*Reflection of Elastic Waves from Cylindrical Surfaces**

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0. Introduction. We shall consider the propagation of elastic waves and their reflection from a rigid cylinder having an arbitrary smooth curve as its cross section, according to the linear theory of elasticity. The propagation of a large class of waves can be described by means of the “progressing wave” expansion, which is explained in Section 1. The special case of time-independent progressing elastic waves was previously considered by Karal and Keller [1]. In Section 2 we show how to find the two reflected progressing waves, one a shear wave and the other a compressional wave, which result when an incident progressing wave hits a rigid cylinder. In Section 3 we calculate the reflected waves in detail for the special but important case of time independent progressing waves, which includes time harmonic waves. Although the method we use could be applied to inhomogeneous media, we shall treat only the homogeneous case.

The theory of progressing waves involves geometrical optics concepts such as rays, phase functions, amplitude functions, transport equations, laws of reflection, etc. Thus it is an asymptotic theory, valid for the short wavelength or high frequency components of a wave motion. When applied to transient waves or pulses, it is valid for the rapidly varying portions of them, such as the first motions. The theory of scalar progressing waves is given in detail by Lewis [2]. The diffraction of scalar progressing waves by the edge of a thin screen was considered by Ahluwalia and Lewis [3]. The reflection of scalar time-harmonic waves from a cylinder was treated by Keller, Lewis and Seckler [4], and their results apply also to time independent progressing waves. The reflection of time-

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independent progressing elastic waves from a rigid cylinder was considered by Resende [5] in an unpublished thesis, of which the present work is an extension. We shall utilize the methods and some results of references [1], [2] and [4] but nevertheless the present work can be read independently.

Progressing wave solutions of general hyperbolic equations have been discussed by Courant–Hilbert [6] and Ludwig [7], and have been studied in acoustics by Friedlander [8], [9]. Special reflection problems for elastic waves have been considered by various authors. In particular Gilbert [10] determined the leading term (*i.e.* the geometrical optics term) in the reflection of a time harmonic wave from a rigid or free cylinder.

In the final section we discuss our results and the use of the present method to solve other problems.

1. Progressing waves. In a homogeneous isotropic medium the displacement vector $\mathbf{u}$ can be written in terms of a scalar potential v_1 and a vector potential $\mathbf{v}_2$ in the form

$$(1.1) \quad \mathbf{u} = \nabla v_1 + \nabla \times \mathbf{v}_2.$$

The potentials satisfy the wave equations

$$(1.2) \quad L_i v_i = 0, \quad j = 1, 2.$$

Here L_i and the propagation speed c_i are defined by

$$(1.3) \quad L_i = \Delta - c_i^{-2} \partial_t^2, \quad c_1^2 = (\lambda + 2\mu)/\rho, \quad c_2^2 = \mu/\rho.$$

The constants ρ , λ and μ are the density and the two Lamé constants of the medium respectively.

A progressing wave is a solution v_i of (1.2) of the form

$$(1.4) \quad v_i(t, \mathbf{x}) = \sum_{m=0}^{\infty} e_i^m[s_i(t, \mathbf{x})] z_i^m(t, \mathbf{x})$$

The functions e_i^m are assumed to be related by

$$(1.5) \quad e_i^m(s) = \frac{d}{ds} e_i^{m+1}(s).$$

Thus they are all iterated integrals of $e_i^0(s)$, which determines the wave form. Upon substituting (1.4) into (1.2) and using (1.5), we find that (1.2) is satisfied provided the phase function s_i and the amplitude functions z_i^m satisfy the equations

$$(1.6) \quad (\nabla s_i)^2 - c_i^{-2} (\partial_t s_i)^2 = 0$$

$$(1.7) \quad 2[\nabla s_i \cdot \nabla z_i^m - c_i^{-2} (\partial_t s_i)(\partial_t z_i^m)] + z_i^m L_i s_i = -L_i z_i^{m-1},$$

$$m = 0, 1, 2, \dots; \quad z_i^{-1} \equiv 0.$$

Eq. (1.6) is the characteristic equation of (1.2), and (1.7) are the transport equations.

These equations can be solved by means of the characteristic curves of (1.6) which are also called the bicharacteristics of (1.2). They are straight lines which can be written in terms of a parameter σ in the form

$$(1.8) \quad t = \tilde{t} + c_i^{-1}(\sigma - \tilde{\sigma})$$

$$(1.9) \quad \mathbf{x} = \tilde{\mathbf{x}} + \mathbf{U}_i(\sigma - \tilde{\sigma}), \quad \mathbf{U}_i^2 = 1.$$

Here $\tilde{t}$, $\tilde{\mathbf{x}}$, $\tilde{\sigma}$ and the unit vector $\mathbf{U}_i$ are constants on each bicharacteristic. The straight lines (1.9) in $\mathbf{x}$ space are called rays. Both (1.6) and (1.7) can be written as linear ordinary differential equations along bicharacteristics and therefore they can be solved explicitly with the results [2]

$$(1.10) \quad s_i[t(\sigma), \mathbf{x}(\sigma)] = s_i[\tilde{t}, \tilde{\mathbf{x}}]$$

$$(1.11) \quad z_i^m[t(\sigma), \mathbf{x}(\sigma)] = \left[\frac{(\tilde{\sigma} + \rho_{i1})(\tilde{\sigma} + \rho_{i2})}{(\sigma + \rho_{i1})(\sigma + \rho_{i2})} \right]^{1/2} z_i^m(\tilde{\sigma}) \\ + \frac{c_i}{2 \partial_t s_i(\tilde{\sigma})} \int_{\tilde{\sigma}}^{\sigma} \left[\frac{(\sigma' + \rho_{i1})(\sigma' + \rho_{i2})}{(\sigma + \rho_{i1})(\sigma + \rho_{i2})} \right]^{1/2} L_i z_i^{m-1}(\sigma') d\sigma', \\ m = 0, 1, 2, \dots$$

