The Use of Eigenfunction Expansions in the General Solution of Three-Dimensional Crack Problems¹

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Introduction. Recent interest in the exploration of the stress distribution in bodies containing flaws or cracks can be evidenced by the sizable volume of literature that has built up. Most of the work on the subject has been based on the plane theory of elasticity. It is to be expected, however, that in many instances the deviations from the two-dimensional theory, due to the three-dimensional character of the stress distribution near the crack boundary, can be of importance.

Methods for the solution of linear elasticity equations are generally dictated by the topology of the region under consideration, and are developed for their intended purposes. For example, although the symbolic method of Lur'e [1] and the Fourier-Bessel expansion technique of Green [2] are adequate for handling non-singular thick plate problems, they are not suitable for solving three-dimensional problems with geometrically-induced singularities. Because of the lack of a systematic way of treating crack problems in three dimensions, only a few effective solutions are available and these have been restricted to a particular crack geometry under simple types of loading. A non-trivial crack geometry, which has received some attention in the past, is that of an ellipse as a surface of discontinuity, the penny-shaped crack problem of Sneddon [3] being a special case. Based on known properties of the classical potential functions, Green and Sneddon [4] discussed the problem of an elliptical crack whose faces are loaded normally, while Kassir and Sih [5] solved the same problem with uniform shear loads. The complementary problem of a flat crack covering the outside of an ellipse was also investigated by Kassir and Sih [6].

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The three-dimensional crack problem to be considered in this paper is, in many ways, motivated by the work of Williams [7-9] and Sih *et al* [10-12]. It was Williams [7] who first conceived the idea of eigenfunction expansions and employed it to analyze semi-infinite crack and wedge problems in two variables. He established a series solution for the in-plane stresses

$$(1) \quad \begin{aligned} \sigma_r &= \sum_{n=0}^{\infty} r^{\lambda_n-1} \left[(\lambda_n + 1) F_n + \frac{d^2 F_n}{d\theta^2} \right], \\ \sigma_\theta &= \sum_{n=0}^{\infty} r^{\lambda_n-1} \lambda_n (\lambda_n + 1) F_n, \\ \tau_{r\theta} &= - \sum_{n=0}^{\infty} r^{\lambda_n-1} \lambda_n \frac{dF_n}{d\theta}, \end{aligned}$$

where

$$(2) \quad F_n(\theta) = a_n \sin(\lambda_n + 1)\theta + b_n \cos(\lambda_n + 1)\theta + c_n \sin(\lambda_n - 1)\theta + d_n \cos(\lambda_n - 1)\theta,$$

and the eigenvalues λ_n are obtainable from the stress and/or displacement conditions specified on the edges of the crack or wedge. The constant coefficients a_n , b_n , etc. in Williams' analysis, however, were not found until Sih and Rice [10, 11] recognized that once the general form of the solution is known in r and θ , the constants in equation (2) can be determined from the Boussinesq's solution in two dimensions. In this way, a number of dissimilar (or similar) media problems involving concentrated forces applied to the surfaces of semi-infinite and finite cracks were solved. In fact, Loeber and Sih [12] have established the equivalence between the method of eigenfunction expansions and the well-known Riemann-Hilbert formulation as presented by Muskhelishvili.

It is the central purpose of this paper to incorporate a third dimension into the eigenfunction expansion method and to provide a general solution of the three-dimensional stress and displacement fields for the case of a plane crack embedded in an elastic solid. Unlike the planar problem, where homogeneous relations were obtained between the independent constants in equation (2), the three-dimensional analysis leads to recurrence relations between the coefficients, which are no longer constants, but functions of the third variable. Two individual sets of results are considered. The first set contains integer powers, say n , of r and the second set contains $n + \frac{1}{2}$ powers of r , where r is the radial distance measured from the leading edge of the crack. These results may be used to answer some of the pressing questions concerning the three-dimensional character of the stress distribution in the vicinity of the crack front.

Preliminary considerations. Since the chief interest in the solution to the crack problem is centered on the disturbances local to the leading edge of the crack, it is essential to select a coordinate system that will exhibit the singular

character of the stresses in a natural manner. To this end, the crack problem will be formulated in circular cylindrical coordinates (r, θ, z) , related to the Cartesian system according to

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z.$$

The elastic solid is assumed to be infinitely extended in the x -, y - and z -directions, and contains a crack in the form of a half-plane with the z -axis along the crack edge. The sides of the crack coincide with the surfaces $\theta = \pm\pi$, where

$$0 \leq r < \infty, \quad -\pi \leq \theta \leq \pi, \quad -\infty < z < \infty.$$

Within the framework of the linear theory of elasticity, the equations of equilibrium in cylindrical coordinates must be solved for the displacement components (u_r, v_θ, w_z) :

$$(3) \quad \begin{aligned} \frac{\partial \vartheta}{\partial r} + (1 - 2\nu) \left[\left(\nabla^2 - \frac{1}{r^2} \right) u_r - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right] &= 0, \\ \frac{1}{r} \frac{\partial \vartheta}{\partial \theta} + (1 - 2\nu) \left[\left(\nabla^2 - \frac{1}{r^2} \right) v_\theta + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right] &= 0, \\ \frac{\partial \vartheta}{\partial z} + (1 - 2\nu) \nabla^2 w_z &= 0. \end{aligned}$$

In equations (3), ν is Poisson's ratio, ϑ is the dilatation given by

$$\vartheta = \frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{u_r}{r} + \frac{\partial w_z}{\partial z},$$

and the Laplace operator is defined as

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

The boundary conditions will be written in terms of the stresses, which are related to the displacements through the Hooke's law for an isotropic body:

$$(4) \quad \begin{aligned} \sigma_r &= \lambda \vartheta + 2\mu \frac{\partial u_r}{\partial r}, & \tau_{r\theta} &= \mu \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} \right), \\ \sigma_\theta &= \lambda \vartheta + 2\mu \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{u_r}{r} \right), & \tau_{\theta z} &= \mu \left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r} \frac{\partial w_z}{\partial \theta} \right), \\ \sigma_z &= \lambda \vartheta + 2\mu \frac{\partial w_z}{\partial z}, & \tau_{zr} &= \mu \left(\frac{\partial w_z}{\partial r} + \frac{\partial u_r}{\partial z} \right). \end{aligned}$$

The Lamé's coefficients are denoted by λ and μ .

