*A New Type of Multidimensional Singular Linear Integral Equation**

K. D. ABHYANKAR** & A. L. FYMAT†

Communicated by I. N. SNEDDON

I. Introduction. In the theory of radiative transfer through a plane-parallel atmosphere, the optical properties of which vary with optical depth, one has to solve the following system of nonlinear singular integral equations [3]:

$$(1a) \quad X^*(\alpha, \beta; \mu) = 1 + \int_0^1 \left[\int_\alpha^\beta X^*(\alpha, \beta'; \mu) X^*(\alpha, \beta'; \mu') \Omega(\beta') \right. \\ \left. \cdot \exp \left\{ -(\beta - \beta') \left(\frac{1}{\mu} + \frac{1}{\mu'} \right) \right\} d\beta' \right] \psi(\mu') \frac{d\mu'}{\mu},$$

$$(1b) \quad Y^*(\alpha, \beta; \mu) = \exp \left\{ \frac{-(\beta - \alpha)}{\mu} \right\} + \int_0^1 \left[\int_\alpha^\beta Y^*(\alpha, \beta'; \mu) X^*(\alpha, \beta'; \mu') \Omega(\beta') \right. \\ \left. \cdot \exp \left\{ \frac{-(\beta - \beta')}{\mu'} \right\} d\beta' \right] \psi(\mu') \frac{d\mu'}{\mu},$$

$$(1c) \quad X(\alpha, \beta; \mu) = 1 + \int_0^1 \left[\int_\alpha^\beta Y^*(\alpha, \beta'; \mu) Y^*(\alpha, \beta'; \mu') \Omega(\beta') d\beta' \right] \psi(\mu') \frac{d\mu'}{\mu},$$

$$(1d) \quad Y(\alpha, \beta; \mu) = \exp \left\{ \frac{-(\beta - \alpha)}{\mu} \right\} + \int_0^1 \left[\int_\alpha^\beta X^*(\alpha, \beta'; \mu) Y^*(\alpha, \beta'; \mu') \Omega(\beta') \right. \\ \left. \cdot \exp \left\{ \frac{-(\beta - \beta')}{\mu} \right\} d\beta' \right] \psi(\mu') \frac{d\mu'}{\mu},$$

where the positive constants α and β are the optical depths at the boundaries of the atmosphere, μ and μ' are the cosines of the angles of emergence and incidence, respectively, measured from the outward normal, $\Omega(\beta')$ is the variable

* This paper presents the results of one phase of research carried out at the Jet Propulsion Laboratory, California Institute of Technology, under Contract No. NAS 7-100, sponsored by the National Aeronautics and Space Administration.

** Senior Postdoctoral Resident Research Associate of NRC-NASA. On leave of absence from Osmania University, Hyderabad, India.

† Also at Department of Meteorology, University of California at Los Angeles.

albedo for single scattering lying in the range $[0, 1]$, and $\psi(\mu)$ is a non-negative characteristic function determined by the law of scattering which satisfies

$$\Omega(\beta') \int_0^1 \psi(\mu) d\mu = \int_0^1 \Psi(\mu) d\mu \leq \frac{1}{2}.$$

We are interested in the real solutions of system (1) which are continuous and bounded in the intervals $0 \leq \mu \leq 1$ and $\beta \geq 0$. Such solutions are known for the case $\Omega(\beta) \equiv \Omega_0$ (constant) such that $0 \leq \Omega_0 \leq 1$. In this situation, the distinction between X^* and X and Y^* and Y disappears [3]; when $\tau = (\beta - \alpha)$ is finite, they are known as the X and Y functions of Chandrasekhar [1]; and when τ tends to infinity, $Y(\tau; \mu)$ approaches zero uniformly, and $X(\tau, \mu)$ becomes $H(\mu)$, the Chandrasekhar H -function. The assumptions regarding $\Omega(\beta')$ and $\psi(\mu)$ ensure that the functions X , Y , X^* , Y^* and H are non-negative.

A cursory examination of system (1) shows that equation (1a), which contains $X^*(\alpha, \beta; \mu)$ only, can be solved independently of the other equations of the system. Once $X^*(\alpha, \beta; \mu)$ is known, equation (1b) becomes a linear integral equation in $Y^*(\alpha, \beta; \mu)$, the solution of which poses no particular problem. Then, by direct substitution of the solutions for $X^*(\alpha, \beta; \mu)$ and $Y^*(\alpha, \beta; \mu)$ in equations (1c) and (1d), we can determine $X(\alpha, \beta; \mu)$ and $Y(\alpha, \beta; \mu)$. We have, therefore, to concentrate on the solution of equation (1a) only.

