Removable Sets for Quasiconformal Mappings

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It is well known that an isolated boundary point is a removable singularity for quasiconformal mappings. This means that if D is a domain in the euclidean n -space R^n and if $b \in D$, then every quasiconformal mapping f of $D - \{b\}$ can be extended to a quasiconformal mapping of D . For $n = 2$ the proof is given in [3, p. 43], and the general case can be treated by a similar method. The purpose of this note is to show that this result remains true if the point b is replaced by a set of $(n - 1)$ -dimensional measure zero. For $n = 2$ this is well known, see [3, p. 216]. However, the 2-dimensional proof cannot be directly generalized to higher dimensions. Our proof is based on some results of Callender and Rešetnyak.

We first introduce some notations and terminology. Suppose that D is a domain in R^n . A mapping $f : D \rightarrow R^n$ is said to be ACL (absolutely continuous on lines) if for every $1 \leq i \leq n$ and for almost every line L , parallel to the x_i -axis, f is absolutely continuous on every compact interval of $L \cap D$. Note that f is not required to be continuous. An ACL-mapping f is said to be ACL^p , $p \geq 1$, if (1) f is locally integrable, (2) the partial derivatives $\partial_i f(x)$ exist a.e. and (3) these partial derivatives are locally L^p -integrable. In the literature, the class of ACL^p -mappings is often denoted by W_p^1 or by H_p^1 . If f is ACL^p , we denote by $f'(x)$ the linear mapping of R^n into R^n , defined by $f'(x)e_x = \partial_i f(x)$, where $e_1, \dots, e_n$ are the coordinate unit vectors. Furthermore, we use the notations

$$|f'(x)| = \max_{|h|=1} |f'(x)h|, \quad J(x, f) = \det f'(x).$$

Thus $f'(x)$ and $J(x, f)$ are defined a.e. in D .

Following Rešetnyak [5], we say that an ACL^n -mapping $f : D \rightarrow R^n$ is of bounded distortion if there is a finite constant K such that

$$(1) \quad |f'(x)|^n \leq KJ(x, f)$$

a.e. in D . If, in addition, f is a homeomorphism onto a domain $D' \subset R^n$, it is called quasiconformal. A quasiconformal mapping satisfies also a condition

$$(2) \quad J(x, f) \leq K \left(\min_{|h|=1} |f'(x)h| \right)^n$$

a.e. in D , not necessarily with the same K . The smallest K for which (1) or (2) is true is called the outer dilatation or the inner dilatation of f , respectively, and it is denoted by $K_o(f)$ or $K_I(f)$, respectively.

We also consider mappings in the compactified n -space $\bar{R}^n = R^n \cup \{\infty\}$. If D and D' are domains in $\bar{R}^n$, a homeomorphism $f: D \rightarrow D'$ is called quasiconformal if the restriction f_1 of f to $D - \{\infty, f^{-1}(\infty)\}$ is quasiconformal. The dilatations of f are defined to be equal to the dilatations of f_1 .

For the basic properties of n -dimensional quasiconformal mappings, see [2], [7] and [8].

Theorem. *Suppose that D is a domain in $\bar{R}^n$, that $E \subset D$ is closed with respect to D and that the $(n - 1)$ -dimensional Hausdorff measure of E is zero. Then every quasiconformal mapping f of $D - E$ has a unique continuous extension g to D . Moreover, g is a quasiconformal mapping, and $K_o(g) = K_o(f)$, $K_I(g) = K_I(f)$.*

Proof. Since E has no interior points, the uniqueness of g is clear. To prove the existence of a continuous extension we must show that the limit of f exists at every point of E . Since this assertion is local, we may assume that $D \neq \bar{R}^n$ and that $f(D - E) \neq \bar{R}^n$. Performing auxiliary inversions in a sphere, if necessary, we may assume that $D \subset R^n$ and that f is bounded.

We extend f to $h: D \rightarrow R^n$ by setting $h(x) = 0$ for $x \in E$. We show that h is of bounded distortion. First, since f is ACL and since the projection of E onto any coordinate hyperplane has $(n - 1)$ -dimensional measure zero, h is ACL. Next, since f is differentiable a.e. in $D - E$ and since $m(E) = 0$, h is differentiable a.e. in D . Furthermore, $|h'(x)|^n = |f'(x)|^n \leq K_o(f)J(x, f) = K_o(f)J(x, h)$ a.e. in D . Since h is bounded and a.e. continuous, it is integrable. Finally, using Lebesgue's theorem and the fact that $J(x, f)$ is a.e. the general derivative of the set function $A \rightarrow m(fA)$, we obtain

$$\int_D |h'(x)|^n dm(x) \leq K_o(f) \int_{D-E} J(x, f) dm(x) \leq K_o(f)m(f(D - E)).$$

Hence $|h'|$ is L^n -integrable. Consequently, h is of bounded distortion.

By a result of Rešetnyak [5, Theorem 1], there exists a continuous mapping $g: D \rightarrow R^n$ such that $g(x) = h(x)$ a.e. in D . This result was also proved by Callender [1], although he did not formulate it explicitly. Since $D - E$ is open and since f is continuous, $g(x) = f(x)$ for $x \in D - E$. The existence of a continuous extension g is thus proved. Moreover, g is of bounded distortion.

We next show that if G is a domain such that $\bar{G}$ is a compact subset of D , then the topological index ([4, p. 124]) $\mu(y, g, G)$ is at most one for every $y \in \bar{R}^n - g\partial G$. If y is not in $g\bar{G}$, $\mu(y, g, G) = 0$. Suppose that $y \in gG - g\partial G$. If $g^{-1}(y)$ meets E , $g^{-1}(y) \subset E$, since otherwise there would exist a sequence of points x_i in $D - E$ such that $x_i \rightarrow x_0 \in E \cap g^{-1}(y)$ and $x_i = f^{-1}(f(x_i)) \rightarrow f^{-1}(y) \in D - E$. Let Q be the y -component of $\bar{R}^n - g\partial G$. Since $g^{-1}Q$ is a neighborhood of $g^{-1}(y)$, it contains a point $z \in G - E$. Then $g^{-1}(g(z)) = \{z\}$. Choose a

connected neighborhood U of z such that $\bar{U} \subset D - E$. Since f is a sense-preserving homeomorphism, $\mu(y, g, G) = \mu(f(z), g, G) = \mu(f(z), f, U) = 1$.

From a result of Rešetnyak [6, Corollary 2 to Theorem 11] it follows that g is a homeomorphism. Since the condition (1) is simultaneously valid for f and g , $K_o(g) = K_o(f)$. Similarly $K_I(g) = K_I(f)$.

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