On the Conformal Equivalence of Compact 2-Dimensional Manifolds

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In this note we obtain a new proof of the fact that any orientable compact non-simply connected 2-manifold with a given smooth (C^∞) Riemannian metric is conformally equivalent to a manifold of constant nonpositive curvature. The proof proceeds by means of the theory of nonlinear elliptic partial differential equations and the calculus of variations. It is interesting to note that the proofs given here are in the spirit of the famous paper of Poincaré [1]. At the end of the paper we show how to modify our proof in the simply connected case to prove conformal equivalence to a sphere. Since the isometry classes of 2-manifolds with constant curvature are well-known, we obtain a new proof of the uniformization theorem for closed Riemann surfaces.

1. Let $\mathfrak{M}$ be a compact 2-dimensional orientable manifold with a given smooth (C^∞) Riemannian metric $ds^2 = g_{11} dx_1^2 + 2g_{12} dx_1 dx_2 + g_{22} dx_2^2$ and with Gaussian curvature $R(x)$. Then if $\mathfrak{M}$ is non-simply connected, by the Gauss-Bonnet formula

$$(1) \quad \int_{\mathfrak{M}} R(x) dS \leq 0$$

with equality holding if and only if the genus of $\mathfrak{M}$ is 1. Furthermore, if we set $\tilde{g}_{ij} = e^{2\sigma} g_{ij}$ ($i, j = 1, 2$) (*i.e.* we consider a conformally equivalent metric $\tilde{g}_{ij}$ determined by the conformal mapping $e^{2\sigma}$) then we obtain the following equation for σ

$$(2) \quad \Delta\sigma - R(x) + R'e^{2\sigma} = 0$$

where Δ is the Laplace Beltrami operator associated with g_{ij} and R' is the Gaussian curvature of the manifold $\mathfrak{M}$ with the Riemannian metric $\tilde{g}_{ij}$. (Indeed

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in terms of isothermal coordinates (u, v) with $g_{11} = g_{22} = \lambda(u, v)$ and $g_{12} = 0$, $R = -\frac{1}{2}\lambda \{(\log \lambda)_{uu} + (\log \lambda)_{vv}\}$, see [2, p. 153]. Hence (2) follows by setting $\lambda' = \exp(2\sigma)\lambda$.

In this connection we shall prove the following result.

Theorem 1. *Every orientable compact (C^∞) 2-dimensional Riemannian manifold which is not simply connected is conformally equivalent to a Riemannian manifold with constant nonpositive curvature $\bar{R}$. Also $\bar{R} = 0$ if and only if $\int_{\mathfrak{M}} R(x) dS = 0$.*

Before proving this theorem, we recall a few basic results.

By $W_{1,2}(\mathfrak{M})$ we denote the class of all functions $u(x)$ defined over $\mathfrak{M}$ which are square-integrable over $\mathfrak{M}$ and possess square-integrable derivatives defined over $\mathfrak{M}$. Let $\bar{W}_{1,2}(\mathfrak{M})$ denote the class of functions $u(x)$ in $W_{1,2}(\mathfrak{M})$ with mean value 0 over $\mathfrak{M}$, i.e. $\int_{\mathfrak{M}} u(x) = 0$. $W_{1,2}(\mathfrak{M})$ is a Hilbert space with respect to the inner product

$$(u, v) = \int_{\mathfrak{M}} uv dS + \sum_{i,j=1}^2 \int_{\mathfrak{M}} g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dS.$$

$\bar{W}_{1,2}(\mathfrak{M})$ is a closed linear subspace of $W_{1,2}(\mathfrak{M})$ with respect to the inner product

$$D[u, v] \equiv (u, v)_{1,2} = \sum_{i,j} \int_{\mathfrak{M}} g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dS.$$

Also we set $D[u] = D[u, u] = \|u\|_{1,2}^2$, and $\int_{\mathfrak{M}} u^2 = \|u\|_{0,2}^2$. Furthermore, for $u \in \bar{W}_{1,2}(\mathfrak{M})$

$$\|u\|_{0,2} \leq K \|u\|_{1,2}.$$

A recent result [3] in this connection is the following: for $u \in \bar{W}_{1,2}(\mathfrak{M})$, $\int_{\mathfrak{M}} e^{cu} < \infty$ for all real numbers $c < \infty$. Thus $e^u \in L_p(\mathfrak{M})$ for all $p < \infty$. Furthermore, if $u_n \rightarrow u$ weakly in $\bar{W}_{1,2}(\mathfrak{M})$, then $\int_{\mathfrak{M}} e^{u_n} \rightarrow \int_{\mathfrak{M}} e^u$, i.e. the functional $G(u) = \int_{\mathfrak{M}} e^u$ is weakly continuous in $\bar{W}_{1,2}(\mathfrak{M})$.