Construction of series solution. For the solution of the half-plane crack problem, it turns out to be very convenient to expand the displacement vector in the double series

$$(5) \quad 2\mu u = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} r^{\lambda_m+n} \mathbf{U}_n^{(m)}(\theta, z; \lambda_m),$$

The eigenvalues λ_m ($m = 0, 1, 2, \dots$) as powers of r are assumed to be constants, and the components of $\mathbf{U}_n^{(m)}$ ($U_n^{(m)}, V_n^{(m)}, W_n^{(m)}$) are functions of θ and z only. From equations (4), the corresponding stress components follow immediately. They are

$$(6) \quad \begin{aligned} (1 - 2\nu)\sigma_r &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} r^{\lambda_m+n-1} \\ &\quad \cdot \left\{ [(1 - \nu)(\lambda_m + n) + \nu] U_n^{(m)} + \nu \left[\frac{\partial V_n^{(m)}}{\partial \theta} + \frac{\partial W_{n-1}^{(m)}}{\partial z} \right] \right\}, \\ (1 - 2\nu)\sigma_{\theta} &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} r^{\lambda_m+n-1} \\ &\quad \cdot \left\{ [1 + \nu(\lambda_m + n - 1)] U_n^{(m)} + (1 - \nu) \frac{\partial V_n^{(m)}}{\partial \theta} + \nu \frac{\partial W_{n-1}^{(m)}}{\partial z} \right\}, \end{aligned}$$

and similar expressions for the other stresses. In eqs. (6), $U_n^{(m)}, V_n^{(m)}$, and $W_n^{(m)}$ are zero for $n < 0$. The required properties of differentiability have been presumed throughout the foregoing computations.

In order to gain some knowledge of the structure of the solution in θ and z , equations (5) are inserted into the displacement equations of equilibrium given by equations (3). This yields a system of three simultaneous partial differential equations in three unknown functions as

$$(7) \quad \begin{aligned} (1 - 2\nu) \frac{\partial^2 U_n^{(m)}}{\partial \theta^2} \\ + 2(1 - \nu)[(\lambda_m + n)^2 - 1] U_n^{(m)} + (\lambda_m + n - 3 + 4\nu) \frac{\partial V_n^{(m)}}{\partial \theta} \\ = -(\lambda_m + n - 1) \frac{\partial W_{n-1}^{(m)}}{\partial z} - (1 - 2\nu) \frac{\partial^2 U_{n-2}^{(m)}}{\partial z^2} \end{aligned}$$

and two similar equations in which $m \geq 0$ and $n \geq 0$. Equations (7) may also be regarded as recurrence relations expressing $U_n^{(m)}, V_n^{(m)}$ and $W_n^{(m)}$ in terms of their previous values. It is clear that for each value of n , a product solution in θ and z can be found from equations (7). For $n = 0$, the solution takes the form

$$(8) \quad \begin{aligned} U_0^{(m)} &= B_1^{(m)}(z) \cos(\lambda_m + 1)\theta + B_2^{(m)}(z) \sin(\lambda_m + 1)\theta \\ &\quad + C_1^{(m)}(z) \cos(\lambda_m - 1)\theta + C_2^{(m)}(z) \sin(\lambda_m - 1)\theta, \\ V_0^{(m)} &= -B_1^{(m)}(z) \sin(\lambda_m + 1)\theta + B_2^{(m)}(z) \cos(\lambda_m + 1)\theta \\ &\quad + \frac{\lambda_m + 3 - 4\nu}{\lambda_m - 3 + 4\nu} [-C_1^{(m)}(z) \sin(\lambda_m - 1)\theta + C_2^{(m)}(z) \cos(\lambda_m - 1)\theta], \\ W_0^{(m)} &= A_1^{(m)}(z) \cos \lambda_m \theta + A_2^{(m)}(z) \sin \lambda_m \theta \end{aligned}$$

in which $A_j^{(m)}(z)$, $B_j^{(m)}(z)$ and $C_j^{(m)}(z)$ ($j = 1, 2$) are arbitrary functions of z . Similarly, solutions corresponding to the remaining values of n may be written down. These details will be deferred.

For further development of the solution, it is crucial to apply the free crack surface conditions

$$(9) \quad \sigma_\theta = \tau_{\theta r} = \tau_{\theta z} = 0 \quad \text{for} \quad \theta = \pm\pi$$

so as to evaluate the eigenvalues λ_m . Without loss in generality, λ_m will be determined from the solution of equations (7) for $n = 0$, *i.e.*, equations (8). Making use of equations (6), (8) and (9), there results in six conditions

$$\begin{aligned} B_1^{(m)}(z) \cos(\lambda_m + 1)\pi \pm B_2^{(m)}(z) \sin(\lambda_m + 1)\pi \\ + \frac{\lambda_m + 1}{\lambda_m - 3 + 4\nu} [C_1^{(m)}(z) \cos(\lambda_m - 1)\pi \pm C_2^{(m)}(z) \sin(\lambda_m - 1)\pi] = 0, \\ B_1^{(m)}(z) \sin(\lambda_m + 1)\pi \pm B_2^{(m)}(z) \cos(\lambda_m + 1)\pi \\ + \frac{\lambda_m - 1}{\lambda_m - 3 + 4\nu} [C_1^{(m)}(z) \sin(\lambda_m - 1)\pi \pm C_2^{(m)}(z) \cos(\lambda_m - 1)\pi] = 0, \\ (10) \quad A_1^{(m)}(z) \sin \lambda_m \pi \pm A_2^{(m)}(z) \cos \lambda_m \pi = 0, \end{aligned}$$

