*Existence and Uniqueness of Solutions to a Nonlinear Boundary Value Problem*¹

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Communicated by GARRETT BIRKHOFF

Often in questions of the existence and uniqueness of solutions of the two point boundary value problem

$$(1) \quad y'' + f(t, y, y') = 0$$

$$(2) \quad y(t_1) = r_1$$

$$(3) \quad y(t_2) = r_2$$

the conditions on $f(t, y, y')$ are such that uniqueness of solutions, *i.e.* existence of at most one solution, is easy to establish but existence of at least one solution is more difficult, see [1]–[2], for example. Recently Lasota and Opial [3] showed that under the assumption

- (c) For each point $(t_0, x_0, u_0) \in D$, the domain of definition of f , there exists one and only solution $x(t) = x(t; t_0, x_0, u_0)$ of equation (1) defined on (a, b) such that $x(t_0) = x_0, x'(t_0) = u_0$,

that the existence of at most one solution of the boundary value problem (1) (2) (3) for every pair $(t_1, r_1)(t_2, r_2)$ of $(a, b) \times R$ implies that there exists at least one solution of (1) (2) (3). Here we remove the uniqueness requirement for solutions of initial value problems but assume that all solutions are continuable over (a, b) . This is important because it removes the need for any sort of Lipschitz type restriction on the nonlinearity.

Theorem. Suppose $f(t, y, y')$ is continuous on $(a, b) \times R \times R$ and that solutions of (1) with initial conditions at $t_0 \in (a, b)$ can be continued over (a, b) . If there exists at most one solution to the boundary value problem (1) (2) (3) for every choice $(t_i, r_i) \in (a, b) \times R, i = 1, 2$, there exists exactly one solution of (1) (2) (3).

Remark. Actually, the proof can be made to work on $[a, b]$ or on $(a, b]$.

Proof. Let $y(t, r, u)$ be a solution of equation (1) which satisfies $y(t_1) = r_1, y'(t_1) = u$. Define a point to set mapping Γ from R into R by associating with

¹ Supported by NSF Grant GP-5599.

each $u \in R$ all of the points $y(t_2, r_1, u)$ (since solutions may not be unique there may be more than one solution satisfying $y(t_1) = r_1, y'(t_1) = u$.) This mapping is upper semicontinuous and takes points into continua by Kenesers theorem. It follows easily that $\Gamma(R)$ is connected and the idea of the proof is to show $\Gamma(R) = R$.

Suppose that $\Gamma(R)$ is a proper subset of R . We show first that in this case $\Gamma(R)$ is open. Suppose not, i.e., suppose there is an $\eta \in \Gamma(R)$ which is not an interior point. Since $\Gamma(R)$ is connected, η must be an extreme point; for definiteness let $\eta = \sup \Gamma(R)$. We note that for each $z \in \Gamma(R)$, there is a unique solution such that $x(t_1) = r_1, x(t_2) = z$. (In the terminology of [4] the lower inverse of Γ, Γ^- , is a single valued mapping). Two such solutions would contradict the assumed uniqueness of boundary value problems. Let α be the slope at t_1 of the unique solution $x(t)$ with $x(t_1) = r_1, x(t_2) = \eta$. Let $\beta > \alpha$. Then every solution $y(t)$, with $y(t_1) = r_1, y'(t_1) = \beta$ in the solution funnel $F(r_1, \beta, t)$ satisfies $y(t_2) > \eta = x(t_2)$ by the uniqueness of solutions of the boundary value problem, contradicting the maximality of η . Thus $\Gamma(R)$ is an open interval.

Let $p_0 = \sup \Gamma(R)$ and suppose $p_0 < \infty$. From the above discussion $p_0 \notin \Gamma(R)$. Let $p_n \rightarrow p_0, p_n \in \Gamma(R)$. $p_n \in \Gamma(R)$ means that there exists a solution $x_n(t)$ which satisfies $x_n(t_1) = r_1, x_n(t_2) = p_n$. As noted above this $x_n(t)$ is the unique solution of (1) satisfying these conditions. Because of this uniqueness the arguments given by Lasota and Opial [3] may be followed through the selection of the sequence $\{s_n\}$ and a subsequence of $\{x_n(t)\}$ such that $\lim_{n \rightarrow \infty} s_n = s_0, \lim_{n \rightarrow \infty} x_n(s_n) = x_0$ and $\lim_{n \rightarrow \infty} x'_n(s_n) = u_0$.

Let K be a closed ball in E^3 of radius ϵ about (s_0, x_0, u_0) where $s_0 + \epsilon < b$. For large $n, (s_n, x_n(s_n), x'_n(s_n)) \in K$. By [5, Theorem 3.7] there exists a $\beta(K)$ such that $[(x_n(t)^2 + (x'_n(t))^2)]^{1/2} < \beta(K)$ for all $t \in [t_1, s_0 + \epsilon]$. Let $x(t)$ be any solution with $x(t_1) = r_1, x'(t_1) = \beta(K)$. Then $x(t) > x_n(t)$ for $t_1 < t \leq t_2$ and in particular $x(t_2) > x_n(t_2)$ or $x(t_2) \geq p_0$, a contradiction. A similar proof shows that $\inf \Gamma(R) = -\infty$.

Added in proof. This theorem has also been proved by K. Schrader in a paper to appear in the *Journal of Differential Equations*.

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Date communicated: JULY 19, 1968