In a recent paper [2] we have developed a perturbation method for solving equation (1a) when $\Omega(\beta')$ differs from a constant value Ω_0 by a small amount throughout the range of β' . We have expressed $\Omega(\beta')$ and $X^*(\alpha, \beta; \mu)$ in the forms

$$(2) \quad \Omega(\beta') = \Omega_0[1 + \omega(\beta')], \quad (\omega \ll 1),$$

and

$$(3) \quad X^*(\alpha, \beta'; \mu) = X_0^*(\alpha, \beta'; \mu)[1 + \chi^*(\alpha, \beta'; \mu)], \quad (\chi^* \ll 1),$$

where $X_0^*(\alpha, \beta'; \mu)$ is the value of $X^*(\alpha, \beta'; \mu)$ for $\Omega(\beta') \equiv \Omega_0$. On substituting equations (2) and (3) in equation (1a) and neglecting the quadratic term in χ^* , we get, after subtracting the equation for $X_0^*(\alpha, \beta; \mu)$,

$$(4) \quad \chi^*(\alpha, \beta; \mu) = \Omega_0 \int_0^1 X_0^*(\alpha, \beta; \mu') \cdot \left[\int_\alpha^\beta \frac{X_0^*(\alpha, \beta'; \mu) X_0^*(\alpha, \beta'; \mu')}{X_0^*(\alpha, \beta; \mu) X_0^*(\alpha, \beta; \mu')} \{ \omega(\beta') + [1 + \omega(\beta')] \chi^*(\alpha, \beta'; \mu) \right. \\ \left. + \chi^*(\alpha, \beta'; \mu') \} \exp \left\{ -(\beta - \beta') \left(\frac{1}{\mu} + \frac{1}{\mu'} \right) \right\} d\beta' \right] \psi(\mu') \frac{d\mu'}{\mu'}.$$

Equation (4) is a linearized form of equation (1a). In solving it we came across a new type of multidimensional singular linear integral equation that appears to be of general interest. This integral equation is presented and discussed in the following section.

II. The integral equation.

Definition. On multiplying equation (4) by $[1 + \omega(\beta)]$, interchanging μ and μ' , and adding the resulting equations, we obtain

$$\begin{aligned}
 (5) \quad & [1 + \omega(\beta)][\chi^*(\alpha, \beta; \mu) + \chi^*(\alpha, \beta; \mu')] \\
 &= \Omega(\beta) \int_0^1 X_0^*(\alpha, \beta; \mu') \frac{\psi(\mu')}{\mu'} \\
 &\quad \cdot \left[\int_\alpha^\beta \frac{X_0^*(\alpha, \beta'; \mu) X_0^*(\alpha, \beta'; \mu')}{X_0^*(\alpha, \beta; \mu) X_0^*(\alpha, \beta; \mu')} \{ \omega(\beta') + [1 + \omega(\beta')][\chi^*(\alpha, \beta'; \mu) \right. \\
 &\quad \left. + \chi^*(\alpha, \beta'; \mu')] \} \exp \left(-(\beta - \beta') \left(\frac{1}{\mu} + \frac{1}{\mu'} \right) \right) d\beta' \right] d\mu' \\
 &\quad + \Omega(\beta) \int_0^1 X_0^*(\alpha, \beta; \mu) \frac{\psi(\mu)}{\mu} \\
 &\quad \cdot \left[\int_\alpha^\beta \frac{X_0^*(\alpha, \beta'; \mu) X_0^*(\alpha, \beta'; \mu')}{X_0^*(\alpha, \beta; \mu) X_0^*(\alpha, \beta; \mu')} \{ \omega(\beta') + [1 + \omega(\beta')][\chi^*(\alpha, \beta'; \mu) \right. \\
 &\quad \left. + \chi^*(\alpha, \beta'; \mu')] \} \exp \left\{ -(\beta - \beta') \left(\frac{1}{\mu} + \frac{1}{\mu'} \right) \right\} d\beta' \right] d\mu.
 \end{aligned}$$

