*An Invariance Principle for Conditioned Sums of Independent Random Variables**

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1. Introduction and notation. In this paper, we establish an invariance principle for processes which are derived from sums of independent random variables in the domain of attraction of a stable law by conditioning on the event that the n th partial sum lies in a given interval. Following this, we will show how this invariance principle may be used to obtain a quick proof of the weak convergence of normalized empirical distribution functions to the Brownian bridge.

By applying certain continuous functionals to the conditioned processes, one may also obtain the existence of limits of conditional distributions associated with sums of independent, identically distributed random variables. Many specific limit theorems of this type were discussed by Dwass and Karlin [2] in 1963. They obtained explicit limiting distributions under the assumption of the existence of the limits involved and in many cases they were able to establish the existence of these limits. The present work was motivated by the investigations of Dwass and Karlin, and serves to give a general framework within which one may establish the existence of some of the limits considered by them.

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The following is the specific situation to be investigated. Let $\xi_1, \xi_2, \dots$ be a sequence of independent, identically distributed, random variables with common characteristic function $g(s)$, which are in the domain of attraction of a non-degenerate stable law. That is, if we let $S_n = \xi_1 + \dots + \xi_n$, there are constants $b_n > 0$ and a_n so that $(S_n - a_n)/b_n$ converges in distribution to a random variable $X(1)$ which has characteristic function

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$$f(s) = \exp \left[ids - c |s|^\alpha \left\{ 1 + i\beta \frac{s}{|s|} \omega(s, \alpha) \right\} \right],$$

$$\omega(s, \alpha) = \begin{cases} \tan \frac{\pi}{2} \alpha & \text{if } \alpha \neq 1 \\ \frac{2}{\pi} \log |s| & \text{if } \alpha = 1 \end{cases}$$

where d, c, β and α are constants satisfying $c > 0$, $-1 \leq \beta \leq 1$, and $0 < \alpha \leq 2$.

Since $|f(s)| = e^{-c|s|^\alpha}$ and $c > 0$, the distribution of $X(1)$ has a density $p(u)$ which is uniformly continuous on the whole line. For future reference, we now fix a real number x_0 which satisfies $p(x_0) > 0$, and define a function $\eta(t)$, which will occur frequently in the following pages, by

$$\eta(t) = \begin{cases} d(t - t^{1/\alpha}) & \text{if } \alpha \neq 1 \\ \frac{2c\beta}{\pi} t \log t & \text{if } \alpha = 1 \end{cases}, \quad 0 \leq t \leq 1.$$

We will consider weak convergence in the space $D[0, 1]$ of functions on $[0, 1]$ which are right-continuous and have no discontinuities of the second kind. For a description of the space and its metric topology, one may refer to Chapter 3 of Billingsley [1]. In the present context, the word "measurable" refers to measurability with respect to the topological Borel field of $D[0, 1]$. For each n , let $X_n(t)$ be the random element of $D[0, 1]$ defined by

$$X_n(t) = \begin{cases} \frac{S_{[(n+1)t]} - \frac{[(n+1)t]}{n} a_n}{b_n} & \text{for } 0 \leq t < 1 \\ \frac{S_n - a_n}{b_n} & \text{for } t = 1 \end{cases}$$

where $[\cdot]$ is the greatest integer function. Notice that for each n , $X_n(t)$ is continuous at each irrational t and at 0 and 1.

If the distribution of ξ_i is of lattice type, we define the lattice which supports S_n by $D_n = \{(2m\pi + na)/t_0; m \text{ is an integer}\}$, where t_0 is the smallest positive t such that $|g(t)| = 1$ and a satisfies $g(t_0) = e^{ia}$. Let $\{\gamma_n\}$ be a sequence of non-negative constants which (a) are bounded away from zero and infinity if ξ_i does not have a lattice distribution, or (b) are bounded away from infinity and satisfy $D_n \cap \{x; |x - (a_n + b_n x_0)| \leq \gamma_n\} \neq \emptyset$ for all sufficiently large n if ξ_i has a lattice distribution. The processes which we will consider in this paper are obtained from X_n by conditioning on the event $\{|X_n(1) - x_0| \leq \gamma_n/b_n\}$.