2. Proof of Theorem 1. The proof will be obtained in the following manner, first for the case $\int_{\mathfrak{M}} R(x) < 0$.

Step I. Solve the variational problem:

(P_1) Minimize $F(w) = D[w]/4 + \int_{\mathfrak{M}} R_a w + \int_{\mathfrak{M}} e^w$, over the class of functions $X = W_{1,2}(\mathfrak{M})$. Here R_a is the mean value of $R(x)$ over $\mathfrak{M}$, i.e. $\int_{\mathfrak{M}} R_a = \int_{\mathfrak{M}} R(x)$, and v is a smooth function to be determined.

Step II. Let $2u = w_0 + v_0$ where w_0 is any solution of (P_1) and v_0 is a solution of the equation $\frac{1}{2}\Delta v_0 = (R(x) - R_a)$ over $\mathfrak{M}$. Show u satisfies an equation of the form

$$(3) \quad D[u, v] + \int_{\mathfrak{M}} R(x)v + \bar{R} \int_{\mathfrak{M}} e^{2u} v = 0$$

for all $v \in W_{1,2}(\mathfrak{M})$ where $\bar{R}$ is a nonpositive constant.

Step III. Show that any function u satisfying (3) is C^∞ .

Clearly the successful completion of Steps (I–III) provides a proof of Theorem 1. To this end, we prove the following three lemmas.

Lemma A. *The variational problem (P_1) has for $v = v_0$ a solution, provided $\min_{\mathfrak{M}} e^{v_0}$ is sufficiently large.*

Proof. Let $X = \{w \mid w \in W_{1,2}(\mathfrak{M})\}$. Set

$$F(w) = D[w]/4 + \int_{\mathfrak{M}} R_a w + \int_{\mathfrak{M}} e^{v_0} e^w$$

where $D[w]$ denotes the Dirichlet integral over $\mathfrak{M}$ associated with the Laplace Beltrami operator Δ . Let $\min_{\mathfrak{M}} e^{v_0} = e_0$. Clearly $e_0 > 0$. Then $c(X) \equiv \inf_X F(w) \geq \inf_X \int_{\mathfrak{M}} (R_a w + e_0 e^w) dS$. Now as $R_a < 0$, the minimum of the real-valued function $f(x) = R_a x + e_0 e^x$ is attained at $x_0 = \log(-R_a/e_0)$, whence $f(x) \geq R_a(\log(-R_a/e_0) - 1) = f(x_0)$. Thus $c(X)$ is bounded below by $f(x_0)\mu(\mathfrak{M})$ where $\mu(\mathfrak{M})$ is the volume of $\mathfrak{M}$.

We now show that $c(X)$ is attained by some function $w \in X$. For if w_n is a sequence in X with $F(w_n) \rightarrow c(X)$, for sufficiently large n , $F(w_n) \leq c(X) + 1$ so that $D[w_n]/4 \leq c(X) + 1 + |f(x_0)|\mu(\mathfrak{M})$ and

$$(\dagger) \quad \int_{\mathfrak{M}} R_a w_n + \int_{\mathfrak{M}} e^{v_0} e^{w_n} \leq c(X) + 1.$$

The above two inequalities imply that the sequence $\|w_n\|^2 = D[w_n] + \int_{\mathfrak{M}} w_n^2$ is uniformly bounded. Indeed, as $D[w_n]$ is automatically uniformly bounded by the first mentioned inequality, it is sufficient to find a uniform bound for the sequence $\int_{\mathfrak{M}} w_n^2$. To this end, we recall the well-known inequality for all $u \in W_{1,2}(\mathfrak{M})$,

$$\int_{\mathfrak{M}} u^2 \leq K \left\{ D[u] + \left| \int_{\mathfrak{M}} u \right|^2 \right\}.$$