Putting

$$(6) \quad \Phi(\beta; \mu, \mu') = [1 + \omega(\beta)][\chi^*(\alpha, \beta; \mu) + \chi^*(\alpha, \beta; \mu')],$$

$$\begin{aligned}
 (7) \quad & f(\beta; \mu, \mu') = \Omega(\beta) \int_0^1 X_0^*(\alpha, \beta; \mu') \frac{\psi(\mu')}{\mu'} \\
 &\quad \cdot \left[\int_\alpha^\beta \frac{X_0^*(\alpha, \beta'; \mu) X_0^*(\alpha, \beta'; \mu')}{X_0^*(\alpha, \beta; \mu) X_0^*(\alpha, \beta; \mu')} \omega(\beta') \exp \left\{ -(\beta - \beta') \left(\frac{1}{\mu} + \frac{1}{\mu'} \right) \right\} d\beta' \right] d\mu' \\
 &\quad + \Omega(\beta) \int_0^1 X_0^*(\alpha, \beta; \mu) \frac{\psi(\mu)}{\mu} \\
 &\quad \cdot \left[\int_\alpha^\beta \frac{X_0^*(\alpha, \beta'; \mu) X_0^*(\alpha, \beta'; \mu')}{X_0^*(\alpha, \beta; \mu) X_0^*(\alpha, \beta; \mu')} \omega(\beta') \exp \left\{ -(\beta - \beta') \left(\frac{1}{\mu} + \frac{1}{\mu'} \right) \right\} d\beta' \right] d\mu,
 \end{aligned}$$

$$(8) \quad K(\beta', \beta; \mu, \mu') = \frac{X_0^*(\alpha, \beta'; \mu) X_0^*(\alpha, \beta'; \mu')}{X_0^*(\alpha, \beta; \mu) X_0^*(\alpha, \beta; \mu')} \exp \left\{ -(\beta - \beta') \left(\frac{1}{\mu} + \frac{1}{\mu'} \right) \right\},$$

and

$$(9) \quad \lambda(\beta; \mu) = \Omega(\beta) X_0^*(\alpha, \beta; \mu) \psi(\mu),$$

equation (5) becomes

$$\begin{aligned}
(10) \quad \Phi(\beta; \mu, \mu') &= f(\beta; \mu, \mu') \\
&+ \int_0^1 \lambda(\beta; \mu) \left[\int_\alpha^\beta K(\beta', \beta; \mu, \mu') \Phi(\beta'; \mu, \mu') d\beta' \right] \frac{d\mu}{\mu} \\
&+ \int_0^1 \lambda(\beta; \mu') \left[\int_\alpha^\beta K(\beta', \beta; \mu, \mu') \Phi(\beta'; \mu, \mu') d\beta' \right] \frac{d\mu'}{\mu'}.
\end{aligned}$$

Replacing β' , β , α , μ , and μ' by ξ , x , x_0 , y , and z , respectively, we obtain the following general integral equation:

$$\begin{aligned}
(11) \quad \Phi(x, y, z) &= f(x, y, z) + \int_0^1 \lambda(x, y) \left[\int_{x_0}^x K(\xi, x, y, z) \Phi(\xi, y, z) d\xi \right] \frac{dy}{y} \\
&+ \int_0^1 \lambda(x, z) \left[\int_{x_0}^x K(\xi, x, y, z) \Phi(\xi, y, z) d\xi \right] \frac{dz}{z}.
\end{aligned}$$

Equation (11) is a multidimensional mixed Volterra-Fredholm type of singular integral equation of the second kind. We shall call $K(\xi, x, y, z)$ the inner kernel and $\lambda(x, y)$ or $\lambda(x, z)$ the outer kernel of this equation. The solution of equation (11) will depend upon the form of these two kernels. In radiative-transfer problems they have the following properties:

(1) Inner kernel:

- a. symmetry in y and z : $K(\xi, x, y, z) = K(\xi, x, z, y)$;
- b. degeneracy (Pincherle-Goursat type):
 $K(\xi, x, y, z) = k_1(x, y, z) k_2(\xi, y, z)$;
- c. convolution (Faltung type): only for $x \rightarrow \infty$ when $X_0^*(x_0, x, y) \rightarrow H_0(y)$
 $K(\xi, x, y, z) = L(x - \xi, y, z)$.