In these equations $t(\sigma)$ and $\mathbf{x}(\sigma)$ denote the right sides of (1.8) and (1.9) and we have written $z_i^m(\sigma)$ to denote $z_i^m[t(\tilde{\sigma}), \mathbf{x}(\tilde{\sigma})]$. In (1.11), ρ_{i1} and ρ_{i2} , which are independent of σ , are defined in terms of a three parameter family of bicharacteristics $t(\sigma, \sigma_1, \sigma_2, \sigma_3)$, $\mathbf{x}(\sigma, \sigma_1, \sigma_2, \sigma_3)$ with parameters $\sigma_1, \sigma_2, \sigma_3$. The Jacobian of the transformation from $\sigma, \sigma_1, \sigma_2, \sigma_3$ to $t, \mathbf{x}$ is, with a_i independent of σ ,

$$(1.12) \quad \partial(t, \mathbf{x})/\partial(\sigma, \sigma_1, \sigma_2, \sigma_3) = a_i^{-1}(\sigma + \rho_{i1})(\sigma + \rho_{i2}).$$

These quantities ρ_{i1} and ρ_{i2} are the principal radii of curvature of the wavefront $s_i(\tilde{t}, \mathbf{x}) = \text{constant}$ at the point $\sigma = 0$ on the ray. In two space dimensions one radius, say ρ_{i2} , may be taken to be infinite and the other can be denoted by ρ_i . In this case a two parameter family of bicharacteristics is used to define ρ_i . The initial values $s_i(\sigma)$ and $z_i^m(\sigma)$ on each bicharacteristic are arbitrary. The wave v_i is given by (1.4) with s_i given by (1.10) and z_i^m by (1.11).

We shall now specialize the preceding results to a wave in two space dimensions. We represent the bicharacteristics by (1.8) and (1.9) with $\tilde{\sigma} = 0$, $\tilde{\mathbf{x}} = \mathbf{y}(\eta)$, $\tilde{t} = \tau$, and $\mathbf{U}_i = \mathbf{U}_i(\tau, \eta)$. Then σ denotes distance along a ray measured from the fixed curve $\mathbf{y}(\eta)$ and (1.8), and (1.9) become

$$(1.13) \quad t = \tau + c_i^{-1}\sigma, \quad \mathbf{x} = \mathbf{y}(\eta) + \sigma \mathbf{U}_i(\tau, \eta).$$

This is a two parameter family of bicharacteristics with τ and η as the parameters. We choose η to be arclength along the curve $\mathbf{y}(\eta)$ so that $(d\mathbf{y}/d\eta)^2 = 1$. We next define the angle $\phi_i(\tau, \eta)$ between the normal to the curve $\mathbf{y}(\eta)$ and the unit vector $\mathbf{U}_i(\tau, \eta)$ along a ray by

$$(1.14) \quad \mathbf{U}_i(\tau, \eta) \cdot \frac{d\mathbf{y}(\eta)}{d\eta} = \sin \phi_i(\tau, \eta).$$

Then the Jacobian of the transformation (1.13) from σ, τ, η to $t, \mathbf{x}$ can be written in the form

$$(1.15) \quad \frac{\partial(t, \mathbf{x})}{\partial(\sigma, \tau, \eta)} = a_i^{-1}(\sigma + \rho_i).$$

Now a_i and the radius of curvature ρ_i are given in terms of ϕ and the signed curvature $\kappa(\eta)$ of the curve $\mathbf{y}(\eta)$ by

$$(1.16) \quad a_i^{-1} = \partial_\eta \phi_i + \kappa(\eta) + c_i^{-1}(\partial_\tau \phi_i) \sin \phi_i, \quad \rho_i = a_i \cos \phi_i.$$

If $\theta(\eta)$ is the angle between the tangent $d\mathbf{y}(\eta)/d\eta$ to the curve $\mathbf{y}(\eta)$ and the positive x_1 -axis, then $\kappa(\eta) = d\theta(\eta)/d\eta$.

In the two dimensional case (1.11) becomes

$$(1.17) \quad z_i^m(\sigma) = \left[\frac{\rho_i}{\sigma + \rho_i} \right]^{1/2} z_i^m(0) + \frac{c_i}{2 \partial_\tau s_i(0)} \int_0^\sigma \left[\frac{\sigma' + \rho_i}{\sigma' + \rho_i} \right]^{1/2} L_i z_i^{m-1}(\sigma') d\sigma'.$$

We shall now prove by induction that in two dimensions $z_i^m(\sigma)$, given by (1.17), has the form

$$(1.18) \quad z_i^m(\sigma) = \sum_{q=0}^{3m} F_{iq}^m(\tau, \eta) [\sigma + \rho_i(\tau, \eta)]^{-q-1/2}.$$

For $m = 0$ this follows from (1.17), since $z_i^{-1} = 0$ by definition. We now assume that (1.18) holds for $m - 1$ and use it in the integrand in (1.17). We use for the operator L_i its expression in terms of σ, τ and η , given by (A.3) of the appendix. Then we find that $z_i^m(\sigma)$ is indeed given by (1.18) which completes the induction. At the same time we obtain a recursion formula for the coefficients $F_{iq}^m(\tau, \eta)$ in (1.18), which determines them in terms of $F_{i0}^0(\tau, \eta)$.

