On the Relation Between Period and Energy in Periodic Dynamical Systems

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1. Introduction. Let $\Phi = \{\phi_t\}$ be a 1-parameter group acting on a manifold M , and let X denote the infinitesimal generator of Φ . We assume that Φ has period P ; i.e., $\Phi_{P(x)}(x) = x$ for all x in M where P is some positive function on M of class C^1 . We employ the standard notations of differential geometry, (cf. [2, 4]) except that we let $X\omega$ represent the Lie derivative of the differential k -form ω with respect to X . Hence

$$(1) \quad X\omega = \frac{d}{dt} \phi_t^* \omega \phi_t \big|_{t=0} = X \rfloor \omega + d(X \rfloor \omega)$$

where ϕ_t^* is the map which ϕ_t induces on $\wedge^k T^*(M)$ and $X \rfloor$ is the anti-derivation defined by setting $X \rfloor f = 0$ if f is a function, and setting $X \rfloor \omega = \langle X, \omega \rangle =$ standard inner product if ω is a 1-form. In local coordinates, if $X = \sum \xi^\alpha (\partial/\partial x^\alpha)$ and $\omega = \sum \omega_{\alpha_1, \dots, \alpha_k} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_k}$, then $X \rfloor \omega = \sum (-1)^{i+1} \xi^{\alpha_i} \omega_{\alpha_1, \dots, \alpha_k} dx^{\alpha_1} \wedge \dots \wedge \widehat{dx^{\alpha_i}} \wedge \dots \wedge dx^{\alpha_k}$. Hereafter, we assume that all manifolds, maps and functions, etc., appearing in the discussion are of class C^1 .

Generalizing previously known results, Lewis [1] proved the following theorem.

Theorem. *Suppose there exists a closed 2-form θ such that $X \rfloor \theta = -dH$ for some function H . Then $dP \wedge dH = 0$.*

For dynamical systems of the Hamiltonian type, such a 2-form exists and H is the Hamiltonian of the system. Hence the theorem asserts that for systems of this type, P and H are functionally dependent in a neighborhood of any point x for which either $dP(x) \neq 0$ or $dH(x) \neq 0$. In this paper we shall give another proof of this theorem, and using the machinery of the proof we shall obtain more definite information concerning the functional dependence of P on H and certain other orbit parameters. We shall also construct a recipe for computing P (at least up to a constant factor) which involves obtaining a solution to a differential equation of the type $X^2 f = g$.

2. We define an operator L on differential forms as follows: For any form ω , let $(L\omega)(x) = (1/P(x)) \int_0^{P(x)} \phi_t^* \omega \phi_t(x) dt$. Then L has the following basic properties:

$$(A) \quad LX\omega = XL\omega = -\frac{dP}{P} \wedge (X \lrcorner \omega),$$

$$(B) \quad (dL - Ld)\omega = \frac{dP}{P} \wedge (\omega - L\omega).$$

Formula (B) is obtained by a direct calculation in local coordinates. The proof of (A) is deferred to the end of the paper.

Lewis' theorem is an easy consequence of (A). Let θ be a 2-form satisfying the conditions of the theorem. Then $X\theta = X \lrcorner d\theta + d(X \lrcorner \theta) = 0 - d^2H = 0$. Hence, from (A), $0 = LX\theta = (dP/P) \wedge dH$.

The following corollary is actually closer to the original statement of Lewis' theorem.

Corollary. *Suppose Φ is a 1-parameter group acting on a manifold M (whose action is not necessarily periodic) whose infinitesimal generator X satisfies the relation $X \lrcorner \theta = -dH$ for some closed 2-form θ and function H . Suppose that M' is a submanifold of M which is closed under Φ , and that the action of Φ on M' is periodic with period P' . Then if H' denotes the restriction of H to M' , $dP' \wedge dH' = 0$.*

Proof. For x' in M' , let $X'(x)$ denote the vector $X(x)$ considered as a vector on M' . Let $\theta' = i^* \theta$ where $i: M' \rightarrow M$ is the embedding of M' into M . The remainder of the proof consists in verifying from general principles that $X' \lrcorner \theta' = -dH'$ and $d\theta' = 0$.