(2) Outer kernel:

- a. in the case

$$\Omega(x) \equiv \Omega_0 : \lim_{x \rightarrow \infty} \lambda(x, y) = \lambda^*(y).$$

Operator form and solution. Defining the singular operator C by

$$\begin{aligned}
(12) \quad C\{g(\xi, y, z)\} &= \int_0^1 \lambda(x, y) \left[\int_{x_0}^x K(\xi, x, y, z) g(\xi, y, z) d\xi \right] \frac{dy}{y} \\
&+ \int_0^1 \lambda(x, z) \left[\int_{x_0}^x K(\xi, x, y, z) g(\xi, y, z) d\xi \right] \frac{dz}{z},
\end{aligned}$$

we can write equation (11) in the operator form

$$(13) \quad (1 - C)\{\Phi(\xi, y, z)\} = f(x, y, z),$$

where 1 is the identity operator. The formal solution of equation (13) is given by the Neumann-type series

$$(14) \quad \Phi(x, y, z) = \sum_{n=0}^{\infty} C^n \{f(\xi, y, z)\},$$

where C^n is the operator C repeated n times. If the series on the right-hand side of equation (14) converges, it would represent the solution of equation (11). The conditions for convergence will have to be studied separately for various forms of the kernels K and λ .

Because of the symmetry of the operator C with respect to the variables y and z , it is obvious that if K and f are symmetric in these variables, Φ will also be symmetric in y and z .

III. The radiative-transfer problem.

Solution. As an illustration, we shall consider the solution of equation (11) for the radiative-transfer problem in which the functions f , K , and λ are defined by equations (7), (8), and (9), respectively. Let

$$(15) \quad \Phi_N(x, y, z) = C^N \{f(\xi, y, z)\};$$

then,

$$(16) \quad \Phi_0(x, y, z) = f(x, y, z) = f_0(x, y) + f_0(x, z).$$

By a repeated use of the operator C , it is seen that

$$(17) \quad \Phi_N(x, y, z) = f_N(x, y) + f_N(x, z), \quad (N = 1, 2, \dots),$$

where

$$(18) \quad f_N(x, y) = k_N(x, y) + g_N(x, y),$$

and

$$(19) \quad k_N(x, y) = \Omega(x) \int_0^1 X_0^*(x_0, x, z) \psi(z) \cdot \left[\int_{x_0}^x f_{N-1}(\xi, y) \frac{X_0^*(x_0, \xi, y) X_0^*(x_0, \xi, z)}{X_0^*(x_0, x, y) X_0^*(x_0, x, z)} \exp \left\{ -(x - \xi) \left(\frac{1}{y} + \frac{1}{z} \right) \right\} d\xi \right] \frac{dz}{z},$$

$$(20) \quad g_N(x, y) = \Omega(x) \int_0^1 X_0^*(x_0, x, z) \psi(z) \cdot \left[\int_{x_0}^x f_{N-1}(\xi, z) \frac{X_0^*(x_0, \xi, y) X_0^*(x_0, \xi, z)}{X_0^*(x_0, x, y) X_0^*(x_0, x, z)} \exp \left\{ -(x - \xi) \left(\frac{1}{y} + \frac{1}{z} \right) \right\} d\xi \right] \frac{dz}{z}.$$

Then, equation (14) gives

$$(21) \quad \Phi(x, y, z) = \sum_{n=0}^{\infty} [f_n(x, y) + f_n(x, z)].$$

Hence, we can write

$$(22) \quad \Phi(x, y, z) = F(x, y) + F(x, z),$$

where

$$(23) \quad F(x, y) = \sum_{n=0}^{\infty} f_n(x, y).$$

Thus, in general, if $f(x, y, z)$ has the symmetric form of equation (16), $\Phi(x, y, z)$ also has the same symmetric form.

Convergence. In the radiative-transfer problem we can discuss the convergence of the series (14) in the following manner. From equation (7) we see that

$$(24) \quad f(x, y, z) = C\{\omega(\xi)\}.$$

Now, let

$$(25) \quad s_1 \leq \omega(\xi) \leq s_2, \quad (s_1, s_2 = \text{constants}),$$

and let s_m be the larger of $|s_1|$ and $|s_2|$. Then, from the equation

$$(26) \quad \begin{aligned} C\{\omega(\xi)\} &= \Omega(x) \int_0^1 \left[\int_{x_0}^x \omega(\xi) \frac{X_0^*(x_0, \xi, y) X_0^*(x_0, \xi, z)}{X_0^*(x_0, x, y) X_0^*(x_0, x, z)} \exp \left\{ -(x - \xi) \left(\frac{1}{y} + \frac{1}{z} \right) \right\} d\xi \right] \\ &\quad \cdot \left[X_0^*(x_0, x, z) \psi(z) \frac{dz}{z} + X_0^*(x_0, x, y) \psi(y) \frac{dy}{y} \right], \end{aligned}$$