We shall omit it because it is lengthy. The special case of this recursion formula for time-independent progressing waves will be given in Section 3. Now the wave v_i is given by (1.4) with s_i given by (1.10) and z_i^m given (1.18). In terms of the parameters σ, τ, η these waves can be written

$$(1.19) \quad v_i(t, \mathbf{x}) = \sum_{m=0}^{\infty} e_i^m[s_i(\tau, \mathbf{y}(\eta))] \sum_{q=0}^{3m} F_{iq}^m(\tau, \eta) [\sigma + \rho_i(\tau, \eta)]^{-q-1/2}.$$

Here t and $\mathbf{x}$ are given by (1.13).

2. Reflection of progressing waves. Let $\mathbf{x} = \mathbf{y}(\eta)$ be the equation of a stationary rigid cylindrical surface C upon which is incident the two dimensional progressing wave

$$(2.1) \quad v_0(t, \mathbf{x}) = \sum_{m=0}^{\infty} e_0^m[s_0(t, \mathbf{x})] z_0^m(t, \mathbf{x}).$$

We shall write the reflected field in the form (1.1) with v_1 and v_2 given by (1.4). Then we shall adjust the arbitrary quantities in these progressing waves so that the total displacement vanishes on the cylinder. For definiteness we shall assume

that the incident wave (2.1) is a compressional wave, *i.e.* a wave like v_1 in Section 1, although the case of an incident shear wave can be treated in a very similar way. In the rest of this work, $\mathbf{x} = (x_1, x_2)$ denotes a point in two dimensional space.

The reflected field will have no displacement in the x_3 direction, as we can see from symmetry, so the vector potential will have only an x_3 component. Therefore we shall write $\mathbf{v}_2 = (0, 0, v_2)$. Then the total displacement $\mathbf{u}$ is

$$(2.2) \quad \mathbf{u} = [\partial_{x_1}(v_0 + v_1) + \partial_{x_2}v_2, \partial_{x_3}(v_0 + v_1) - \partial_{x_1}v_2].$$

Now the boundary condition of no displacement on the cylinder becomes

$$(2.3) \quad \partial_{x_1}(v_0 + v_1) + \partial_{x_2}v_2 = 0 \quad \text{on } C,$$

$$(2.4) \quad \partial_{x_3}(v_0 + v_1) - \partial_{x_1}v_2 = 0 \quad \text{on } C.$$

We next substitute (1.4) and (2.1) for v_1, v_2 and v_0 into (2.3) and (2.4). We find that (2.3) and (2.4) will be satisfied provided that the following equations hold:

$$(2.5) \quad e_1^m = e_2^m = e_0^m, \quad m = 0, 1, \dots,$$

$$(2.6) \quad s_1[t, \mathbf{y}(\eta)] = s_2[t, \mathbf{y}(\eta)] = s_0[t, \mathbf{y}(\eta)].$$

$$(2.7) \quad z_1^m \partial_{x_1}s_1 + z_2^m \partial_{x_2}s_2 = -z_0^m \partial_{x_3}s_0 - \partial_{x_1}z_0^{m-1} - \partial_{x_1}z_1^{m-1} - \partial_{x_2}z_2^{m-1} \quad \text{on } C,$$

$$m = 0, 1, \dots,$$

$$(2.8) \quad z_1^m \partial_{x_2}s_1 - z_2^m \partial_{x_1}s_2 = -z_0^m \partial_{x_2}s_0 - \partial_{x_2}z_0^{m-1} - \partial_{x_1}z_1^{m-1} - \partial_{x_2}z_2^{m-1} \quad \text{on } C,$$

$$m = 0, 1, \dots$$

Differentiating (2.6) with respect to t and with respect to η yields

$$(2.9) \quad \partial_t s_1 = \partial_t s_2 \quad \text{on } C.$$

$$(2.10) \quad \nabla s_1 \cdot \frac{d\mathbf{y}}{d\eta} = \nabla s_2 \cdot \frac{d\mathbf{y}}{d\eta} = \nabla s_0 \cdot \frac{d\mathbf{y}}{d\eta} \quad \text{on } C.$$

It follows from the theory of characteristics applied to (1.6) that the ray direction $\mathbf{U}_j$ is the direction of ∇s_j for $j = 1$ and $j = 2$ and also for $j = 0$, since the wave v_0 is of the same type as v_1 . The length of ∇s_i is $c_i^{-1} |\partial_t s_i|$ as we see from (1.6), so we have

$$(2.11) \quad \nabla s_i = c_i^{-1} |\partial_t s_i| \mathbf{U}_i.$$

We now use (2.11) for ∇s_i in (2.10) and make use of (2.9) to obtain

$$(2.12) \quad c_1^{-1} \mathbf{U}_1 \cdot \frac{d\mathbf{y}}{d\eta} = c_2^{-1} \mathbf{U}_2 \cdot \frac{d\mathbf{y}}{d\eta} = c_1^{-1} \mathbf{U}_0 \cdot \frac{d\mathbf{y}}{d\eta} \quad \text{on } C.$$

Upon introducing ϕ_i defined by (1.14), we obtain from (2.12) the two equations

$$(2.13) \quad \sin \phi_1 = \sin \phi_0, \quad c_2^{-1} \sin \phi_2 = c_1^{-1} \sin \phi_0.$$

These are the laws of reflection of elasticity. From the definitions of ϕ_i , if we choose the normal pointing into the cylinder, then $\pi/2 < \phi_1 < \pi$, $\pi/2 < \phi_2 < \pi$ and $0 < \phi_0 < \pi/2$. (See Figure 1.) It follows from (2.13) that

$$(2.14) \quad \phi_1 = \pi - \phi_0, \quad \phi_2 = \pi - \sin^{-1} \left[\left(\frac{\mu}{\lambda + 2\mu} \right)^{1/2} \sin \phi_0 \right].$$

These results determine the direction of the reflected bicharacteristics and of their projections into x -space, the reflected rays. Then s_1 and s_2 are constant along these bicharacteristics with the constant value $s_0[t, y(\eta)]$ given by (2.6). In this way s_1 and s_2 are determined at all points of the reflected waves.