2. The invariance principle. The two most important tools used in the proof of the invariance principle for the conditioned processes are

- (a) the invariance principle for the original processes X_n , and

(b) a ratio limit theorem for the distributions of the relevant partial sums.

The first of these is an immediate consequence of a result of Skorokhod, while the second follows from a local limit law of Charles Stone. We consider first the invariance principle for the processes X_n .

Theorem 1. X_n converges weakly to X in $D[0, 1]$, where $X(t)$ is a temporally homogeneous process with independent increments which has no fixed times of discontinuity and whose one-dimensional distributions are the same as those of $\eta(t) + t^{1/\alpha}X(1)$.

Proof. It is known that there is a process $X(t)$ in $D[0, 1]$ with the properties mentioned above. (See [4], for example. Alternatively, its existence can be derived easily from Theorem 15.7 of [1].) For a fixed t , $0 \leq t < 1$, the characteristic function of $X_n(t)$ is given by

$$\exp \left\{ -i \frac{[(n+1)t]}{n} \frac{a_n}{b_n} s \right\} g^{[(n+1)t]} \left(\frac{s}{b_n} \right)$$

which converges to $[f(s)]^t$ uniformly for s in compact sets, since $(S_n - a_n)/b_n$ converges in distribution to $X(1)$. As a consequence of the identity

$$\begin{aligned} idst - ct |s|^\alpha \left\{ 1 + i\beta \frac{s}{|s|} \omega(s, \alpha) \right\} \\ = is\eta(t) + idst^{1/\alpha} - ct |s|^\alpha \left\{ 1 + i\beta \frac{s}{|s|} \omega(st^{1/\alpha}, \alpha) \right\}, \end{aligned}$$

we may conclude that the characteristic function of $X(t)$ is $[f(s)]^t$. So, the one-dimensional distributions of $X_n(t)$ converge to the one-dimensional distributions of $X(t)$. An application of Skorokhod's theorem (Theorem 2.7 of [5]) yields the conclusion that X_n converges weakly to X in $D[0, 1]$.

We will use the ratio limit theorem in the following form.

Theorem 2. As $n \rightarrow \infty$,

$$\frac{P\left(|X_n(t) - u| \leq \frac{\gamma_n}{b_n}\right)}{P\left(|X_n(1) - x_0| \leq \frac{\gamma_n}{b_n}\right)} \rightarrow \frac{p\left(\frac{u - \eta(t)}{t^{1/\alpha}}\right)}{t^{1/\alpha} p(x_0)}.$$

The above convergence is uniform for t bounded away from zero and for (a) $u \in \mathbb{R}^1$ if ξ_i is nonlattice, and (b) u such that $x_0 - u$ is in the support of the measure $P(X_n(1) - X_n(t) \in dx)$ if ξ_i is lattice.

Proof. In the nonlattice case, we may apply Theorem 1 of [6] immediately in order to write

$$\begin{aligned}
& \frac{P\left(|X_n(t) - u| \leq \frac{\gamma_n}{b_n}\right)}{P\left(|X_n(1) - x_0| \leq \frac{\gamma_n}{b_n}\right)} \\
&= \frac{P\left(\left|S_{\lfloor (n+1)t \rfloor} - \frac{\lfloor (n+1)t \rfloor}{n} a_n - ub_n\right| \leq \gamma_n\right)}{P(|S_n - a_n - x_0 b_n| \leq \gamma_n)} \\
&= \frac{\frac{2\gamma_n}{b_{\lfloor (n+1)t \rfloor}} p\left(\frac{ub_n + \frac{\lfloor (n+1)t \rfloor}{n} a_n - a_{\lfloor (n+1)t \rfloor}}{b_{\lfloor (n+1)t \rfloor}}\right)}{\frac{2\gamma_n}{b_n} p(x_0) + o\left(\frac{1}{b_n}\right)} + o\left(\frac{1}{b_{\lfloor (n+1)t \rfloor}}\right)
\end{aligned}$$