3. Let L_0 denote the restriction of L to C^1 functions. Then $(L_0 f)(x) = (1/P(x)) \int_0^{P(x)} f \circ \phi_t(x) dx$, and it is easy to see that $L_0 f = f$ if f is a constant of motion (i.e., $f \circ \phi_t = f$). Also, if f is any function, $L_0 f$ is a constant of motion. Hence $L_0^2 = L_0$. Now the C^1 constants of motion are the solutions to $Xf = 0$, therefore $XL_0 = 0$. And since

$$(Xf)(x) = \frac{d}{dt} f \circ \phi_t(x) \big|_{t=0},$$

it follows that $L_0 X = 0$. Suppose $X^2 f = 0$. Then since

$$(X^2 f)(x) = \frac{d^2}{dt^2} f \circ \phi_t(x) \big|_{t=0},$$

we have $f \circ \phi_t(x) = a(x)t + f(x)$. But for each fixed x , $|f \circ \phi_t(x)|$ is bounded since the orbits of Φ are closed. Therefore $a(x) = 0$ and f is a constant of motion; i.e., we have shown that $X^2 f = 0$ implies that $Xf = 0$. Summarizing these results, we have the following lemma:

Lemma.

$$\begin{cases} L_0 f = f & \text{iff } Xf = 0 & \text{iff } X^2 f = 0. \\ XL_0 = L_0 X = 0; & L_0^2 = L_0. \end{cases}$$

Remark. It is not true in general that $L^2 = L$.

We now state our main results.

Proposition 1. Suppose that Φ satisfies all of the conditions of the theorem, and in addition, that θ is exact: $\theta = d\alpha$. Then at any point for which $L(X \lrcorner \alpha) \neq 0$,

$$(3) \quad \frac{dP}{P} = \frac{dH - dL(X \lrcorner \alpha)}{L(X \lrcorner \alpha)}.$$

In particular, suppose that Φ is the group associated with a dynamical system of the Hamiltonian type, so that Φ acts on the cotangent bundle of some manifold, (the "phase space" of the system). Then by definition of a Hamiltonian system (cf [3, 4]), there is a function H such that $X \lrcorner d\alpha = -dH$ where α is the canonical 1-form $\sum p_i dq^i$. Let $T = \frac{1}{2} \sum p_i \partial H / \partial p_i$ and let $V = H - T$ (so that in ordinary applications, T is the kinetic energy and V is the potential of the system). The relation $X \lrcorner d\alpha = -dH$ is equivalent to

$$X = \sum \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right),$$

therefore $X \lrcorner \alpha = 2T$, and from (3) we obtain

$$(4) \quad -\frac{dP}{P} = \frac{dL(T - V)}{2L(T)}.$$

In order to use this proposition to compute P we need to compute Lf for certain functions f . The following proposition describes a method for doing this which does not require the integration of the equations of motion.

Proposition 2. For any C^1 function f , $Lf = f - Xf_1$ where f_1 is any solution to the equation $X^2 f_1 = Xf$. Such a solution always exists.

Proofs. To prove Prop 1, we have, from (1), $X\alpha = dg$ where $g = -H + (X \lrcorner \alpha)$. Hence, using (B), we have $LX\alpha = Ldg = dLg - (dP/P)(g - Lg)$. But from (A), we have $LX\alpha = -(dP/P) \wedge (X \lrcorner \alpha)$. Therefore $(dP/P)(g - Lg - (X \lrcorner \alpha)) = dLg$. And since $L(H) = H$, this last equation yields (3).