Let $\theta(\eta)$ be the previously defined angle between $dy/d\eta$, the tangent to C , and the positive x_1 axis. Then we can write $\underline{U}_i$ in the form

$$(2.15) \quad \underline{U}_i = [\sin(\phi_i + \theta), -\cos(\phi_i + \theta)].$$

By using (2.9) and (2.15), we can write (2.11) in the form

$$(2.16) \quad \nabla s_i = c_i^{-1} |\partial_i s_0| [\sin(\phi_i + \theta), -\cos(\phi_i + \theta)].$$

Now we can solve (2.7) and (2.8) for $z''_i(0)$, using (2.16) to eliminate ∇s_i and the

FIGURE 1. An incident ray in the direction $\underline{U}_0$ is shown hitting the curve C which bounds the cross section of a reflecting cylinder. The reflected compressional ray in the direction $\underline{U}_1$ and the reflected shear ray in the direction $\underline{U}_2$ are indicated, as are the directions of the x_1 and x_2 axes. The angle between the tangent $\underline{T}$ to C and the positive x_1 -axis is denoted by θ .

first equation in (2.14) to eliminate ϕ_2 . Then we obtain

$$(2.17) \quad z_1^m(0) = R_1 z_0^m(0) + c_1 [|\partial_t s_0| \cos(\phi_2 + \phi_0)]^{-1} \\ \cdot [(\partial_{x_2} z_0^{m-1} + \partial_{x_2} z_1^{m-1} - \partial_{x_1} z_2^{m-1}) \cos(\phi_2 + \theta) \\ - (\partial_{x_1} z_0^{m-1} + \partial_{x_1} z_1^{m-1} + \partial_{x_2} z_2^{m-1}) \sin(\phi_2 + \theta)],$$

$$(2.18) \quad z_2^m(0) = R_2 z_0^m(0) + c_2 [|\partial_t s_0| \cos(\phi_2 + \phi_0)]^{-1} \\ \cdot [(\partial_{x_2} z_0^{m-1} + \partial_{x_2} z_1^{m-1} - \partial_{x_1} z_2^{m-1}) \sin(\phi_0 - \theta) \\ - (\partial_{x_1} z_0^{m-1} + \partial_{x_1} z_1^{m-1} + \partial_{x_2} z_2^{m-1}) \cos(\phi_0 - \theta)].$$

Here the reflection coefficients R_i are defined by

$$(2.19) \quad R_1 = \frac{\cos(\phi_2 - \phi_0)}{\cos(\phi_2 + \phi_0)}, \quad R_2 = \frac{c_2 \sin 2\phi_0}{c_1 \cos(\phi_2 + \phi_0)}.$$

By using $z_i^m(0)$ given by (2.17) and (2.18) in (1.17), we obtain the $z_i^m(\sigma)$ at all points of the reflected waves. For $m = 0$, we find

$$(2.20) \quad z_i^0(\sigma) = \left[\frac{\rho_i}{\sigma + \rho_i} \right]^{1/2} R_i z_0^0(0).$$

Then (1.18) and (2.20) yield

$$(2.21) \quad F_{i0}^0(\tau, \eta) = \rho_i^{1/2} R_i z_0^0(0).$$

The other F_{i0}^m can be found recursively from F_{i0}^0 , given by (2.21).

We can now write the reflected potentials in the form (1.19) with $e_i^m[s_i(\tau, \mathbf{y}(\eta))]$ replaced by $e_0^m[s_0(\tau, \mathbf{y}[\eta])]$. The reflected bicharacteristics are given by (1.13) with $\mathbf{U}_i$ given by (2.15).

3. Reflection of time-independent progressing waves. A progressing wave of the form (1.4) is called time-independent if z_i^m and s_i are of the form

$$(3.1) \quad z_i^m(t, \mathbf{x}) = z_i^m(\mathbf{x}), \quad s_i(t, \mathbf{x}) = t - c_i^{-1} \psi_i(\mathbf{x}).$$

It follows from (3.1) and (1.6) that ψ_i satisfies the eiconal equation

$$(3.2) \quad (\nabla \psi_i)^2 = 1.$$

From (1.10) and (3.1) it follows that along a bicharacteristic ψ_i is given by

$$(3.3) \quad \psi_i[\mathbf{x}(\sigma)] = \sigma - \bar{\sigma} + \psi_i[\mathbf{x}(\bar{\sigma})].$$

From (3.1) we have $\partial_t s_i = 1$ and $L_i z_i^{m-1} = \Delta z_i^{m-1}$ so (1.11) and (1.17) become slightly simpler. Upon combining these results we find that on a bicharacteristic

$$(3.4) \quad v_i(t, \mathbf{x}) = \sum_{m=0}^{\infty} e_i^m[t - c_i^{-1}(\sigma - \bar{\sigma} + \psi_i\{\mathbf{x}(\bar{\sigma})\})] \\ \cdot \left\{ [J(\bar{\sigma})/J(\sigma)]^{1/2} z_i^m(\sigma) + \frac{c_i}{2} \int_{\bar{\sigma}}^{\sigma} [J(\sigma')/J(\sigma)]^{1/2} \Delta z_i^{m-1}(\sigma') d\sigma' \right\}.$$

The values of t and $\mathbf{x}$ are given by (1.8) and (1.9).

