Distortion Theorems for the Level Curves of Rational Functions and Harmonic Functions

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Communicated by J. B. DIAZ

I. Introduction. The level loci of Green's function and more general harmonic functions have certain properties of smoothness and limitations on distortion. These properties of the level curves and orthogonal trajectories of Green's function, including multiple points, tangents, normals and convexity have been studied by J. L. Walsh in a series of publications [4, 5, 6]. In a recent paper [3] the author studied the distortion and location of the circle of curvature of lemniscates and the level curves of Green's function. It is the object of this paper to extend these results to general rational functions and harmonic functions. The level loci of a harmonic function, are by no means arbitrary as to their shape, but also have certain limitations on distortion. The present investigation establishes best possible geometric bounds for these properties for rational functions, all of whose zeros and poles are contained in disjoint discs. The same bounds are shown to be valid for functions harmonic in a region bounded by sets of curves C_{1i} , $i = 1, \dots, m$ and C_{2i} , $i = 1, \dots, n$; $u(x, y) = 1$ on the set $C_{1i} \subset \text{disc } D_1$ and $u(x, y) = 0$ on $C_{2i} \subset \text{disc } D_2$.

II. Normals and bisectors. The foundation for the theory of the critical points of a rational function of a complex variable z :

$$(1) \quad R(z) = \frac{\prod_{i=1}^m (z - \alpha_i)}{\prod_{i=1}^n (z - \beta_i)}$$

was provided by Bôcher [1] who modified the field of force in Gauss' theorem by admitting both positive and negative masses. The lines of force are the loci $\arg R(z) = k$ which are orthogonal to the level curves R_μ :

$$(2) \quad |R(z)| = \frac{\prod_{i=1}^m |z - \alpha_i|}{\prod_{i=1}^n |z - \beta_i|} = \mu > 0.$$

Our first result is a generalization of Theorem 1 in [3]. (The result was obtained by a different method, for lemniscates only, by Davis and Pollack [2]).

Theorem 1. *The perpendicular bisector of a chord of a level curve (2) cannot separate the finite zeros and finite poles of (1).*

Proof. Suppose the contrary. Let z_a and z_b lie on (2). Assume all the zeros α_i , $i = 1, \dots, m$, to lie to the same side of the bisector as z_a , and all the poles β_i , $i = 1, \dots, n$, to the opposite side, then

$$\frac{\prod_{i=1}^m |z_a - \alpha_i|}{\prod_{i=1}^n |z_a - \beta_i|} < \frac{\prod_{i=1}^m |z_b - \alpha_i|}{\prod_{i=1}^n |z_b - \beta_i|},$$

and z_a and z_b cannot lie on (2).

If we let z_a and z_b approach each other and coincide at P , we obtain:

Theorem 2. *The normal to the level curve (2) at P cannot separate the finite zeros and poles of (1).*

Further results can be deduced by specializing to the case that all zeros and poles of a rational function are contained in disjoint discs. In the following corollary the possibility that all the poles and zeros lie to one side of the bisector is ruled out.

Corollary 1. *Let all $\alpha_i \in D_1$, $i = 1, \dots, m$ and all $\beta_i \in D_2$, $i = 1, \dots, n$, $D_1 \cap D_2 = \emptyset$; then the normal to (2) at the internal center of similitude of the two circles C_1 and C_2 bounding D_1 and D_2 must intersect D_1 and D_2 .*

The configuration of Corollary 1 is invariant under linear transformations and we obtain the following result due to Walsh [4, p. 277].

Corollary 2. *Let $\alpha_i \in D_1$, $i = 1, \dots, m$ and $\beta_i \in D_2$, $i = 1, \dots, n$, $D_1 \cap D_2 = \emptyset$; then the normal to (2) at a point P exterior to $D_1 \cup D_2$ must lie in the circular sector bounded by the two circles through P which are transversely tangent to C_1 and C_2 .*

