A Lemma of H. Tong and Some of Its Consequences

MARY POWDERLY

Communicated by E. R. LORCH

We present here a proof of a lemma of H. Tong, which does not depend on the Axiom of Choice or any of its equivalents, and some well known results which are easy or immediate consequences of this lemma. Some of these results were announced in [1]. A much broader study was indicated in [1]. However, because of recent interest in this subject, it seems appropriate to publish some results at present, leaving further consequences and generalizations for another occasion.

Tong's Lemma. *Let g be a mapping from a subset of $P(X)$ (the set of all subsets of X) into X . Let $A \subset X$ be a well-ordered set with the property that $g(S_A(x)) = x$ for every $x \in A$ (where $S_A(x)$ is the (possibly empty) segment in A consisting of all the elements of A which are $< x$). Then the union U of all A satisfying the above conditions can be well-ordered in such a way that $g(S_U(x)) = x$ for every $x \in U$. Moreover, if $g(U)$ exists, then $g(U) \in U$.*

Remarks. We note that if $A_1 \neq A_2$ and both A 's satisfy the conditions on A given above, then one is a segment of the other (this will be shown in the proof below). It is also useful to note the conditions (on the well-ordered set U): (1) $g(S_U(x)) = x$ for every $x \in U$, and (2) If $g(U)$ exists, then $g(U) \in U$. The fact that these two conditions determine U uniquely can be readily seen.

Proof. (This proof and the proof of Corollary 1 are due to H. Tong.) Let A_1 and A_2 be sets satisfying the hypothesis. Let B be the set of all $x \in A_1 \cap A_2$ such that $S_{A_1}(x) = S_{A_2}(x)$. If $B \neq A_i$ for $i = 1, 2$, let a_i be the least element of $A_i - B$. Thus $S_{A_i}(a_i) = B$. Consequently, $g(B) = g(S_{A_i}(a_i)) = a_i$. Therefore, $a_1 = a_2$. This contradicts the definition of a_i . Thus, $B = A_1$ or A_2 . It follows that if $A_1 \neq A_2$, then one is a segment of the other.

We now define an ordering on U . Let x_1 and $x_2 \in U$. There exist A_i such that $x_i \in A_i$. The above result implies that one of the A 's must contain both x_1 and x_2 . Provide x_1 and x_2 with the ordering obtained in that A . As a consequence of the same result above, the ordering is independent of A , and it is easy to see that the U is thus well-ordered.

Now consider any $x \in U$. There exists an A such that $x \in A$. Then $S_A(x) = S_U(x)$ and $g(S_U(x)) = g(S_A(x)) = x$. The last sentence in the statement of the lemma is obvious.

Corollary 1—Zorn's Lemma. *If every well-ordered set of a partially ordered set P has an upper bound in P , then P has a maximal element.*

Proof. Let B be a well-ordered set of the partially ordered set such that there exists an upper bound u of B with $u \notin B$. Let B^* be the set of all such upper bounds of B . Let f be a choice function defined over all such B^* with $f(B^*) \in B^*$. Define $g(B) = f(B^*)$. Then any upper bound of U (U is a well-ordered set; therefore, the hypothesis for Zorn's Lemma is applicable) is a maximal element. Otherwise there would be an upper bound x_0 of U such that $g(U) = x_0 \notin U$.

Corollary 2—Hausdorff's Theorem. *Every partially ordered set contains a maximal chain.*

Proof. Let P be the partially ordered set. Let C be the class of all chains contained in P . Order C by set inclusion. Let $B \subset C$ such that there exists an upper bound u of B with $u \in C$, but $u \notin B$. Let B^* be the set of all such upper bounds of B . Let f be a choice function defined over all such B^* with $f(B^*) \in B^*$. Define $g(B) = f(B^*)$. Then $\bigcup_{c \in U} c$ is an upper bound of U . Moreover, $\bigcup_{c \in U} c$ is a maximal element; otherwise, there would be an upper bound $x_0 (\in C)$ of U such that $g(U) = x_0 \notin U$.

Corollary 3—Transfinite Recursion Theorem. *If W is a well-ordered set, and if f is a sequence function of type W in a set X , then there exists a unique function $U(a)$ from W into X such that $U(a) = f(U^a)$ for each a in W .*

The terminology used in the statement and proof of this corollary follows that of Halmos [2; p. 70].

Proof. Let $\sigma^{(a)}$ be a sequence of type a in X defined for all $a \in W$. The sequences $\sigma^{(a)}$ are thought of as subsets of $W \times X$. Let f be a sequence function of type W in X such that $f(\sigma^{(a)}) = x$ with $x \in X$. Define $g(\sigma^{(a)}) = (a, f(\sigma^{(a)}))$. Thus g maps a subset of $P(W \times X)$ into $W \times X$. g determines a U (in the notation of Tong's Lemma). It follows from the fact that $g(S_U(\alpha)) = \alpha$ and the definition of g that members of U are of the form $(a, f(U^a))$. Thus U may be viewed as the function $U(a)$ such that $U(a) = f(U^a)$. The uniqueness of $U(a)$ follows from the unique way in which U is obtained.

The central role that Tong's Lemma plays becomes evident not only from the above corollaries, but also from the use of similar arguments to produce Zermelo's Well-Ordering Theorem, Tukey's Lemma, and other so-called maximal principles. The details for these proofs will not be given here as indicated above. Some of the ideas in the proof of Tong's Lemma are similar to some in Zermelo's first proof of the Well-Ordering Theorem [3]. Kneser [4] was the first to give a proof of Corollary 1 based on the ideas of Zermelo. However, the importance of the

generalizations of the lemma above (which was independent from Zermelo's work), and its versatility as a unifying proof method were first pointed out by H. Tong in a communication to the present writer in 1965. These points will be developed further in future publications.

BIBLIOGRAPHY

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Fairfield University
Fairfield, Connecticut
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