where $b_n o(1/b_n) \rightarrow 0$ uniformly for $u \in R^1$. Now, it is a known and easily verifiable consequence of the convergence of $(S_n - a_n)/b_n$ to $X(1)$ that

$$b_{\lfloor (n+1)t \rfloor}/b_n \rightarrow t^{1/\alpha} \quad \text{and} \quad \left(a_{\lfloor (n+1)t \rfloor} - \frac{\lfloor (n+1)t \rfloor}{n} a_n\right)/b_n \rightarrow \eta(t)$$

uniformly for $t \in [\delta, 1]$, where δ is any positive number. So,

$$\frac{\left(ub_n + \frac{\lfloor (n+1)t \rfloor}{n} a_n - a_{\lfloor (n+1)t \rfloor}\right)}{b_{\lfloor (n+1)t \rfloor}} \rightarrow \frac{u - \eta(t)}{t^{1/\alpha}}$$

uniformly for t bounded away from zero and u in a compact set. Since $p(x)$ is uniformly continuous and $p(x) \rightarrow 0$ as $|x| \rightarrow \infty$,

$$p\left(\frac{ub_n + \frac{\lfloor (n+1)t \rfloor}{n} a_n - a_{\lfloor (n+1)t \rfloor}}{b_{\lfloor (n+1)t \rfloor}}\right) \rightarrow p\left(\frac{u - \eta(t)}{t^{1/\alpha}}\right)$$

uniformly for t bounded away from zero and all real u . Since the γ_n 's are bounded away from zero, we see that the theorem holds in the nonlattice case.

Now consider the second case, in which the ξ_i 's have a lattice distribution. Let $\{z_i^n; i = 1, \dots, N_n\}$ be the set of elements of $D_n \cap \{x; |x - (a_n + b_n x_0)| \leq \gamma_n\}$, which is finite and non-empty because of the way the γ_n 's were selected. Then, by Theorem 1 of [6], we may conclude that

$$P(|S_n - a_n - x_0 b_n| \leq \gamma_n) = \sum_{i=1}^{N_n} P(S_n = z_i^n) = \sum_{i=1}^{N_n} \left\{ \frac{2\pi}{b_n t_0} p\left(\frac{z_i^n - a_n}{b_n}\right) + o\left(\frac{1}{b_n}\right) \right\}$$

where $b_n o(1/b_n) \rightarrow 0$ uniformly for $i = 1, \dots, N_n$. Using the uniform continuity of $p(x)$ and the fact that $|z_i^n - a_n - b_n x_0| \leq \gamma_n$, we have that

$$(1) \quad \frac{b_n}{N_n} P(|S_n - a_n - x_0 b_n| \leq \gamma_n) = \frac{2\pi}{t_0} p(x_0) + o(1).$$

Let $\mu_i^n = z_i^n + b_n(u - x_0) + a_n([(n+1)t]/n - 1)$, where u is any real number such that $x_0 - u$ is in the support of $P(X_n(1) - X_n(t) \in dx)$. Then $\{\mu_i^n; i = 1, \dots, N_n\}$ is exactly the set of elements of

$$D_{[(n+1)t]} \cap \left\{x; \left|x - \frac{[(n+1)t]}{n} a_n - ub_n\right| \leq \gamma_n\right\}.$$

So, by Theorem 1 of [6],

$$\begin{aligned} P\left(\left|S_{[(n+1)t]} - \frac{[(n+1)t]}{n} a_n - ub_n\right| \leq \gamma_n\right) \\ = \sum_{i=1}^{N_n} P(S_{[(n+1)t]} = \mu_i^n) = \sum_{i=1}^{N_n} \left\{ \frac{2\pi}{t_0 b_{[(n+1)t]}} p\left(\frac{\mu_i^n - a_{[(n+1)t]}}{b_{[(n+1)t]}}\right) + o\left(\frac{1}{b_{[(n+1)t]}}\right) \right\} \end{aligned}$$

where $b_{[(n+1)t]} o(1/b_{[(n+1)t]}) \rightarrow 0$ uniformly for u such that $x_0 - u$ is in the support of $P(X_n(1) - X_n(t) \in dx)$ and for $i = 1, \dots, N_n$.