to solve for the six unknown functions $A_j^{(m)}(z)$, $B_j^{(m)}(z)$ and $C_j^{(m)}(z)$ ($j = 1, 2$). For a non-trivial solution, the determinant of the coefficients of these functions must vanish, and hence λ_m are found to be the roots of the characteristic-value equation

$$\sin 2\pi\lambda_m = 0$$

which renders

$$(11) \quad \lambda_m = \frac{m}{2}, \quad m = 0, 1, 2, \dots$$

The negative values of m have been excluded in equation (11) so that the boundedness conditions of the displacements are not violated as $r \rightarrow 0$.

By virtue of equation (11), the double-series representation of each component of the displacement vector in equation (5) can be reduced to a single power series in r , *i.e.*,

$$\begin{aligned} 2\mu u_r &= \sum_{n=0}^{\infty} r^{n/2} f_n(\theta, z), \\ (12) \quad 2\mu v_\theta &= \sum_{n=0}^{\infty} r^{n/2} g_n(\theta, z), \\ 2\mu w_z &= \sum_{n=0}^{\infty} r^{n/2} h_n(\theta, z). \end{aligned}$$

(Depending on the nature of the eigenvalues λ_m , the double-series representation

of the displacement vector must, in general, be retained. This point will become evident in the analysis of the wedge problem later on.) For the same reason, the expressions for the stresses in equations (6) may be simplified and they become

$$\begin{aligned}
 (1 - 2\nu)\sigma_r &= \sum_{n=0}^{\infty} r^{n/2-1} \left\{ \left[\frac{n}{2} - \left(\frac{n}{2} - 1 \right) \nu \right] f_n + \nu \left[\frac{\partial g_n}{\partial \theta} + \frac{\partial h_{n-2}}{\partial z} \right] \right\}, \\
 (1 - 2\nu)\sigma_{\theta} &= \sum_{n=0}^{\infty} r^{n/2-1} \left\{ \left[1 + \left(\frac{n}{2} - 1 \right) \nu \right] f_n + (1 - \nu) \frac{\partial g_n}{\partial \theta} + \nu \frac{\partial h_{n-2}}{\partial z} \right\} \\
 (13) \quad (1 - 2\nu)\sigma_z &= \sum_{n=0}^{\infty} r^{n/2-1} \left\{ \nu \left[\left(\frac{n}{2} + 1 \right) f_n + \frac{\partial g_n}{\partial \theta} \right] + (1 - \nu) \frac{\partial h_{n-2}}{\partial z} \right\}, \\
 2\tau_{r\theta} &= \sum_{n=0}^{\infty} r^{n/2-1} \left[\frac{\partial f_n}{\partial \theta} + \left(\frac{n}{2} - 1 \right) g_n \right], \\
 2\tau_{\theta z} &= \sum_{n=0}^{\infty} r^{n/2-1} \left(\frac{\partial h_n}{\partial \theta} + \frac{\partial g_{n-2}}{\partial z} \right), \\
 2\tau_{zr} &= \sum_{n=0}^{\infty} r^{n/2-1} \left(\frac{n}{2} h_n + \frac{\partial f_{n-2}}{\partial z} \right).
 \end{aligned}$$

The basic unknowns f_n , g_n and h_n on the knowledge of which depends the solution of the crack problem must now be found from the linear equations in the classical theory of elasticity. This is the objective of the next section.

Determination of displacement functions. Consider now the substitution of equations (12) into the three equations of equilibrium in terms of displacements. Assuming that the resulting expressions will hold for arbitrary values of r , it is found that

$$\begin{aligned}
 (1 - 2\nu) \frac{\partial^2 f_n}{\partial \theta^2} + 2(1 - \nu) \left(\frac{n^2}{4} - 1 \right) f_n + \left(\frac{n}{2} - 3 + 4\nu \right) \frac{\partial g_n}{\partial \theta} \\
 &= - \left(\frac{n}{2} - 1 \right) \frac{\partial h_{n-2}}{\partial z} - (1 - 2\nu) \frac{\partial^2 f_{n-4}}{\partial z^2}, \\
 2(1 - \nu) \frac{\partial^2 g_n}{\partial \theta^2} + (1 - 2\nu) \left(\frac{n^2}{4} - 1 \right) g_n + \left(\frac{n}{2} + 3 - 4\nu \right) \frac{\partial f_n}{\partial \theta} \\
 (14) \quad &= - \frac{\partial^2 h_{n-2}}{\partial \theta \partial z} - (1 - 2\nu) \frac{\partial^2 g_{n-4}}{\partial z^2}, \\
 (1 - 2\nu) \left(\frac{\partial^2 h_n}{\partial \theta^2} + \frac{n^2}{4} h_n \right) &= - \frac{n}{2} \frac{\partial f_{n-2}}{\partial z} - \frac{\partial^2 g_{n-2}}{\partial \theta \partial z} - 2(1 - \nu) \frac{\partial^2 h_{n-4}}{\partial z^2}.
 \end{aligned}$$

