \pi_k = \pi_k \circ \varphi_*^{-1}$$

and

$$(B) \quad \varphi \circ \pi_{k+1} = \pi_k$$

for all $k \geq 1$.

1. Lemma. Let $\{U_i : i \in I\}$ be any collection of subsets of X , and k a positive integer. Then $X = \bigcup_i U_i$ if and only if $X_\infty = \bigcup_i \pi_k^{-1}(U_i)$.

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Proof. If there exists $x_* \in X_\infty - \bigcup_I \pi_k^{-1}(U_i)$ then $\pi_k x_* \in X - \bigcup_I U_i$.

Conversely, suppose there exists $x \in X - \bigcup_I U_i$. Let $x_j = \varphi^{k-j}(x)$ for $j = 1, 2, \dots, k$. Since φ is onto, there exists x_{k+1} such that $\varphi(x_{k+1}) = x$. Assuming x_{k+n} is defined there is x_{k+n+1} such that $\varphi(x_{k+n+1}) = x_{k+n}$ and by induction we have defined the sequence $(x_1, x_2, \dots, x_k, \dots) = x_*$ which is in X . Since $\pi_k x_* = x$, $x_* \notin \bigcup_I \pi_k^{-1}(U_i)$. Thus $x_* \in X_\infty - \bigcup_I \pi_k^{-1}(U_i)$.

2. Corollary. *The collection $\mathfrak{U}$ of open subsets of X is a minimal cover if X if and only if $\pi_k^{-1}(\mathfrak{U})$ is a minimal open cover of X_∞ .*

3. Corollary. *If $\mathfrak{U}$ is an open cover of X then for any $k \geq 1$, $N(\mathfrak{U}) = N(\pi_k^{-1}\mathfrak{U})$.*

Proof. If $\{U_i : 1 \leq j \leq N(\mathfrak{U})\}$ is a finite minimal sub cover of $\mathfrak{U}$ then by 2, $\{\pi_k^{-1}(U_i) : 1 \leq j \leq N(\mathfrak{U})\}$ is a minimal cover of X_∞ and hence $N(\mathfrak{U}) \geq N(\pi_k^{-1}(\mathfrak{U}))$. By using the same argument with $\{\pi_k^{-1}(U_i) : 1 \leq j \leq N(\pi_k^{-1}(\mathfrak{U}))\}$ a finite minimal sub cover of $\pi_k^{-1}(\mathfrak{U})$ one gets the reverse inequality.

4. Lemma. *Given any open cover $\mathfrak{U}$ of X , $h(\varphi, \mathfrak{U}) = h(\varphi_*^{-1}, \pi_k^{-1}(\mathfrak{U}))$.*

Proof. It follows from (A) that $\varphi_*^p(\pi_k^{-1}\mathfrak{U}) = \pi_k^{-1}(\varphi^{-p}\mathfrak{U})$ for all $p \geq 0$. Thus

$$V_{p=0}^{n-1} \varphi_*^p(\pi_k^{-1}\mathfrak{U}) = \pi_k^{-1}(V_{p=0}^{n-1} \varphi^{-p}\mathfrak{U}),$$

and by 3 and the definition of the entropy H of a cover we have

$$\frac{1}{n} H(V_{p=0}^{n-1} \varphi_*^p(\pi_k^{-1}\mathfrak{U})) = \frac{1}{n} H(V_{p=0}^{n-1} \varphi^{-p}\mathfrak{U}).$$

Taking the limit as $n \rightarrow \infty$ gives the desired result.