Thus $\int_{\mathfrak{M}} w_n^2$ is uniformly bounded, provided we can show that the sequence $|w_{n_a}|$ is uniformly bounded; where w_{n_a} denotes the mean value of w_n over $\mathfrak{M}$, i.e. $\int_{\mathfrak{M}} w_{n_a} = \int_{\mathfrak{M}} w$. Suppose $\{|w_{n_a}|\}$ or some subsequence tends to infinity, and write

$$w_n = w_{n_a} + w_{n_0} \quad \text{where} \quad \int_{\mathfrak{M}} w_{n_0} = 0.$$

Then $w_{n_0} \in \bar{W}_{1,2}(\mathfrak{M})$ and the sequence $\{\|w_{n_0}\|_{1,2}\}$ is uniformly bounded, so that by the result of [3] mentioned above $\int_{\mathfrak{M}} e^{w_{n_0}}$ are uniformly bounded. Furthermore, suppose $e_0 = \min_{\mathfrak{M}} e^{v_0} > |R_a|\mu(\mathfrak{M})$. Then as $e^x \geq 1 + x$ for all x , we obtain from the inequality [†]

$$\int_{\mathfrak{M}} R_a w_n + \int_{\mathfrak{M}} e_0(1 + w_n) \leq c(X) + 1,$$

i.e.

$$(-|R_a| \mu(\mathfrak{M}) + e_0)w_{n_a} + \int_{\mathfrak{M}} e_0 \leq c(X) + 1.$$

Thus w_{n_a} cannot tend to $+\infty$, as $n \rightarrow \infty$. On the other hand w_{n_a} cannot tend to $-\infty$, as $n \rightarrow \infty$; because the inequality $(\dagger)$ implies

$$-|R_a| w_{n_a} \mu(\mathfrak{M}) \leq -|R_a| w_{n_a} \mu(\mathfrak{M}) + e^{w_{n_a}} \int_{\mathfrak{M}} e^v e^{w_{n_a}} \leq c(X) + 1.$$

As $W_{1,2}(\mathfrak{M})$ is a Hilbert space and $\|w_n\|$ are uniformly bounded, $\{w_n\}$ has a weakly convergent subsequence which we relabel $\{w_n\}$ with weak limit $\bar{w}$. We shall show $F(\bar{w}) = c(X)$. As $w_n \rightarrow \bar{w}$ weakly in $W_{1,2}(\mathfrak{M})$, $\int_{\mathfrak{M}} e^{w_n} \rightarrow \int_{\mathfrak{M}} e^{\bar{w}}$ and

$$D[w_n]/4 + \int_{\mathfrak{M}} R_a(x)w_n + \int_{\mathfrak{M}} e^v e^{w_n} \rightarrow c(X).$$

Thus as $D[w]$ is a weakly lower semi-continuous functional on $W_{1,2}(\mathfrak{M})$, $D[\bar{w}] \leq \liminf D[w_n]$. So $F(\bar{w}) \leq c(X) = \inf_x F(w)$, which guarantees that $F(\bar{w}) = c(X)$.

Standard techniques of the calculus of variations guarantee that the Gateaux derivative of $F(\bar{w})$ is zero for all directions $v \in W_{1,2}(\mathfrak{M})$. Hence $u = \bar{w}$ satisfies an integral identity of the form

$$(4) \quad D[u, v]/2 + \int_{\mathfrak{M}} R_a v + \int_{\mathfrak{M}} e^v e^{u} v = 0 \quad \text{for all } v \in W_{1,2}(\mathfrak{M}).$$

Lemma B. *There is a function $u(x) \in W_{1,2}(\mathfrak{M})$ which satisfies (3) for all $v \in W_{1,2}(\mathfrak{M})$, with $\bar{R} = -1$ if $\int_{\mathfrak{M}} R(x) < 0$.*

Proof. Let $2u = \bar{w} + v_0$ where $\bar{w}$ is the function in $W_{1,2}(\mathfrak{M})$ satisfying (4) described in Lemma A and v_0 is a smooth solution of the equation

$$(5) \quad \frac{1}{2} \Delta w - (R(x) - R_a) = 0,$$

i.e.