This system of three partial differential equations is similar to that of equations (7) and can be solved for the functions f_n , g_n and h_n ($n = 0, 1, 2, \dots$). The solution for $n = 0$ is

$$\begin{aligned}
f_0 &= {}^{(1)}B_0 \cos \theta + {}^{(2)}B_0 \sin \theta \\
&\quad + (3 - 4\nu)[{}^{(1)}C_0 \theta \sin \theta - {}^{(2)}C_0 \theta \cos \theta], \\
(15) \quad g_0 &= -[{}^{(1)}B_0 + {}^{(1)}C_0] \sin \theta + [{}^{(2)}B_0 + {}^{(2)}C_0] \cos \theta \\
&\quad + (3 - 4\nu)[{}^{(1)}C_0 \theta \cos \theta + {}^{(2)}C_0 \theta \sin \theta], \\
h_0 &= {}^{(1)}A_0 + {}^{(2)}A_0 \theta,
\end{aligned}$$

while for $n = 2$, it takes the form

$$\begin{aligned}
(16) \quad f_2 &= {}^{(1)}B_2 \cos 2\theta + {}^{(2)}B_2 \sin 2\theta + {}^{(1)}C_2 - \frac{1}{4(1 - \nu)} {}^{(2)}A'_0 \theta, \\
g_2 &= -{}^{(1)}B_2 \sin 2\theta + {}^{(2)}B_2 \cos 2\theta + {}^{(2)}C_2, \\
h_2 &= {}^{(1)}A_2 \cos \theta + {}^{(2)}A_2 \sin \theta - {}^{(1)}C'_0 \theta \sin \theta + {}^{(2)}C'_0 \theta \cos \theta.
\end{aligned}$$

In the sequel, the quantities such as ${}^{(i)}A_{4m}$, ${}^{(i)}B_{4m}$, etc. for $i = 1, 2$ are to be understood as arbitrary functions of z , and primes (e.g. ${}^{(i)}A'_{4m}$) will denote differentiation with respect to z .

At this point, it is convenient to separate the unknown functions into two groups distinguished by subscripts being even and odd integers as follows:

$$(A) \quad f_{4m}, f_{4m+2}, \quad \text{for } m = 1, 2, \dots$$

$$(B) \quad f_{4m+1}, f_{4m+3}, \quad \text{for } m = 0, 1, 2, \dots$$

Each group will be separated into two classes as shown above.

The solution for the first class of group (A) for $m \geq 1$ can be written out in the form

$$\begin{aligned}
(17) \quad f_{4m} &= {}^{(1)}B_{4m} \cos (2m + 1)\theta + {}^{(2)}B_{4m} \sin (2m + 1)\theta \\
&\quad + (2m - 3 + 4\nu)[{}^{(1)}C_{4m} \cos (2m - 1)\theta + {}^{(2)}C_{4m} \sin (2m - 1)\theta] \\
&\quad - {}^{(1)}\alpha_{4m} \theta \sin \theta + {}^{(2)}\alpha_{4m} \theta \cos \theta \\
&\quad + \sum_{k=0}^{m-1} [{}^{(1)}\alpha_{4m} \cos (2k + 1)\theta + {}^{(2)}\alpha_{4m} \sin (2k + 1)\theta], \\
g_{4m} &= -{}^{(1)}B_{4m} \sin (2m + 1)\theta + {}^{(2)}B_{4m} \cos (2m + 1)\theta \\
&\quad + (2m + 3 - 4\nu)[-{}^{(1)}C_{4m} \sin (2m - 1)\theta + {}^{(2)}C_{4m} \cos (2m - 1)\theta] \\
&\quad + {}^{(1)}\beta_{4m} \theta \cos \theta + {}^{(2)}\beta_{4m} \theta \sin \theta \\
&\quad + \sum_{k=0}^{m-1} [-{}^{(1)}\beta_{4m} \sin (2k + 1)\theta + {}^{(2)}\beta_{4m} \cos (2k + 1)\theta], \\
h_{4m} &= {}^{(1)}A_{4m} \cos 2m\theta + {}^{(2)}A_{4m} \sin 2m\theta + {}^{(1)}\gamma_{4m} + {}^{(2)}\gamma_{4m} \theta \\
&\quad + \sum_{k=0}^{m-2} [{}^{(1)}\gamma_{4m} \cos 2(k + 1)\theta + {}^{(2)}\gamma_{4m} \sin 2(k + 1)\theta].
\end{aligned}$$

The additional coefficients

$${}^{(i)}_k \alpha_{4p}, {}^{(i)}_k \beta_{4p}, {}^{(i)}_k \gamma_{4p}, \quad i = 1, 2; p \geq 1$$

in equations (17) for the various values of k are also functions of z and they can be expressed in terms of ${}^{(i)}A_{2p}, {}^{(i)}B_{2p}$, etc.

The solution for terms in the other classes is written out in full in [13]. The equations giving the α, β, γ are also contained in that report [13], and will be used later in this paper. To illustrate without writing the equations out in full, they first give

$$(18) \quad {}^{(i)}_i \alpha_4, {}^{(i)}_i \beta_4, {}^{(i)}_i \gamma_4, \quad i = 1, 2 \quad \text{and} \quad j = 0, 1$$

in terms of derivatives of

$${}^{(i)}A_2, {}^{(i)}C_2, {}^{(i)}B_0, {}^{(i)}C_0$$

with respect to z . Then the equations of the second class of group (A) give

$${}^{(i)}_i \alpha_6, {}^{(i)}_i \beta_6, {}^{(i)}_i \gamma_6, \quad i = 1, 2 \quad \text{and} \quad j = 0, 1$$

in terms of derivatives of

$${}^{(i)}A_4, {}^{(i)}C_4, {}^{(i)}A_2, {}^{(i)}B_2, {}^{(i)}C_2, {}^{(i)}A_0, {}^{(i)}B_0, {}^{(i)}C_0.$$

When m is increased to 2, the equations of the first class are again obtained which give the corresponding α, β, γ in terms of the A, B, C and the previously calculated α, β, γ . Then the equations of class two give similar results. In this way, the solution switches between classes as m is further increased. The same type of equations for group (B) are also given in [13].