III. Curvature. Without loss of generality we can choose the coordinate system so that the point $P(x, y)$ is the origin, and let the normal to the level curve (2) coincide with the negative y -axis. By writing equation (1) in rectangular coordinates and taking the logarithmic derivative, and substituting in the usual formula for the curvature K , we obtain for the curvature K of R_μ at $P(x, y) = (0, 0)$:

$$(3) \quad K = -\frac{R_{xx}}{R_y} = \frac{\sum_{i=1}^m \left(\frac{1}{r_i^2} - \frac{2x_i^2}{r_i^4} \right) - \sum_{i=1}^n \left(\frac{1}{R_i^2} - \frac{2X_i^2}{R_i^4} \right)}{\sum_{i=1}^m \frac{y_i}{r_i^2} - \sum_{i=1}^n \frac{Y_i}{R_i^2}},$$

$$\begin{aligned}\alpha_i &= x_i + iy_i, & |\alpha_i| &= r_i, & i &= 1, \dots, m, \\ \beta_i &= X_i + iY_i, & |\beta_i| &= R_i, & i &= 1, \dots, n.\end{aligned}$$

This choice of coordinate system and notation will be used throughout. Our first result on the curvature of R_μ is the following:

Theorem 3. *Let all the zeros α_i lie in a half plane Q_1 , boundary q_1 and the poles β_i lie in a half plane Q_2 , boundary q_2 , q_2 parallel to q_1 , $Q_1 \cap Q_2 = \emptyset$. Let P be an arbitrary point in the strip between Q_1 and Q_2 . Denote by n the normal to (2) at P . Let $A_1 = n \cap q_1$, and $A_2 = n \cap q_2$. Then the radius of curvature of (2) at P is numerically greater than the smaller of the distances PA_1 and PA_2 .*

Proof. Write the equations for Q_1 and Q_2 in the form $y > ax + b$, $b > 0$, and $Y < aX - B$, $B > 0$. It follows that:

$$-R_\nu > b \sum_{i=1}^m \frac{1}{r_i^2} + B \sum_{i=1}^n \frac{1}{R_i^2}.$$

Without loss of generality assume $b < B$; then we wish to prove $|K| < 1/b$.

The proof is divided into two cases, $K > 0$, i.e. $R_{xx} > 0$, and $K < 0$, i.e. $R_{xx} < 0$, and the desired result follows in each case from substitution in (3).

Our main result concerns the precise upper and lower bounds for K when the α_i lie in a disc D_1 and the β_i lie in a disc D_2 , $D_1 \cap D_2 = \emptyset$. This theorem is an extension of the central result of [2, Th. 3, pp. 61–65] which states the bounds for the curvature of lemniscates whose poles are contained in a disc.

Theorem 4. *Let R_μ be the level curve of a rational function (2) whose zeros α_i lie in a disc D_1 and whose poles β_i lie in a disc D_2 , $D_1 \cap D_2 = \emptyset$. Let P be an arbitrary point exterior to $D_1 \cup D_2$. Draw the four circles through P tangent to D_1 or D_2 , and orthogonal to the normal n to (2) at P . Let A and B denote the intersections of the circles tangent to D_1 with n , and C and D the intersections of the circles tangent to D_2 with n . Let the directed distances PA , PB , PC , and PD be denoted by $1/a$, $1/b$, $1/c$, and $1/d$, $a \leq b$, $d \leq c$. Then the following bounds are obtained for the curvature K of (2):*

$$(4a) \quad K \leq K_{\max} = b + c, \quad \text{for } b - d \leq 2(b - c),$$

$$(4b) \quad K \leq K_{\max} = 4b - d - c - 2[2(b - c)(b - d)]^{1/2}, \quad \text{for } b - d \geq 2(b - c),$$

$$(4c) \quad K \geq K_{\min} = d + a, \quad \text{for } b - d \leq 2(a - d),$$

$$(4d) \quad K \geq K_{\min} = 4d - a - b + 2[2(a - d)(b - d)]^{1/2}, \quad \text{for } b - d \geq 2(a - d).$$

The proof is based on a study of a rational function with only one pair of zeros and one pair of poles; the extremal configuration is then reduced to that case.

In the following lemma the bounds (4a) to (4d) are established for this special case.