For a two dimensional time-independent progressing wave, (1.18) and (1.19) simplify because $F_{i,q}^m$ is independent of τ and s_i is given by (3.1). It is convenient to set $F_{i,q}^m(\eta) = (c_i)^m f_{i,q}^m(\eta)$ and then (1.18) and (1.19) become

$$(3.5) \quad z_i^m(\sigma) = (c_i)^m \sum_{q=0}^{3m} f_{i,q}^m(\eta) [\sigma + \rho_i(\eta)]^{-q-1/2},$$

$$(3.6) \quad v_i(t, \mathbf{x}) = \sum_{m=0}^{\infty} (c_i)^m e_0^m [t - c_i^{-1}(\sigma + \psi_i\{\mathbf{y}(\eta)\})] \cdot \sum_{q=0}^{3m} f_{i,q}^m(\eta) [\sigma + \rho_i(\eta)]^{-q-1/2}.$$

In writing (3.6) we have made use of (2.5). The values of t and $\mathbf{x}$ are given by (1.13). It also follows from (3.1) that $\mathbf{U}_i(\eta)$ is independent of τ and therefore $\phi_i(\eta)$, $b_i(\eta)$ and $\rho_i(\eta)$ are also independent of τ .

To obtain the recursion formula for the $f_{i,q}^m$ in (3.5) we insert z_i^m into (1.17). Upon using the facts that $\partial_i s_i = 1$ and $L_i z_i^{m-1} = \Delta z_i^{m-1}$, and the representation of Δ in (A.10) we obtain

$$(3.7) \quad f_{i,q}^m = \frac{1}{2q} [a_i \partial_\eta (a_i \partial_\eta f_{i,q-1}^{m-1}) + 2(q-1)b_i a_i \partial_\eta f_{i,q-2}^{m-1} + (q-1/2)^2 f_{i,q-1}^{m-1} \\ + (q-3/2)a_i (\partial_\eta b_i) f_{i,q-2}^{m-1} + b_i^2 (q-5/2)(q-1/2) f_{i,q-3}^{m-1}], \\ q, m = 1, 2, \dots; \quad j = 1, 2.$$

Here b_i is defined by

$$(3.8) \quad b_i = a_i (\sin \phi_i - \partial_\eta \rho_i).$$

In addition

$$(3.9) \quad f_{i,q}^m = 0 \quad \text{for } q < 0, \quad q > 3m.$$

The result (3.5) and the recursion formulas (3.7)–(3.9) were obtained by Keller, Lewis and Seckler [4] who used slightly different independent variables. To obtain (3.7) from their (18.4) we must replace their ∂_θ by $a_i \partial_\eta$ and their b by our b_i .

Since (3.7) does not hold for $q = 0$, we set $\sigma = 0$ in (3.5) and solve the resulting equation for $f_{i,0}^m$ to obtain

$$(3.10) \quad f_{i,0}^m(\eta) = (c_i)^{-m} \rho_i^{1/2} z_i^m(0, \eta) - \sum_{q=1}^{3m} f_{i,q}^m(\eta) \rho_i^{-q}(\eta).$$

Now $z_i^m(0)$ is given by (2.17) and (2.18) in which $\partial_i s_i = 1$. However these equations involve $\partial_{x_i} z_i^{m-1}(0)$ and $\partial_{x_a} z_i^{m-1}(0)$, which we wish to replace by expressions involving $f_{i,q}^{m-1}$. To this end we differentiate the expression (3.5) for $z_i^m(\sigma)$ with respect to x_i , $i = 1, 2$ and then set $\sigma = 0$ to get

$$(3.11) \quad \partial_{x_i} z_i^m(0) = -(\partial_{x_i} \sigma) \sum_{q=0}^{3m} (q+1/2) f_{i,q}^m \rho_i^{-q-3/2} \\ + (\partial_{x_i} \eta) \sum_{q=0}^{3m} [\rho_i^{-q-1/2} \partial_\eta f_{i,q}^m - (q+1/2) f_{i,q}^m \rho_i^{-q-3/2} \partial_\eta \rho_i].$$

The derivatives $\partial_{x_i}\sigma$ and $\partial_{x_i}\eta$ can be obtained from the ray equation (1.13) in which $\mathbf{U}_i$ is given by (2.15). Since this equation involves j , the derivatives depend upon j . These derivatives are calculated in the appendix and are given by (A.4), (A.5), (A.7) and (A.8) in which we must set $\sigma = 0$ and $\partial_r\phi = 0$. Then we find

$$(3.12) \quad \partial_{x_1}\sigma = \sin(\phi_i + \theta) - a_i\rho_i^{-1} \cos(\phi_i + \theta) \sin\phi_i,$$

$$(3.13) \quad \partial_{x_2}\sigma = -\cos(\phi_i + \theta) - a_i\rho_i^{-1} \sin(\phi_i + \theta) \sin\phi_i,$$

$$(3.14) \quad \partial_{x_1}\eta = a_i\rho_i^{-1} \cos(\phi_i + \theta), \quad \partial_{x_2}\eta = a_i\rho_i^{-1} \sin(\phi_i + \theta).$$