The set of equations consisting of (15) through (17) and those mentioned in the report (equations (15) through (40) in [13]) constitute a general solution of the three-dimensional equations of linear elasticity for the half-plane crack problem. The functions f_n, g_n, h_n ($n = 0, 1, 2, \dots$) may be put into equations (12) and (13) for the determination of displacements and stresses at all interesting points both near to and far from the edge of the crack.

Three-dimensional stress distribution. Having established the expressions for the displacement functions, it is in order to calculate the three-dimensional stresses for each value of n in equations (13). The following notation will be adopted:

$$\sigma_r = \sum_{n=0}^{\infty} (\sigma_r)_n, \quad \sigma_\theta = \sum_{n=0}^{\infty} (\sigma_\theta)_n, \quad \text{etc.}$$

First, consider the case of $n = 0$, where f_0, g_0 and h_0 are given by equations (15). Upon substitution of the appropriate stress components into the free crack surface boundary conditions in equation (9), there results

$$(19) \quad {}^{(2)}A_0 = {}^{(i)}C_0 = 0, \quad i = 1, 2$$

and hence equations (15) are simplified accordingly. These considerations lead to the vanishing of all stresses for $n = 0$, *i.e.*,

$$(20) \quad (\sigma_r)_0 = (\sigma_\theta)_0 = \dots = (\tau_{rz})_0 = 0.$$

For the purpose of illustrating the use of the recurrence formulas, stress solutions for $n = 1, 2, 3, 4$ will be worked out.

Group (A). Even integers of n .

1. $n = 2$. From equation (9), it is not difficult to show that

$$(21) \quad {}^{(2)}A_2 = - {}^{(2)}B'_0, \quad {}^{(2)}B_2 = 0, \quad {}^{(1)}C_2 = (1 - 2\nu) {}^{(1)}B_2 - \nu {}^{(1)}A'_0,$$

and thus

$$\begin{aligned} f_2 &= {}^{(1)}B_2 \cos 2\theta + (1 - 2\nu) {}^{(1)}B_2 - \nu {}^{(1)}A'_0, \\ g_2 &= - {}^{(1)}B_2 \sin 2\theta + {}^{(2)}C_2, \\ h_2 &= {}^{(1)}A_2 \cos \theta - {}^{(2)}B'_0 \sin \theta. \end{aligned}$$

It is apparent from equations (13) that for $n = 2$, the stresses are independent of r :

$$\begin{aligned} (\sigma_r)_2 &= {}^{(1)}B_2(1 + \cos 2\theta), \\ (\sigma_\theta)_2 &= {}^{(1)}B_2(1 - \cos 2\theta), \\ (\sigma_z)_2 &= 2\nu {}^{(1)}B_2 + (1 + \nu) {}^{(1)}A'_0, \\ (\tau_{r\theta})_2 &= - {}^{(1)}B_2 \sin 2\theta, \\ (\tau_{\theta z})_2 &= -\frac{1}{2} [{}^{(1)}A_2 + {}^{(1)}B'_0] \sin \theta, \\ (\tau_{rz})_2 &= \frac{1}{2} [{}^{(1)}A_2 + {}^{(1)}B'_0] \cos \theta. \end{aligned} \quad (22)$$

2. $n = 4$. Now, applying equations (19), (21) and (18) in [13], the quantities in equations (18) become

$$\begin{aligned} -{}_0^{(1)}\alpha_4 &= {}_0^{(1)}\beta_4 = \frac{1}{4} [{}^{(1)}A'_2 + (1 - 2\nu) {}^{(1)}B'_0'], \\ -{}_0^{(2)}\alpha_4 &= {}_0^{(2)}\beta_4 = -\frac{1}{2}\nu {}^{(2)}B'_0, \\ {}_1^{(1)}\alpha_4 &= {}_1^{(1)}\beta_4 = 0, \end{aligned} \quad (23)$$

and

$$\begin{aligned} {}_0^{(1)}\gamma_4 &= 0, \\ {}_1^{(1)}\gamma_4 &= -\frac{1}{2} [{}^{(1)}B'_2 + {}^{(1)}A'_0'], \quad {}_1^{(2)}\gamma_4 = 0. \end{aligned}$$

These coefficients determine f_4 , g_4 and h_4 in equations (17) which in turn give $(\sigma_\theta)_4$, $(\tau_{r\theta})_4$ and $(\tau_{\theta z})_4$. The conditions in equation (9) are thus satisfied by taking

$$\begin{aligned} (24) \quad {}^{(2)}A_4 &= -\frac{1}{2} {}^{(2)}C'_2, \quad {}^{(1)}C_4 = -\frac{1}{3} {}^{(1)}B_4 \\ &\quad - \frac{1}{12} [{}^{(1)}A'_2 + {}^{(1)}B'_0'], \quad {}^{(2)}C_4 = -{}^{(2)}B_4. \end{aligned}$$

This leads to

$$\begin{aligned}
 f_4 &= {}^{(1)}B_4[\cos 3\theta + \tfrac{1}{3}(1 - 4\nu) \cos \theta] \\
 &\quad + {}^{(2)}B_4[\sin 3\theta + (1 - 4\nu) \sin \theta] \\
 &\quad - \tfrac{1}{6}[(1 + 2\nu) {}^{(1)}A'_2 + (1 - \nu) {}^{(1)}B'_0'] \cos \theta + \tfrac{1}{2}\nu {}^{(2)}B'_0' \sin \theta, \\
 g_4 &= -{}^{(1)}B_4[\sin 3\theta - \tfrac{1}{3}(5 - 4\nu) \sin \theta] \\
 &\quad + {}^{(2)}B_4[\cos 3\theta - (5 - 4\nu) \cos \theta] \\
 &\quad + \tfrac{1}{6}[(1 - 2\nu) {}^{(1)}A'_2 + (1 + \nu) {}^{(1)}B'_0'] \sin \theta - \tfrac{1}{2}\nu {}^{(2)}B'_0' \cos \theta, \\
 h_4 &= {}^{(1)}A_4 \cos 2\theta - \tfrac{1}{2} {}^{(2)}C'_2 \sin 2\theta - \tfrac{1}{2}[{}^{(1)}B'_2 + {}^{(1)}A'_0'],
 \end{aligned}$$