5. Theorem. $h(\varphi_*^{-1}) = h(\varphi)$.

Proof. First $h(\varphi) \leq h(\varphi_*^{-1})$ follows either from Theorem 5 of [1] or from the equation

$$h(\varphi) = \sup \{h(\varphi_*^{-1}, \pi_1^{-1}(\mathfrak{U})) : \mathfrak{U} \text{ open cover of } X\}.$$

To obtain the reverse inequality suppose $\mathfrak{B}^*$ is any open cover of X_∞ . Since sets of the form $\pi_k^{-1}(U)$ where k ranges over the positive integers and U ranges over the open sets in X form a base for the topology in X_∞ , (see [2] Lemma 3.12) there is a cover $\mathfrak{B}$ of X_∞ such that $\mathfrak{B}$ refines $\mathfrak{B}^*$, i.e., every set in $\mathfrak{B}$ is a subset of some set from $\mathfrak{B}^*$, and $\mathfrak{B}$ consists of sets of the form $\pi_k^{-1}(U)$.

Let $\mathfrak{B}'_m = \{\pi_{i_1}^{-1}(U_1), \dots, \pi_{i_m}^{-1}(U_m)\}$ be a finite minimal sub cover of $\mathfrak{B}$ where $1 \leq i_1 \leq i_2 \leq \dots \leq i_m$. From (B), we have $\varphi^j \circ \pi_{k+j} = \pi_k$ for all $k \geq 1$ and all $j \geq 0$. Thus $\varphi^{i_m-i_j} \circ \pi_{i_j} = \pi_{i_m}$ for $j = 1, 2, \dots, m$, which implies that

$$\pi_{i_j}^{-1}(U_j) = \pi_{i_m}^{-1}(\varphi^{-i_m+i_j}(U_j)) \quad \text{for } 1 \leq j \leq m.$$

Thus if $\mathfrak{U} = \{\varphi^{-i_m+i_j}(U_j) : 1 \leq j \leq m\}$ we have $\mathfrak{B}_m = \pi_{i_m}^{-1}(\mathfrak{U})$. By 1, $\mathfrak{U}$ is an open cover of X and by 4, $h(\varphi_*^{-1}, \mathfrak{B}_m) = h(\varphi, \mathfrak{U})$. Since $\mathfrak{B}_m$ refines $\mathfrak{B}$ and $\mathfrak{B}$ refines $\mathfrak{B}^*$ it follows that $h(\varphi_*^{-1}, \mathfrak{B}_m) \geq h(\varphi_*^{-1}, \mathfrak{B}^*)$. Thus $h(\varphi_*^{-1}, \mathfrak{B}^*) \leq h(\varphi)$ and since $\mathfrak{B}^*$ was an arbitrary cover of X_∞ , $h(\varphi_*^{-1}) \leq h(\varphi)$.

II. The entropy of the solenoid. Let S denote the circle group and p be any positive integer. Let φ denote the map of S onto S defined by $\varphi(z) = z^p$. The p -adic solenoid is the inverse limit space S_∞ and if φ_* denotes the shift on S_∞ we have from Theorem 5 that φ and φ_* have the same topological entropy.

Theorem. $h(\varphi_*) = \log p$.

Proof. If I is any interval in S then $\varphi^{-1}(I)$ consists of p disjoint intervals each of whose lengths is p^{-1} times the length of I and whose end points are the p th roots of the end points of I . Notice also that if $k > 2p$ then any interval in S of length less than or equal to $1/k$ can contain at most one p th root of any point in S since the distance between any two p th roots of the same number is greater than or equal to $1/p$.

Now, let k be a fixed integer such that $k > 2p$. For m any integer let $\mathcal{Q}_m$ denote the open cover of S which consists of all open intervals in S of length less than or equal to m^{-1} .