We now use (3.12)–(3.14) in (3.11) to obtain $\partial_{x_i}z_i^m(0)$ in terms of f_{ia}^m , $q = 0, \dots, 3m$. Then we use $\partial_{x_i}z_i^{m-1}(0)$ in (2.17) and (2.18) to obtain

$$(3.15) \quad z_1^m(0) = R_1 z_0^m(0) + c_1 \sec(\phi_2 + \phi_0) \\ \cdot \{-\partial_{\sigma_s} z_0^{m-1} + A_1^{m-1} \cos(\phi_2 - \phi_1) - B_1^{m-1} \sin(\phi_2 - \phi_1) - B_2^{m-1}\},$$

$$(3.16) \quad z_2^m(0) = R_2 z_0^m(0) + c_2 \sec(\phi_2 + \phi_0) \\ \cdot \{\partial_{r_i} z_0^{m-1} - A_2^{m-1} \cos(\phi_2 + \phi_0) - B_2^{m-1} \sin(\phi_2 + \phi_0) + B_1^{m-1}\}.$$

The quantities A_i^{m-1} and B_i^{m-1} are defined by

$$(3.17) \quad A_i^{m-1}(\eta) = \sum_{q=0}^{3(m-1)} (q+1/2) f_{ia}^{m-1} \rho_i^{-q-3/2},$$

$$(3.18) \quad B_i^{m-1}(\eta) = \sum_{q=0}^{3(m-1)} (q+1/2) f_{ia}^{m-1} \rho_i^{-q-5/2} + a \sum_{q=0}^{3(m-1)} \rho_i^{-q-3/2} \partial_{\eta} f_{ia}^{m-1}.$$

In addition $A_i^{-1} = B_i^{-1} = 0$. In (3.15) $\partial_{\sigma_s} z_0^{m-1}$ denotes the derivative of z_0^{m-1} with respect to σ along a reflected shear ray and in (3.16) $\partial_{r_i} z_0^{m-1}$ is the derivative of z_0^{m-1} in the direction normal to a reflected compressional ray. In terms of x_1 and x_2 derivatives,

$$(3.19) \quad \partial_{\sigma_s} z_0^{m-1} = \sin(\phi_2 + \theta) \partial_{x_1} z_0^{m-1} - \cos(\phi_2 + \theta) \partial_{x_2} z_0^{m-1},$$

$$(3.20) \quad \partial_{r_i} z_0^{m-1} = \cos(\phi_1 + \theta) \partial_{x_1} z_0^{m-1} + \sin(\phi_1 + \theta) \partial_{x_2} z_0^{m-1}.$$

The f_{ia}^m can now be obtained recursively by using (3.7) for $q, m = 1, 2, \dots$ and (3.10) for $q = 0$ with $z_i^m(0)$ given by (3.15)–(3.18). Then $z_i^m(\sigma)$ is given by (3.5), $v_i(t, \mathbf{x})$ is given by (3.6) and $\mathbf{x}$ is given by (1.13) in which $\mathbf{U}_i(\eta)$ is given by (2.15). The total displacement $\mathbf{u}$ is now given by (2.2) in which v_0 is given by (2.1) while v_1 and v_2 are given by (3.6) in the manner just described.

To express these potentials at a point $\mathbf{x}$ it is convenient to denote by σ_1 and η_1 the values of σ and η on a reflected compressional ray at $\mathbf{x}$ and by σ_2 and η_2 the corresponding quantities on a reflected shear ray at $\mathbf{x}$. Thus $\mathbf{y}(\eta_i)$ is the point of reflection of a ray of type j which passes through $\mathbf{x}$ and σ_i is the distance along this ray from $\mathbf{y}(\eta_i)$ to $\mathbf{x}$. Then we can write

$$\begin{aligned}
(3.21) \quad v_0(t, \mathbf{x}) + v_1(t, \mathbf{x}) &= \sum_{m=0}^{\infty} (c_2)^m e_0^m [t - c_2^{-1} \psi_0(\mathbf{x})] z_0^m(\mathbf{x}) \\
&\quad + \sum_{m=0}^{\infty} (c_1)^m e_0^m [t - c_1^{-1} \{\psi_0[\mathbf{y}(\eta)] + \sigma_1\}] \sum_{q=0}^{3m} f_{1q}^m(\eta_1) [\sigma_1 + \rho_1(\eta_1)]^{-q-1/2}, \\
(3.22) \quad v_2(t, \mathbf{x}) &= \sum_{m=0}^{\infty} (c_2)^m e_0^m [t - c_2^{-1} \{\psi_0[\mathbf{y}(\eta_2)] + \sigma_2\}] \sum_{q=0}^{3m} f_{2q}^m(\eta_2) [\sigma_2 + \rho_2(\eta_2)]^{-q-1/2}.
\end{aligned}$$

If there are more than one reflected ray of either type through $\mathbf{x}$, we must sum over the corresponding values of σ_1 and η_1 in (3.21) and over the values of σ_2 and η_2 in (3.22).