Lemma 1. *For the set of level curves*

$$(5) \quad |R(z)| = \frac{(z - \alpha_1)^{m_1}(z - \alpha_2)^{m_2}}{(z - \beta_1)^{n_1}(z - \beta_2)^{n_2}},$$

$$m_1 + m_2 = n_1 + n_2 = n, \quad m_1 > 0, \quad m_2 > 0, \quad n_1 > 0, \quad n_2 > 0;$$

$$\alpha_1 \subset D_1, \quad \alpha_2 \subset D_1; \quad \beta_1 \subset D_2, \quad \beta_2 \subset D_2; \quad D_1 \cap D_2 = \emptyset;$$

the bounds (4a), (4b), (4c), and (4d) hold.

Proof. In the proof of Lemma 1 we will invert the z -plane in the unit circle. This device allows simplified formulae for the first and second derivatives of (2) in terms of new coordinates; and a greatly simplified geometric situation in the inverted plane allows easier determination of the extremal configurations.

If we now denote by (x_1, y_1) and (x_2, y_2) the coordinates of the zeros after inversion and by (X_1, Y_1) and (X_2, Y_2) those of the poles after inversion we obtain for the curvature of (5):

$$(6) \quad K = \frac{m_1(y_1^2 - x_1^2) + m_2(y_2^2 - x_2^2) - n_1(Y_1^2 - X_1^2) - n_2(Y_2^2 - X_2^2)}{m_1y_1 + m_2y_2 - n_1Y_1 - n_2Y_2},$$

α_1 and α_2 lie on the boundary C_1 of D_1 and

β_1 and β_2 lie on the boundary C_2 of D_2 .

Let

$$m_1x_1 + m_2x_2 = n_1X_1 + n_2X_2 = nx_0,$$

$$(7) \quad ns_1 = m_1y_1 + m_2y_2, \quad s_1 = MG_1, \quad s_2 = MG_2,$$

$$ns_2 = n_1Y_1 + n_2Y_2.$$

M is the point $(x, 0)$, G_1 is the center of gravity of the two zeros Z_1 and Z_2 ; and G_2 that of the poles P_1 and P_2 . θ_1 is the angle between the x -axis and the line joining Z_1Z_2 ; θ_2 is the angle between the x -axis and the line joining P_1P_2 . Substituting in (6)

$$(8) \quad K = \frac{s_1^2 - \overline{Z_1G_1} \cdot \overline{Z_2G_1} \cos 2\theta_1 - (s_2^2 - \overline{P_1G_2} \cdot \overline{P_2G_2} \cos 2\theta_2)}{s_1 - s_2}.$$

For our choice of coordinate system $(s_1 - s_2) > 0$. Then if we keep s_1 and s_2 fixed and vary θ_1 and θ_2

$$(9) \quad K \leq \frac{s_1^2 + \overline{Z_1G_1} \cdot \overline{Z_2G_1} - s_2^2 + \overline{P_1G_2} \cdot \overline{P_2G_2}}{s_1 - s_2} = f_1(s_1, s_2),$$

$$(10) \quad K \geq \frac{s_1^2 - \overline{Z_1G_1} \cdot \overline{Z_2G_1} - s_2^2 - \overline{P_1G_2} \cdot \overline{P_2G_2}}{s_1 - s_2} = f_2(s_1, s_2).$$

It follows from (9) and (10) that the curvature of a level curve (5) with the poles and zeros in the interior of the discs D_1 and D_2 will lie within the bounds

established for the level curve (5) with the poles and zeros on C_1 and C_2 . Let a , b , c , and d be the intercepts of the line x_0 with the circles C'_1 and C'_2 in the inverted plane, $d < c < a < b$. By direct calculation, similar to the proof of Theorem 3 in [3], we obtain

$$(11) \quad f_1(s_1, s_2) = \frac{-ab + (a + b)s_1 - 2s_2^2 - cd + (c + d)s_2}{s_1 - s_2},$$

$$(12) \quad f_2(s_1, s_2) = \frac{2s_1^2 + ab - (a + b)s_1 + cd - (c + d)s_2}{s_1 - s_2}.$$

1) Upper Limit

$$\frac{\partial f_1(s_1, s_2)}{\partial s_1} = 0 \text{ if } s_2 = \frac{1}{4}((a + b + c + d) \pm [(a + b + c + d)^2 - 8ab - 8cd]^{1/2});$$

we have $\frac{1}{4}[(a + b + c + d) - [(a + b + c + d)^2 - 8ab - 8cd]^{1/2}] > c$, and K is a monotonically increasing function of s_1 , for the allowed position $d < s_2 < c$. Therefore we choose $s_1 = b$.