$$(5') \quad \frac{1}{2} D[w, v] + \int_{\mathfrak{M}} (R(x) - R_a) v = 0 \quad \text{for all } v \in W_{1,2}(\mathfrak{M})$$

such that $e^{u_0} > |R_a| \mu(\mathfrak{M})$. That (5) has such a solution is clear because a necessary and sufficient condition for the solvability of (5) is that $\int_{\mathfrak{M}} (R(x) - R_a) = 0$, and if w_0 is any solution of (5) then $v_0 = w_0 + c$ is also a solution of (5) for any constant c . Now we choose c so that

$$\inf_{\mathfrak{M}} e^{u_0} = \inf_{\mathfrak{M}} e^c e^{w_0} > |R_a| \mu(\mathfrak{M}),$$

$$e^c > (\sup_{\mathfrak{M}} e^{-w_0}) |R_a| \mu(\mathfrak{M}).$$

Then $2u = \bar{w} + v_0$ satisfies

$$D[u, v] + \int_{\mathfrak{M}} R(x)v + \int_{\mathfrak{M}} e^{2u} v = 0 \quad \text{for all } v \in W_{1,2}(\mathfrak{M}).$$

Indeed

$$\begin{aligned}
& \frac{1}{2} D[\bar{w} + v_0, v] + \int_{\mathfrak{M}} R(x)v + \int_{\mathfrak{M}} e^{\bar{v}} e^{*v} \\
&= \frac{1}{2} D[\bar{w}, v] + \left(\frac{1}{2} D[v_0, v] + \int_{\mathfrak{M}} (R(x) - R_a)v \right) + \int_{\mathfrak{M}} R_a v + \int_{\mathfrak{M}} e^{*v} e^{\bar{v}} v \\
&= 0 \quad \text{by virtue of (4) and (5).}
\end{aligned}$$

Finally we demonstrate the smoothness of the function u .

Lemma C. Any solution u of (3) in $W_{1,2}(\mathfrak{M})$ is in $C^\infty(\mathfrak{M})$ provided $R(x) \in C^\infty(\mathfrak{M})$.

Proof. We interpret the solution u of (3) as a weak solution in $W_{1,2}(\mathfrak{M})$ of the linear equation

$$\Delta u = f \quad \text{where} \quad f \in L_p(\mathfrak{M}) \quad \text{for all} \quad p < \infty.$$

Indeed by the results of [3], $f = e^u$ and $\int_{\mathfrak{M}} e^{pu} < \infty$ for all positive $p < \infty$. Thus by the L_p regularity theory (see [4]) $u \in W_{2,p}(\mathfrak{M})$ for all $1 < p < \infty$. So that by the Sobolev Imbedding Theorem, $u(x)$ is of class $C_{1,\alpha}(\mathfrak{M})$ for $\alpha > \frac{1}{2}$ say, i.e. $u(x)$ is continuously differentiable with Hölder exponent $\alpha > \frac{1}{2}$. Then by the Schauder regularity theory (see [5]) $u(x) \in C^\infty(\mathfrak{M})$.

Completion of the proof of Theorem 1. Suppose $\int_{\mathfrak{M}} R(x) = 0$. Then we show that the metric g_{ij} is conformally equivalent to the metric $\bar{g}_{ij}$ with Gaussian curvature zero everywhere on $\mathfrak{M}$.

Equivalently by (2), we show there is a smooth function σ satisfying

$$(6) \quad \Delta \sigma - R(x) = 0 \quad \text{on} \quad \mathfrak{M}.$$

Indeed provided $\int_{\mathfrak{M}} R(x) = 0$ a function σ satisfying (6) can be found (see for example [5]) by minimizing $D[u]/2 + \int_{\mathfrak{M}} R(x)u$ over the class of functions $u \in \bar{W}_{1,2}(\mathfrak{M})$. Integrating over $\mathfrak{M}$, we find (6) is solvable only if $\int_{\mathfrak{M}} R(x) = 0$. Thus if $\int_{\mathfrak{M}} R(x) \neq 0$, the metric g_{ij} is not conformally equivalent to one with Gaussian curvature zero, a fact already clear from the invariance of $\int_{\mathfrak{M}} R(x)$ under diffeomorphism.