and therefore

$$\begin{aligned}
 r^{-1}(\sigma_r)_4 &= 2 {}^{(1)}B_4(\cos 3\theta + \tfrac{1}{3} \cos \theta) \\
 &\quad + 2 {}^{(2)}B_4(\sin 3\theta + \sin \theta) - \tfrac{1}{3}[{}^{(1)}A'_2 + {}^{(1)}B'_0'] \cos \theta, \\
 r^{-1}(\sigma_\theta)_4 &= -2 {}^{(1)}B_4(\cos 3\theta - \cos \theta) \\
 &\quad - 2 {}^{(2)}B_4(\sin 3\theta - 3 \sin \theta), \\
 r^{-1}(\sigma_z)_4 &= 8\nu[\tfrac{1}{3} {}^{(1)}B_4 \cos \theta + {}^{(2)}B_4 \sin \theta] \\
 &\quad + \tfrac{1}{3}[(3 + 2\nu) {}^{(1)}A'_2 - \nu {}^{(1)}B'_0'] \cos \theta \\
 &\quad - (1 + \nu) {}^{(2)}B'_0' \sin \theta, \\
 (25) \quad r^{-1}(\tau_{r\theta})_4 &= -2 {}^{(1)}B_4(\sin 3\theta - \tfrac{1}{3} \sin \theta) \\
 &\quad + 2 {}^{(2)}B_4(\cos 3\theta - \cos \theta) \\
 &\quad + \tfrac{1}{3}[{}^{(1)}A'_2 + {}^{(1)}B'_0'] \sin \theta, \\
 r^{-1}(\tau_{\theta z})_4 &= -[{}^{(1)}A_4 + \tfrac{1}{2} {}^{(1)}B'_2] \sin 2\theta + \tfrac{1}{2} {}^{(2)}C'_2(1 - \cos 2\theta), \\
 r^{-1}(\tau_{zr})_4 &= [{}^{(1)}A_4 + \tfrac{1}{2} {}^{(1)}B'_2] \cos 2\theta - \tfrac{1}{2} {}^{(2)}C'_2 \sin 2\theta \\
 &\quad - \nu {}^{(1)}B'_2 - \frac{1 + \nu}{2} {}^{(1)}A'_0'.
 \end{aligned}$$

It is clear that all terms containing positive integer powers of r remain finite as $r \rightarrow 0$.

Group (B). Odd integers of n .

1. $n = 1$. This portion of the solution will give rise to singularities in the stresses. The vanishing of the stresses at $\theta = \pm\pi$ renders

$$(26) \quad {}^{(1)}A_1 = 0, \quad {}^{(1)}C_1 = {}^{(1)}B_1, \quad {}^{(2)}C_1 = -\tfrac{1}{3} {}^{(2)}B_1,$$

from which it can be deduced that

$$\begin{aligned}
f_1 &= {}^{(1)}B_1 \left[\cos \frac{3\theta}{2} - (5 - 8\nu) \cos \frac{\theta}{2} \right] \\
&\quad + {}^{(2)}B_1 \left[\sin \frac{3\theta}{2} - \frac{1}{3}(5 - 8\nu) \sin \frac{\theta}{2} \right], \\
g_1 &= -{}^{(1)}B_1 \left[\sin \frac{3\theta}{2} - (7 - 8\nu) \sin \frac{\theta}{2} \right] \\
&\quad + {}^{(2)}B_1 \left[\cos \frac{3\theta}{2} - \frac{1}{3}(7 - 8\nu) \cos \frac{\theta}{2} \right], \\
h_1 &= {}^{(2)}A_1 \sin \frac{\theta}{2}.
\end{aligned}$$

The stresses, which become unbounded at $r = 0$, are found to be

$$\begin{aligned}
r^{1/2}(\sigma_r)_1 &= \frac{1}{2} {}^{(1)}B_1 \left(\cos \frac{3\theta}{2} - 5 \cos \frac{\theta}{2} \right) \\
&\quad + \frac{1}{2} {}^{(2)}B_1 \left(\sin \frac{3\theta}{2} - \frac{5}{3} \sin \frac{\theta}{2} \right), \\
r^{1/2}(\sigma_\theta)_1 &= -\frac{1}{2} {}^{(1)}B_1 \left(\cos \frac{3\theta}{2} + 3 \cos \frac{\theta}{2} \right) \\
&\quad - \frac{1}{2} {}^{(2)}B_1 \left(\sin \frac{3\theta}{2} + \sin \frac{\theta}{2} \right), \\
(27) \quad r^{1/2}(\sigma_z)_1 &= -4\nu \left[{}^{(1)}B_1 \cos \frac{\theta}{2} + \frac{1}{3} {}^{(2)}B_1 \sin \frac{\theta}{2} \right], \\
r^{1/2}(\tau_{r\theta})_1 &= -\frac{1}{2} {}^{(1)}B_1 \left(\sin \frac{3\theta}{2} + \sin \frac{\theta}{2} \right) \\
&\quad + \frac{1}{2} {}^{(2)}B_1 \left(\cos \frac{3\theta}{2} + \frac{1}{3} \cos \frac{\theta}{2} \right), \\
r^{1/2}(\tau_{\theta z})_1 &= \frac{1}{4} {}^{(2)}A_1 \cos \frac{\theta}{2}, \\
r^{1/2}(\tau_{zr})_1 &= \frac{1}{4} {}^{(2)}A_1 \sin \frac{\theta}{2}.
\end{aligned}$$