As an illustration of the preceding method, we shall consider the incident plane compressional wave with potential v_0 given by

$$(3.23) \quad v_0(t, x) = e_0^0(t - c_1^{-1}x_1).$$

This is an exact solution of the wave equation $L_1 v_0 = 0$ for any e_0^0 . From (3.23) we see that v_0 represents a time-independent progressing wave with

$$(3.24) \quad \psi_0(\mathbf{x}) = x_1, \quad z_0^m(\mathbf{x}) = \delta_{m0}.$$

The unit vector $\mathbf{U}_0$ in the ray direction is just $\nabla \psi_0 = (1, 0)$, which is along the positive x_1 axis. Thus (1.14) yields $\sin \phi_0(\eta) = dy_1(\eta)/d\eta$. On the other hand from the definition of $\theta(\eta)$, $\cos \theta(\eta) = dy_1/d\eta$. Therefore $\sin \phi_0 = \cos \theta$ so $\phi_0 = \pi/2 - \theta$. Differentiating this relation and recalling that $d\theta/d\eta = \kappa$ yields

$$(3.25) \quad -d\phi_0/d\eta = \kappa(\eta).$$

Then (2.14) and (3.25) yield

$$(3.26) \quad \frac{d\phi_1}{d\eta} = \kappa, \quad \frac{d\phi_2}{d\eta} = -\kappa \frac{c_2 \cos \phi_0}{c_1 \cos \phi_2}.$$

Upon using (3.26) in (1.16) we obtain

$$(3.27) \quad a_1 = (2k)^{-1}, \quad a_2 = c_1 \cos \phi_2 [k(c_1 \cos \phi_2 - c_2 \cos \phi_0)]^{-1}.$$

Let us now calculate $z_i^m(0)$ from (3.15) and (3.16). For $m = 0$, (3.24) yields $z_0^0 = 1$ and since $z_0^{-1} = A_i^{-1} = B_i^{-1} = 0$, we obtain

$$(3.28) \quad z_i^0(0) = R_i.$$

For $m > 0$, $z_0^m = 0$ and $\partial_{\sigma_1} z_0^{m-1} = \partial_{\nu_1} z_0^{m-1} = 0$ so (3.15) and (3.16) become

$$\begin{aligned}
(3.29) \quad z_1^m(0) &= c_1 \sec(\phi_2 + \phi_0) \\
&\quad \cdot \{A_1^{m-1} \cos(\phi_2 - \phi_1) - B_1^{m-1} \sin(\phi_2 - \phi_1) - B_2^{m-1}\},
\end{aligned}$$

$$\begin{aligned}
(3.30) \quad z_2^m(0) &= c_2 \sec(\phi_2 + \phi_0) \\
&\quad \cdot \{-A_2^{m-1} \cos(\phi_2 + \phi_0) - B_2^{m-1} \sin(\phi_2 + \phi_0) + B_1^{m-1}\}.
\end{aligned}$$

By using (1.16) we can replace ρ_i by $a_i \cos \phi_i$ in these equations and use (3.27) for a_i . To complete the determination of the solution we proceed in the manner described after (3.20).

4. Discussion. We have seen that in homogeneous isotropic media, elastic waves can be represented by progressing wave expansions, which are formal solutions of the linear equations of elasticity. Although they are infinite expansions in general, they are most useful when the leading term, or a few terms, suffice to describe a wave accurately. This is the case when the wave consists mainly of components with wavelengths short compared to the radii of curvature of the wavefronts. The leading term of the wave (1.4) on a bicharacteristic (1.8) and (1.9) is

$$(4.1) \quad v_i[\tilde{t} + c_i^{-1}(\sigma - \bar{\sigma}), \tilde{\mathbf{x}} + \mathbf{U}_i(\sigma - \bar{\sigma})] \\ \sim e_0^0[s_i(\tilde{t}, \tilde{\mathbf{x}})] \left[\frac{(\bar{\sigma} + \rho_{i1})(\bar{\sigma} + \rho_{i2})}{(\sigma + \rho_{i1})(\sigma + \rho_{i2})} \right]^{1/2} z_i^0(\tilde{t}, \tilde{\mathbf{x}}).$$

If we measure σ along a ray from $\tilde{\mathbf{x}}$ then $\bar{\sigma} = 0$ and (4.1) becomes

$$(4.2) \quad v_i[\tilde{t} + c_i^{-1}\sigma, \tilde{\mathbf{x}} + \mathbf{U}_i\sigma] \sim e_0^0[s_i(\tilde{t}, \tilde{\mathbf{x}})] \left[\frac{\rho_{i1}\rho_{i2}}{(\sigma + \rho_{i1})(\sigma + \rho_{i2})} \right]^{1/2} z_i^0(\tilde{t}, \tilde{\mathbf{x}}).$$

In two dimensions (4.2) becomes

$$(4.3) \quad v_i[\tilde{t} + c_i^{-1}\sigma, \tilde{\mathbf{x}} + \mathbf{U}_i\sigma] \sim e_0^0[s_i(\tilde{t}, \tilde{\mathbf{x}})] [\rho_i/(\sigma + \rho_i)]^{1/2} z_i^0(\tilde{t}, \tilde{\mathbf{x}}).$$

For the reflected waves in two dimensions, z_i^0 is given by (2.20) provided $\sigma = 0$ at the reflector. When that result is used, (4.3) becomes

$$(4.4) \quad v_i[\tau + c_i^{-1}\sigma, \mathbf{y}(\eta) + \mathbf{U}_i\sigma] \sim e_0^0[s_0(\tau, \mathbf{y}(\eta))] [\rho_i/(\sigma + \rho_i)]^{1/2} R_i z_0^0[\tau, \mathbf{y}(\eta)].$$

Completely analogous results can be obtained in three dimensions and for reflection from free surfaces, interfaces, etc. (See, for example, H. B. Keller [11].)

The progressing wave expansion can also be used to solve various propagation, reflection, refraction, diffraction, radiation and other problems. (See, for example, Keller and Lewis [12].) It can also be improved upon to eliminate its failure at caustics by the introduction of boundary layers or by the uniform method of Ludwig [13].