Remark. Analogous results for nonorientable manifolds $\mathfrak{M}$ can be obtained by replacing $\mathfrak{M}$ by its orientable two-sheeted covering manifold.

3. Addendum. (The simply connected case). If the manifold $\mathfrak{M}$ is simply connected, we indicate the modifications necessary in the proof of Theorem I to show that $\mathfrak{M}$ is conformally equivalent to a manifold with constant positive curvature, i.e. a sphere. This result again follows by finding a C^∞ solution $\sigma(x)$ of (2) (for some $R' > 0$), and to this end we modify the 3 steps mentioned in Section 2 as follows: In place of Step I we solve the variational problem: (P'_1)

Minimize $F(u) = D[u]/2 + \int_{\mathfrak{M}} R(x)u$ over the class X of functions $u \in W_{1,2}(\mathfrak{M})$ subject to the constraint $\int_{\mathfrak{M}} e^{2u} = 1$ (without loss of generality we assume $\mu(\mathfrak{M}) \neq 1$ so that $u(x) \equiv 0 \notin X$). In Step II, we show that any solution of (P'_1) satisfies (3) for some constant $\bar{R}$ (arising here as a Lagrange multiplier). $\bar{R} > 0$ because setting $v = 1$ in (3), the Gauss-Bonnet formula implies $\bar{R} = 4\pi$. Step III is carried over directly by repeating the proof of Lemma C.

Thus the desired results follows by standard techniques, provided we show that the analogue of Lemma A holds; namely

Lemma A'. *The variational problem (P'_1) has a solution.*

Proof. For any $u \in X$, write $u = \bar{u} + u_m$ where u_m is the mean value of u over $\mathfrak{M}$. As $\int_{\mathfrak{M}} e^{2u} = 1$, $u_m = -\frac{1}{2} \log \int_{\mathfrak{M}} e^{2u} (*)$. Hence

$$F(u) = \frac{1}{2} D[\bar{u}] - 2\pi \log \int_{\mathfrak{M}} e^{2\bar{u}} + \int_{\mathfrak{M}} R(x)\bar{u} \equiv G(\bar{u}).$$

To show $c(X) = \inf_X F(u) = \inf_{\bar{u}} G(\bar{u})$ is bounded above $-\infty$, we write $\bar{u} = \|\bar{u}\|_{1,2} v$ where $\|v\|_{1,2} = 1$, and let $\epsilon > 0$ be a number to be chosen later. As

$$\int_{\mathfrak{M}} \exp(2\|\bar{u}\|_{1,2} v) \leq \int_{\mathfrak{M}} \exp(\epsilon \bar{u}_{1,2}^2 + v^2/\epsilon) = \exp(\epsilon \|\bar{u}\|_{1,2}^2) \int_{\mathfrak{M}} \exp(v^2/\epsilon),$$

$$G(\bar{u}) \geq \frac{1}{2} \|\bar{u}\|_{1,2}^2 - 2\pi\epsilon \|\bar{u}\|_{1,2}^2 - 2\pi \log \int_{\mathfrak{M}} \exp(v^2/\epsilon) - K \|\bar{u}\|_{1,2}.$$

Now choose $\epsilon = 1/8\pi$ and note that by the results of [3], the supremum of $\int_{\mathfrak{M}} \exp(8\pi v^2)$ over $\|v\|_{1,2} = 1$ is bounded by a finite constant K_0 . Hence

$$(7) \quad G(\bar{u}) \geq \frac{1}{4} \|\bar{u}\|_{1,2} \{ \|\bar{u}\|_{1,2} - K \} - 2\pi K_0.$$

Equation (7) shows that $c(X)$ is finite, and together with (*) that any minimizing sequence $u_n \in X$ has uniformly bounded norms in $W_{1,2}(\mathfrak{M})$, so that a suitable subsequence has weak limit w . As $\int_{\mathfrak{M}} e^{2u}$ is continuous with respect to weak convergence, $w \in X$; also w solves (P'_1) because $F(w) = c(X)$ by virtue of the lower semicontinuity of $F(u)$ with respect to weak convergence.

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