$$\frac{\partial f_1(s_1, s_2)}{\partial s_2} = 0 \quad \text{for} \quad s_2 = b - \left[\frac{(b - c)(b - d)}{2} \right]^{1/2};$$

we have

$$s_2(K_{\max}) < c, \quad \text{if} \quad b - \frac{(b - c)(b - d)}{2} < c,$$

i.e. $(b - d) \geq 2(b - c)$ and the bounds (4a), (4b) are obtained by substitution of $s_1 = b$ and $s_2 = c$ or $s_2 = b - (b - c)(b - d)/2$ in (11). The proof for the lower bound is similar and the bounds (4c) and (4d) are obtained by substituting $s_2 = d$ and either $s_1 = a$ or

$$s_1 = d + \left[\frac{(a - d)(b - d)}{2} \right]^{1/2} \text{ in (12).}$$

By inspection of formula (4) it can be seen that the maximum is increased and the minimum decreased by increasing b and c and decreasing a and d . If we compare the bounds obtained by leaving the size of the circles C'_1 and C'_2 invariant and changing the position of the line $x = x_0$ intercepting the two circles, it can be seen that the largest variation is obtained if the line $x = x_0$ passes through the centers of C_1 and C_2 , and our theorem remains true if we replace the intercepts a , b , c and d by the intercepts of the same circles moved parallel to the x -axis so that the two centers of the circles and the two centers of gravity G_1 and G_2 lie on the same parallel to the y -axis. By inverting this configuration back to the original plane we obtain the configuration of Theorem 3, and Lemma 1 is proved.

The proof of Theorem 4 follows from Lemma 1 by methods similar to the ones employed in [2]. The rational function (1) with zeros in D_1 and poles in D_2 is represented as a finite product of rational functions (5) with one pair of

zeros and poles and the curvature K of (1) is obtained as the finite sum of terms of the form (6). The extreme values of K will be assumed if all the pairs are coincident and the bounds established in Lemma 1 thus hold for all functions (1).

IV. The level curves of harmonic functions. In [3] it was shown that a close connection exists between the level curves of Green's function and lemniscates; in a similar manner the equipotentials of more general harmonic functions can be shown to possess properties similar to those of rational functions. In this section the properties of the bisectors, normals, and curvature of these equipotentials are derived. As a basis for the proof of these theorems we will use a representation of a harmonic function in terms of its distance to the boundary of a region.

Starting with Green's third identity we obtain the following representation for $u(\xi, \eta)$, harmonic in a region R , exterior to a set of smooth curves, $u = 0$ on C_{1i} , $i = 1, \dots, m$ and $u = 1$ on C_{2i} , $i = 1, \dots, n$ at an arbitrary point $P(\xi, \eta) \in R$. u is continuous in $\bar{R}$ [5, pp. 270–272].

$$(13) \quad \begin{aligned} u(\xi, \eta) &= \int_{C_1} \log r \, d\sigma - \int_{C_2} \log r \, d\sigma, \\ C_1 &= \sum_{i=1}^m C_{1i}, \quad C_2 = \sum_{i=1}^n C_{2i}, \end{aligned}$$

r denotes the distance from P to a boundary point,

$$\begin{aligned} \text{on } C_1 - \frac{1}{2\pi} \frac{\partial u}{\partial n} ds &= d\sigma > 0, \quad 0 \leq \sigma \leq \tau = \int_{C_1} d\sigma = \int_{C_1} d\sigma, \\ \text{on } C_2 - \frac{1}{2\pi} \frac{\partial u}{\partial n} ds &= d\sigma. \end{aligned}$$

It follows from the representation (13) that the level curves of our function $u(x, y)$ can be interpreted as a potential and the orthogonal trajectories as lines of force due to a spread of positive matter σ on the C_{1i} and negative matter $-\sigma$ on the C_{2i} , where the total mass is zero and the inverse distance law holds.