Appendix. Expressions for L_i and Δ in ray coordinates. In the case of two space dimensions, the bicharacteristics are given in terms of the ray coordinates τ, σ, η by (1.13). If in (1.13) we use (2.15) for $\mathbf{U}_i$ we obtain

$$(A.1) \quad \begin{aligned} x_1 &= y_1(\eta) + \sigma \sin(\phi_i + \theta) \\ x_2 &= y_2(\eta) - \sigma \cos(\phi_i + \theta) \\ t &= \tau + c_i^{-1}\sigma. \end{aligned}$$

We wish to express $L_i = \Delta - c_i^{-2} \partial_i^2$ in terms of ray coordinates. To this end we observe that

$$(A.2) \quad \begin{aligned} \partial_{x_i} &= (\partial_{x_i}\sigma)\partial_\sigma + (\partial_{x_i}\tau)\partial_\tau + (\partial_{x_i}\eta)\partial_\eta, \quad i = 1, 2 \\ \partial_t &= (\partial_t\sigma)\partial_\sigma + (\partial_t\tau)\partial_\tau + (\partial_t\eta)\partial_\eta. \end{aligned}$$

Thus

$$(A.3) \quad L_i = \sum_{i=1}^2 [(\partial_{x_i}\sigma)\partial_\sigma + (\partial_{x_i}\tau)\partial_\tau + (\partial_{x_i}\eta)\partial_\eta]^2 - c_i^{-2}[(\partial_t\sigma)\partial_\sigma + (\partial_t\tau)\partial_\tau + (\partial_t\eta)\partial_\eta]^2.$$

To determine the coefficients in (A.3) we differentiate (A.1) with respect to x_i , $i = 1, 2$, and with respect to t and solve the resulting equations to obtain

$$(A.4) \quad \begin{aligned} \partial_{x_i}\sigma &= -c_i \partial_{x_i}\tau = a_i(\partial_\eta\phi_i + \kappa) \sin(\phi_i + \theta) - (\sigma + \rho_i)^{-1}a_i^2 \\ &\quad \cdot [(\partial_\eta\phi_i + \kappa) \cos(\phi_i + \theta) - c_i^{-1} \partial_\tau\phi_i \sin \theta] \sin \phi_i, \end{aligned}$$

$$(A.5) \quad \begin{aligned} \partial_{x_i}\sigma &= -c_i \partial_{x_i}\tau = a_i(\partial_\eta\phi_i + \kappa) \cos(\phi_i + \theta) + (\sigma + \rho_i)^{-1}a_i^2 \\ &\quad \cdot [(\partial_\eta\phi_i + \kappa) \sin(\phi_i + \theta) + c_i^{-1} \partial_\tau\phi_i \cos \theta] \sin \phi_i, \end{aligned}$$

$$(A.6) \quad \partial_t\sigma = a_i \partial_\tau\phi_i \sin \phi_i [1 - (\sigma + \rho_i)^{-1}a_i \cos \phi_i],$$

$$(A.7) \quad \begin{aligned} \partial_{x_i}\eta &= a_i c_i^{-1} \partial_\tau\phi_i \sin(\phi_i + \theta) + (\sigma + \rho_i)^{-1}a_i^2 \\ &\quad \cdot [(\partial_\eta\phi_i + \kappa) \cos(\phi_i + \theta) - c_i^{-1} \partial_\tau\phi_i \sin \theta], \end{aligned}$$

$$(A.8) \quad \begin{aligned} \partial_{x_i}\eta &= -a_i c_i^{-1} \partial_\tau\phi_i \cos(\phi_i + \theta) + (\sigma + \rho_i)^{-1}a_i^2 \\ &\quad \cdot [(\partial_\eta\phi_i + \kappa) \sin(\phi_i + \theta) + c_i^{-1} \partial_\tau\phi_i \cos \theta], \end{aligned}$$

$$(A.9) \quad \partial_t\eta = -a_i \partial_\tau\phi_i [1 - (\sigma + \rho_i)^{-1}a_i \cos \phi_i],$$

$$(A.10) \quad \partial_t\tau = a_i(\partial_\eta\phi_i + \kappa) + (\sigma + \rho_i)^{-1}a_i^2 c_i^{-1} \partial_\tau\phi_i \sin \phi_i \cos \phi_i.$$

When L_i is applied to a function independent of t , it reduces to Δ . If ϕ_i is also independent of τ then

$$(A.11) \quad L_i = \Delta = \sum_{i=1}^2 [(\partial_{x_i}\sigma)\partial_\sigma + (\partial_{x_i}\eta)\partial_\eta]^2.$$

Upon using (A.4), (A.5), (A.7) and (A.8) in (A.11) and simplifying the result, we obtain

$$(A.12) \quad \begin{aligned} \Delta &= (\sigma + \rho_i)^{-2}a_i^2[\partial_\eta^2 - 2 \sin \phi_i \partial_\eta \partial_\sigma + \sin^2 \phi_i \partial_\sigma^2] \\ &\quad + \partial_\sigma^2 + [(\sigma + \rho_i)^{-2}a_i \partial_\eta a_i + a_i^2(\sin \phi_i - \partial_\eta \rho_i)(\sigma + \rho_i)^{-3}]\partial_\eta \\ &\quad + [(\sigma + \rho_i)^{-1} - (\sigma + \rho_i)^{-2}a_i(\partial_\eta a_i \sin \phi_i + \partial_\eta \phi_i a_i \cos \phi_i) \\ &\quad + (\sigma + \rho_i)^{-3}a_i^2(\partial_\eta \rho_i - \sin \phi_i) \sin \phi_i]\partial_\sigma. \end{aligned}$$

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