To prove Prop 2, we use the Fourier expansion $f \circ \phi_t = \sum \hat{f}_n \exp(2\pi i n t / P)$. The functions $\hat{f}_n$ are given by $\hat{f}_n = (1/2\pi) \int_0^P \exp(-2\pi i n t / P) f \circ \phi_t dt$. In particular, $\hat{f}_0 = Lf$. Also, it is easy to show that $X\hat{f}_n = 2\pi i n \hat{f}_n / P$, and that $f = Lf + Xf_1$ where $f_1 = \sum_{n \neq 0} \hat{f}_n / n$. And since, from the lemma, we have $XLf = 0$, it follows that $Xf = X^2 f_1$. Now suppose f_2 is any other function satisfying $X^2 f_2 = Xf$. Then $X^2(f_1 - f_2) = 0$, so that using the lemma again, we get $Xf_2 = Xf_1$. Hence $Lf = f - Xf_1 = f - Xf_2$.

4. In some cases it is possible to use Prop 1 to compute P without either integrating the equations of motion or using the device of Prop 2.

Example. Consider a periodic conservative system arising from a potential V which is positively homogenous of degree k . Using an elementary dimension argument [2, p. 532] it is possible to show that $P = \text{constant} \times |H|^{(2-k)/2k}$. We show how Prop 1 can be used to obtain this result, as well as certain other information. Using vector notation, we have

$$L(T) = \frac{1}{2P} \int_0^P \left\langle \frac{dx}{dt}, \frac{dx}{dt} \right\rangle dt = -\frac{1}{2P} \int_0^P \left\langle x, \frac{d^2x}{dt^2} \right\rangle dt = \frac{1}{2P} \int_0^P \langle x, \nabla V \rangle dt.$$

But $\langle x, \nabla V \rangle = kV$, hence (i) $L(T) = kL(V)/2$. Since $L(T) + L(V) = L(H) = H$, we have for $k \neq -2$, (ii) $L(T) = kH/(2+k)$ and (iii) $L(V) = 2H/(2+k)$. The formula for P given above is an easy consequence of (ii), (iii) and (4). Moreover, using these relations and the fact that $L(T) > 0$, it is easy to establish the following: In order that the action of Φ be periodic on a submanifold of phase space on which $(\alpha) H < 0$, $(\beta) H = 0$, $(\gamma) H > 0$ it is necessary that $(\alpha) -2 < k < 0$, $(\beta) k = -2$, $(\gamma) k > 0$ or $k < -2$.

5. *Proof of (A).* From the definitions, we have $PLX\omega = \phi_i^* \omega \phi_i|_{t=t} - \omega$. Therefore the proof consists in showing that $\phi_i^* \omega \phi_i|_{t=-P} = \omega - dP \wedge (X \lrcorner \omega)$. For ease of exposition, we prove this relation only in the case where ω is a 1-form. Let $\{x^i\}$ be a local coordinate chart, and let $x_i^i = (\phi_i(x))^i$, and $\dot{x}_i^i = \partial x_i^i / \partial t$. Using the summation convention, we write $\omega = \omega_i dx^i$, so that $\phi_i^* \omega \phi_i = [(\partial x_i^i / \partial x^j) \omega_j(x_i)] dx^i$. Now, $x_{P(x)}^i(x) = x^i$, hence $(\partial x_i^i / \partial x^j)|_{t=-P} + \dot{x}_P^i \partial P / \partial x^j = \delta_j^i$. But $\{\dot{x}_P^i\} = \{\dot{x}_0^i\} =$ components of X in the local chart. Hence

$$\phi_i^* \omega \phi_i|_{t=-P} = \left(\delta_j^i - \dot{x}_0^i \frac{\partial P}{\partial x^j} \right) \omega_j dx^j = \omega - (X \lrcorner \omega) dP.$$

Note the the arrangement of factors in the right hand side of (A) is the most convenient one, since the order of the factors can only be inverted if we introduce a factor $(-1)^{k-1}$ where k is the degree of ω . Finally, we note that the relation $LX = XL$ is a consequence of the fact that $XP = 0$.

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