We now show that $V_{i=0}^{n-1} \varphi^{-i} \mathcal{Q}_k = \mathcal{Q}_{kp^{n-1}}$. Let E be any set in $\mathcal{Q}_k \vee \varphi^{-1}(\mathcal{Q}_k)$. Then $E = A \cap B$ where A is an interval of length less than or equal to k^{-1} and B is the inverse image under φ of such a set, say $B = \varphi^{-1}(J)$. Thus $B = \bigcup_{i=0}^{p-1} B_i$ where each B_i is an open interval of length less than or equal to $(kp)^{-1}$ and the B_i are disjoint. Hence $E = \bigcup_{i=0}^{p-1} A \cap B_i$. If more than two of the intervals $A \cap B_i$ are non-empty then two p th roots of one of the end points of J must be contained in A which is impossible since the length of A does not exceed k^{-1} . For the same reason if two of the intervals are non-empty they must be of the form $A \cap B_i$ and $A \cap B_{i+1}$. (We are assuming that the labeling on B_i are such that B_i and B_{i+1} correspond to consecutive p th roots of the end points of J .) Let a_i and b_i denote the end points of B_i . Suppose $j = 0, 1, \dots, p-2$ and $A \cap B_j$ and $A \cap B_{j+1}$ are non-empty. Then b_j and a_{j+1} are contained in A making the distance between b_j and a_{j+1} less than or equal to k^{-1} . Since the distance between a_j and b_j is less than or equal to $(kp)^{-1}$ it follows that the distance between a_j and a_{j+1} does not exceed $k^{-1} + (kp)^{-1} = (p+1)(kp)^{-1} < p^{-1}$. This is a contradiction since a_j and a_{j+1} are consecutive p th roots of an end point of J . If $A \cap B_{p-1}$ and $A \cap B_0$ are both non-empty then A contains b_{p-1} and a_0 and a similar argument leads to a contradiction. Thus E is a single interval of the form $A \cap B_i$ and has length not exceeding $(kp)^{-1}$ so that $\mathcal{Q}_k \vee \varphi^{-1}(\mathcal{Q}_k) \subset \mathcal{Q}_{kp}$.

Next suppose that $J \in \mathcal{Q}_{kp}$ and denote the end points of J by a and b . Then $J \subset \varphi^{-1}(I)$ where I is an interval with end points a^p and b^p so that $J = J \cap \varphi^{-1}(I)$. But the length of I does not exceed k^{-1} and hence $I \in \mathcal{Q}_k$. But $\mathcal{Q}_{kp} \subset \mathcal{Q}_k$ so that $J \in \mathcal{Q}_k$. Thus $J \in \mathcal{Q}_k \vee \varphi^{-1}(\mathcal{Q}_k)$, so that $\mathcal{Q}_{kp} \subset \mathcal{Q}_k \vee \varphi^{-1}(\mathcal{Q}_k)$. From the previous paragraph we have $\mathcal{Q}_{kp} = \mathcal{Q}_k \vee \varphi^{-1}(\mathcal{Q}_k)$.

By the above argument and the equation

$$V_{i=0}^{n-1} \varphi^{-i}(\mathcal{Q}_k) = (V_{i=0}^{n-2} \varphi^{-i}(\mathcal{Q}_k)) \vee \varphi^{-1}(V_{i=0}^{n-2} \varphi^{-i}(\mathcal{Q}_k)),$$

an induction will give that

$$V_{i=0}^{n-1} \varphi^{-i} \mathcal{Q}_k = \mathcal{Q}_{kp^{n-1}} \quad \text{for all } n.$$

To compute $h(\varphi)$ it is now only necessary to note that $\{\mathcal{Q}_k : k > 2p\}$ is a refining sequence of covers so that by property 12 [1], we have

$$(6.1) \quad h(\varphi) = \lim_{k \rightarrow \infty} h(\varphi, \mathcal{Q}_k).$$

Now,

$$\begin{aligned} h(\varphi, \mathcal{Q}_k) &= \lim_{n \rightarrow \infty} 1/n H(V_{i=0}^{n-1} \varphi^{-i} \mathcal{Q}_k) \\ &= \lim_{n \rightarrow \infty} 1/n \log N(\mathcal{Q}_{kp^{n-1}}) \\ &= \lim_{n \rightarrow \infty} 1/n \log (kp^{n-1} + 1) \\ &= \log p, \end{aligned}$$

and equation 6.1 gives the desired result.

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