If R is an infinite region, integration over a large circle has to be added in the boundary of a region over which the integration is performed and an additional term $u(\infty)$ is to be added to the right-hand side of (13).

With the aid of this representation we can prove an analogue of Bôcher's theorem for harmonic functions. Similarly we obtain

Theorem 1'. *Let $u(x, y)$ be a function harmonic in a region R , continuous in $\bar{R}$. R is bounded by mutually disjoint Jordan curves C_1 and C_2 , $u = 0$ on the C_{1i} , $i = 1, \dots, m$, $u = 1$ on the C_{2i} , $i = 1, \dots, n$. Then the perpendicular bisector of a chord joining two points of a level curve*

$$(14) \quad U_\mu: u(x, y) = \mu$$

cannot separate the C_{1i} and C_{2i} .

Proof. Let $A(x_a, y_a)$ and $B(x_b, y_b)$ be two points on (14). Assume A and all the C_{1i} to the right of the bisector and all the C_{2i} to the left;

then $\log r_{AC_{1i}} < \log r_{BC_{1i}}, i = 1, \dots, m,$
 $\log r_{AC_{2i}} > \log r_{BC_{2i}}, i = 1, \dots, n,$
 $r_{AC_{1i}}(r_{AC_{2i}})$ is the distance from A to any point on $C_{1i}(C_{2i}),$
 $r_{BC_{1i}}(r_{BC_{2i}})$ is the distance from B to any point on $C_{1i}(C_{2i}).$

Substituting in (13), $u(A) < u(B)$ and we obtain a contradiction.

By letting A and B approach each other we obtain the result equivalent to one due to Walsh [5, Cor. 1, p. 277].

Theorem 2'. *The normal to (14) cannot separate the boundaries α and β .*

Theorems 3 and 4 also have their analogues for harmonic functions. The curvature of the level curves of harmonic functions can be calculated by means of representation (13). We will choose the point $P(\xi, \eta)$ to be the origin and let the normal to (14) coincide with the negative y -axis, and we obtain:

$$(15) \quad K = \frac{\left[\int_{C_1} \frac{y^2 - x^2}{r^4} d\sigma - \int_{C_2} \frac{Y^2 - X^2}{R^4} d\sigma \right]}{\int_{C_1} \frac{y}{r^2} d\sigma - \int_{C_2} \frac{Y}{R^2} d\sigma},$$

$$r^2 = x^2 + y^2, \quad R^2 = X^2 + Y^2.$$

Formula (15) corresponds to (3) for the curvature of the level curve of a rational function and the derivation of the bounds on the curvature can be carried out with integrals replacing the finite sums. This connection enables us to derive the following result on the curvature of (14).

Theorem 3'. *Let $u(x, y)$ be a function harmonic in R , bounded by smooth curves C_1 and C_2 . Let all curves C_{1i} on which $u(x, y)$ assumes the value zero lie in a half plane Q_1 bounded by a line q_1 . Let all the C_{2i} be contained in the half plane Q_2 bounded by q_2 , q_2 parallel to q_1 , $Q_1 \cap Q_2 = \emptyset$. Let P be an arbitrary point in the strip exterior to Q_1 and Q_2 , denote by A_1 and A_2 the intersections of the normal n to the curve (14) at P with q_1 and q_2 . Then the radius of curvature of the curve (14) is numerically greater than the smaller of the distances PA_1 or PA_2 .*

Theorem 4'. *Let u be harmonic in a region and assume the value zero on a set of curves C_{1i} contained in a disc D_1 and unity on a set of curves C_{2i} contained in a disc D_2 , $D_1 \cap D_2 = \emptyset$. P exterior to $(D_1 \cup D_2)$. Draw the four circles tangent to D_1 or D_2 through P orthogonal to the normal to $u(x, y)$ at P . Let A and B denote the intersections of the circles tangent to D_1 with the normal, and C and D the intersections of the circles tangent to D_2 with the normal. Let the directed distances PA , PB , PC , and PD be denoted by $1/a$, $1/b$, $1/c$, and $1/d$. Then the curvature of $u(x, y)$ must lie between the limits (4a) to (4d).*

The method of proof is similar to that of Theorem 4, with integrals replacing the finite sums.

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Date communicated: OCTOBER 31, 1968