2. $n = 3$. The relations necessary to meet the conditions in equation (9) are

$$(28) \quad {}^{(1)}A_3 = -\frac{2}{3}(7 - 8\nu) {}^{(1)}B'_1, \quad {}^{(1)}C_3 = -{}^{(1)}B_3, \quad {}^{(2)}C_3 = -\frac{1}{5} {}^{(2)}B_3 - \frac{1}{15} {}^{(2)}A'_1$$

which may be used to give

$$\begin{aligned}
f_3 &= {}^{(1)}B_3 \left[\cos \frac{5\theta}{2} + (3 - 8\nu) \cos \frac{\theta}{2} \right] \\
&\quad + {}^{(2)}B_3 \left[\sin \frac{5\theta}{2} + \frac{1}{5}(3 - 8\nu) \sin \frac{\theta}{2} \right] \\
&\quad - \frac{2}{15}(1 + 4\nu) {}^{(2)}A'_1 \sin \frac{\theta}{2}, \\
g_3 &= - {}^{(1)}B_3 \left[\sin \frac{5\theta}{2} - (9 - 8\nu) \sin \frac{\theta}{2} \right] \\
&\quad + {}^{(2)}B_3 \left[\cos \frac{5\theta}{2} - \frac{1}{5}(9 - 8\nu) \cos \frac{\theta}{2} \right] \\
&\quad - \frac{4}{15}(1 - 2\nu) {}^{(2)}A'_1 \cos \frac{\theta}{2}, \\
h_3 &= 2 {}^{(1)}B'_1 \left[-\frac{1}{3}(7 - 8\nu) \cos \frac{3\theta}{2} + \cos \frac{\theta}{2} \right] \\
&\quad + {}^{(2)}A_3 \sin \frac{3\theta}{2} + \frac{2}{3} {}^{(2)}B'_1 \sin \frac{\theta}{2},
\end{aligned}$$

and the stresses become

$$\begin{aligned}
r^{-1/2}(\sigma_r)_3 &= \frac{3}{2} {}^{(1)}B_3 \left(\cos \frac{5\theta}{2} + 3 \cos \frac{\theta}{2} \right) \\
&\quad + \frac{3}{2} {}^{(2)}B_3 \left(\sin \frac{5\theta}{2} + \frac{3}{5} \sin \frac{\theta}{2} \right) - \frac{1}{5} {}^{(2)}A'_1 \sin \frac{\theta}{2}, \\
r^{-1/2}(\sigma_\theta)_3 &= -\frac{3}{2} {}^{(1)}B_3 \left(\cos \frac{5\theta}{2} - 5 \cos \frac{\theta}{2} \right) \\
&\quad - \frac{3}{2} {}^{(2)}B_3 \left(\sin \frac{5\theta}{2} - \sin \frac{\theta}{2} \right), \\
r^{-1/2}(\sigma_z)_3 &= 12\nu \left[{}^{(1)}B_3 \cos \frac{\theta}{2} + \frac{1}{5} {}^{(2)}B_3 \sin \frac{\theta}{2} \right] \\
&\quad + \frac{1}{5}(5 + 4\nu) {}^{(2)}A'_1 \sin \frac{\theta}{2}, \\
(29) \quad r^{-1/2}(\tau_{r\theta})_3 &= -\frac{3}{2} {}^{(1)}B_3 \left(\sin \frac{5\theta}{2} - \sin \frac{\theta}{2} \right) \\
&\quad + \frac{3}{2} {}^{(2)}B_3 \left(\cos \frac{5\theta}{2} - \frac{1}{5} \cos \frac{\theta}{2} \right) \\
&\quad - \frac{1}{10} {}^{(2)}A'_1 \cos \frac{\theta}{2},
\end{aligned}$$

$$\begin{aligned}
r^{-1/2}(\tau_{\theta z})_3 &= \frac{3}{4} {}^{(2)}A_3 \cos \frac{3\theta}{2} + (3 - 4\nu) {}^{(1)}B_1' \left(\sin \frac{3\theta}{2} + \sin \frac{\theta}{2} \right) \\
&\quad + \frac{1}{2} {}^{(2)}B_1' \left[\cos \frac{3\theta}{2} - \frac{2}{3}(3 - 4\nu) \cos \frac{\theta}{2} \right], \\
r^{-1/2}(\tau_{rz})_3 &= - {}^{(1)}B_1' \left[(3 - 4\nu) \cos \frac{3\theta}{2} + (1 - 4\nu) \cos \frac{\theta}{2} \right] \\
&\quad + \frac{3}{4} {}^{(2)}A_3 \sin \frac{3\theta}{2} + \frac{1}{2} {}^{(2)}B_1' \left[\sin \frac{3\theta}{2} - \frac{2}{3}(1 - 4\nu) \sin \frac{\theta}{2} \right].
\end{aligned}$$

The same procedure may be followed for the development of the recurrence relations for the stress components. (Because of space limitation, these relationships will not be given here, but will be reported in a subsequent communication.) The coefficients

$${}^{(i)}A_n, {}^{(i)}B_n, i = 1, 2; \quad n = 1, 2, \dots$$

in equations (22), (25), (27) and (29) depend on the variable z and they are to be determined from the loading conditions of the problem under consideration.