we can write (since X_0^* , ψ , and $\Omega(x)/\Omega_0 = 1 + \omega(x)$ are non-negative functions)

$$(26) \quad \begin{aligned} |C\{\omega(\xi)\}| &\leq s_m \Omega_0 (1 + s_2) \int_0^1 \left[\int_{x_0}^x \frac{X_0^*(x_0, \xi, y) X_0^*(x_0, \xi, z)}{X_0^*(x_0, x, y) X_0^*(x_0, x, z)} \right. \\ &\quad \cdot \exp \left\{ -(x - \xi) \left(\frac{1}{y} + \frac{1}{z} \right) \right\} d\xi \left. \right] \\ &\quad \cdot \left[X_0^*(x_0, x, z) \psi(z) \frac{dz}{z} + X_0^*(x_0, x, y) \psi(y) \frac{dy}{y} \right]. \end{aligned}$$

Then, using equation (1a), we have

$$(27) \quad |C\{\omega(\xi)\}| \leq s_m (1 + s_2) \left[2 - \left\{ \frac{1}{X_0^*(x_0, x, y)} + \frac{1}{X_0^*(x_0, x, z)} \right\} \right]$$

or, since $X_0^*(x_0, x, y)$ is a monotonically increasing function of y ,

$$(28) \quad |C\{\omega(\xi)\}| \leq 2s_m (1 + s_2) \left[1 - \frac{1}{X_0^*(x_0, x, 1)} \right].$$

Now, suppose that

$$(29) \quad |C^n\{\omega(\xi)\}| \leq s_m [r(x_0, x)]^n,$$

where

$$(30) \quad r(x_0, x) = 2(1 + s_2) \left[1 - \frac{1}{X_0^*(x_0, x, 1)} \right],$$

is a non-negative function. Then, since

$$(31) \quad r(x_0, \xi) \leq r(x_0, x), \quad (x_0 \leq \xi \leq x),$$

it can be verified by induction that the inequality (29) is true for any n . Therefore, the right-hand side of equation (14) satisfies the condition:

$$(32) \quad \left| \sum_{n=0}^{\infty} C^n \{f(\xi, y, z)\} \right| = \left| \sum_{n=1}^{\infty} C^n \{\omega(\xi)\} \right| \leq \sum_{n=1}^{\infty} |C^n \{\omega(\xi)\}| \leq \sum_{n=1}^{\infty} s_n [r(x_0, x)]^n.$$

Hence, the series (14) will be absolutely convergent if $r(x_0, x) < 1$. This condition of convergence can alternatively be written as

$$(33) \quad \frac{1}{X_0^*(x_0, x, 1)} > \frac{1 + 2s_2}{2(1 + s_2)},$$

which depends only on the upper limit s_2 of $\omega(\xi)$. In the case of a semi-infinite atmosphere $X_0^*(x_0, x, 1)$ is replaced by $H_0(1)$ in this condition.

The condition of convergence (33) has to be investigated separately for each law of scattering, $\psi(z)$. As an illustration, we have considered the case of isotropic scattering, $\psi(z) = \frac{1}{2}$, in a semi-infinite atmosphere for which the H -functions have been tabulated for all values of Ω_0 ([1], p. 125). From this table, it is found that for any given value of s_2 , the condition of convergence (33) is satisfied whenever $\Omega_0 \leq \Omega_0(s_2)$. The function $\Omega_0(s_2)$ for s_2 between -0.30 and $+0.20$ is drawn in Fig. 1, which shows that we have convergence provided Ω_0 does not

FIGURE 1. The curve $\Omega_0(s_2)$ which limits the domain of convergence of the series (14) for the case $\psi(z) = 1/2$.

lie in the hatched region. It is interesting to note that the region of convergence is larger for negative values of s_2 and that, in particular, we get convergence for all values of Ω_0 when $s_2 \leq -0.238$.

REFERENCES

- [1] S. CHANDRASEKHAR, *Radiative Transfer*, Clarendon Press, Oxford, 1950.
- [2] A. L. FYMAT & K. D. ABHYANKAR, Theory of radiative transfer in inhomogeneous atmospheres: I. Perturbation method, *Astrophys. Journal*, (in press).
- [3] S. UENO, Diffuse reflection and transmission in a finite inhomogeneous atmosphere, *Astrophys. Journal*, **132** (1960) 729.

Jet Propulsion Laboratory
California Institute of Technology
Date communicated: DECEMBER 2, 1968