Of particular interest is the behavior of the stress solution in the limit as the distance r tends to zero. The terms that approach high, mathematically infinite, values at $r = 0$ are those shown in equations (27). In contrast to the two-dimensional solution, all six components of the stress tensor are present in the immediate vicinity of the crack front, and they all vary as the inverse square root of the radial distance r . The coefficients ${}^{(i)}A_1, {}^{(i)}B_1$ ($i = 1, 2$), which vary along the leading edge of the crack, may be physically interpreted as quantities that reflect the redistribution of the three-dimensional stresses in a solid due to the introduction of a plane crack. In fact, equations (27) represent the most general stress state near the crack border and include the two-dimensional problems of plane extension [8] and longitudinal shear [14] as special cases:

1. *Plane extension.* If the coefficients in equations (27) are independent of z and such that

$${}^{(2)}A_1 = 0, \quad {}^{(1)}B_1 = -\frac{k_1}{2(2)^{1/2}} = \text{const.}, \quad {}^{(2)}B_1 = \frac{3k_2}{2(2)^{1/2}} = \text{const.},$$

then the two-dimensional problem of plane strain is recovered. The parameters k_1 and k_2 [15] are known, respectively, as the crack-tip stress-intensity factors for loads applied symmetrically and skew-symmetrically with respect to the line crack.

2. *Longitudinal Shear.* On the other hand, if

$${}^{(2)}A_1 = 2(2)^{1/2}k_3 = \text{const.}, \quad {}^{(i)}B_1 = 0, \quad i = 1, 2,$$

the stress state in equations (27) represents that of a crack under anti-plane shear deformation and k_3 is the corresponding crack-tip stress-intensity factor.

It should be noted in passing that equations (27) also display the three-dimensional character of the singular stresses *interior* to a thick plate containing a through crack. On the surface layers, where the crack penetrates through the plate, singularities of order different from $r^{-1/2}$ may exist. Nevertheless, the present analysis does indicate that the crack problem solution of Hartranft and Sih [16], based on the Reissner's theory of plate bending, indeed possesses the correct functional relationship of the in-plane stresses to r and θ for values of $|z| < h/2$, where z is the thickness coordinate and h the plate thickness. For further discussion of the thickness problem, the reader may refer to the work of Sih *et al* [17].

Wedge problems in three-dimensions. As mentioned earlier, the eigenfunction expansion method outlined here may also be used for examining the three-dimensional stress distribution near a V-shaped notch in an infinite solid. Similar to the two-dimensional case [7], unbounded stresses may occur for certain notch angles depending on the boundary conditions specified on the radial edges of the notch. The three possible combinations of boundary conditions are

1. *Free-Free.*

$$(30) \quad \sigma_\theta = \tau_{\theta r} = \tau_{\theta z} = 0, \quad \text{at} \quad \theta = \pm\alpha.$$

2. *Clamped-Clamped.*

$$(31) \quad u_r = v_\theta = w_z = 0, \quad \text{at} \quad \theta = \pm\alpha.$$

3. *Clamped-Free.*

$$(32) \quad \begin{aligned} \sigma_\theta = \tau_{\theta r} = \tau_{\theta z} = 0, \quad \text{at} \quad \theta = +\alpha, \\ u_r = v_\theta = w_z = 0, \quad \text{at} \quad \theta = -\alpha. \end{aligned}$$

The boundaries of the notch or wedge are described by

$$0 \leq r < \infty, \quad -\alpha \leq \theta \leq \alpha, \quad -\infty < z < \infty.$$

To fix ideas, consider the case of free-free edges. The six boundary conditions in equations (30) yield the same expressions as those in equations (10) provided π is replaced by α . However, there results four sets of eigenvalues $\lambda_m^{(i)}$ ($i = 1, 2, \dots, 4$):

$$(33) \quad \begin{aligned} \sin 2\lambda_m^{(1)}\alpha &= \lambda_m^{(1)} \sin 2\alpha, \\ \sin 2\lambda_m^{(2)}\alpha &= -\lambda_m^{(2)} \sin 2\alpha, \\ \lambda_m^{(3)} &= m\left(\frac{\pi}{\alpha}\right), \\ \lambda_m^{(4)} &= \left(m + \frac{1}{2}\right)\frac{\pi}{\alpha}. \end{aligned}$$

where $m = 0, 1, 2, \dots$. Note that the first two eigenequations in equations (33) coincide with those obtained by Williams [7] for the planar analysis, while the eigenvalues $\lambda_m^{(3)}$ and $\lambda_m^{(4)}$ correspond to those found by Sih [14] for the problem of a sector cylinder under longitudinal shear. In the three-dimensional case, the four sets of $\lambda_m^{(i)}$ occur simultaneously. It follows that the components of the displacement vector are given by

$$(34) \quad \begin{aligned} 2\mu u_r &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{i=1}^4 r^{\lambda_m^{(i)}+n} U_n^{(m)}(\theta, z; \lambda_m^{(i)}), \\ 2\mu v_\theta &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{i=1}^4 r^{\lambda_m^{(i)}+n} V_n^{(m)}(\theta, z; \lambda_m^{(i)}), \\ 2\mu w_z &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{i=1}^4 r^{\lambda_m^{(i)}+n} W_n^{(m)}(\theta, z; \lambda_m^{(i)}). \end{aligned}$$

For this problem, the stress components in equations (6) for each of the four eigenvalues $\lambda_m^{(i)}$ must be summed in i from one to four as it was done in equations (34) for the displacements.

The cases of clamped-clamped and clamped-free edges warrant no further comments, since they may be treated in the same way.

Conclusions. The eigenfunction expansion technique, used previously for analyzing two-dimensional crack and wedge problems, has been extended to the three-dimensional case. The three displacement components valid everywhere in the infinite solid weakened by a half-plane crack are derived in closed form. The pertinent steps for finding the recurrence relations of the stress components are also laid out in detail. In particular, the singular behavior of the three-dimensional stress field near a straight-edged crack is exhibited for the first time.

It is anticipated that further exploitation of the present investigation will provide a knowledge of the structure of the stress coefficients ${}^{(i)}A_n$ and ${}^{(i)}B_n$ for problems of half-plane cracks opened by concentrated normal and shear forces.

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