Note that

$$\begin{aligned} \left| \frac{\mu_i^n - a_{[(n+1)t]}}{b_{[(n+1)t]}} - \frac{(u - \eta(t))}{t^{1/\alpha}} \right| &\leq \frac{\gamma_n}{b_{[(n+1)t]}} \\ &+ \left| \frac{\frac{[(n+1)t]}{n} a_n - a_{[(n+1)t]}}{b_{[(n+1)t]}} + \frac{\eta(t)}{t^{1/\alpha}} + u \left| \frac{b_n}{b_{[(n+1)t]}} - \frac{1}{t^{1/\alpha}} \right| \right|, \end{aligned}$$

so that $(\mu_i^n - a_{[(n+1)t]})/b_{[(n+1)t]} = (u - \eta(t))/t^{1/\alpha} + o(1)(1 + |u|)$, where $o(1) \rightarrow 0$ uniformly for u such that $x_0 - u$ is in the support of $P(X_n(1) - X_n(t) \in dx)$ and for $i = 1, \dots, N_n$. So, we may write

$$\frac{b_{[(n+1)t]}}{N_n} P\left(\left|S_{[(n+1)t]} - \frac{[(n+1)t]}{n} a_n - ub_n\right| \leq \gamma_n\right) = \frac{2\pi}{t_0} p\left(\frac{u - \eta(t)}{t^{1/\alpha}}\right) + o(1).$$

This, together with (1), yields the statement of the theorem in the lattice case.

In the proof of the conditioned invariance principle, we will want to transform random elements of $D[0, 1]$ into random elements of $D[0, t]$ for some $t < 1$ by "looking at the process only up to time t ". We will now make precise the meaning of this transformation. For this purpose, we define a mapping $T_t : D[0, 1] \rightarrow D[0, t]$ by $(T_t x)(s) = x(s)$ if $s < t$ and $(T_t x)(t) = \lim_{s \uparrow t} x(s)$, and prove the following theorem.

Theorem 3. Choose $t \in (0, 1)$, and let $B_t = \{x(\cdot) \in D[0, 1] \mid x(\cdot) \text{ is continuous at } t\}$. Let μ be a probability measure on $D[0, 1]$ such that $\mu(B_t) = 1$. Then there exists a probability measure μ^t on $D[0, t]$ such that $\mu^t(A) = \mu[(T_t^{-1}A) \cap B_t]$ for each measurable subset A of $D[0, t]$.

Proof. For $s \in [0, 1]$, let π_s be the projection on $D[0, 1]$ defined by $\pi_s x = x(s)$. Then π_s is a measurable mapping (see [1]). So, since we may express B_t

as $\{x(\cdot) \in D[0, 1] \mid \pi_t x = \pi_{t-} x\}$, B_t is a measurable subset of $D[0, 1]$. We will now show that T_t is continuous at each point of B_t .

Let $x(\cdot) \in B_t$ and $x_n(\cdot) \in D[0, 1]$ with $x_n(\cdot) \rightarrow x(\cdot)$ in the $D[0, 1]$ topology. Then there exists a sequence of strictly increasing continuous functions λ_n on $[0, 1]$ onto $[0, 1]$ such that $\sup_{0 \leq s \leq 1} |x_n(s) - x(\lambda_n s)| \rightarrow 0$ and $\sup_{0 \leq s \leq 1} |\lambda_n s - s| \rightarrow 0$ as $n \rightarrow \infty$. Let $\epsilon_n = \sup_{0 \leq s \leq 1} |\lambda_n s - s|$. Define a sequence of strictly increasing continuous functions λ'_n on $[0, t]$ onto $[0, t]$ by

$$\lambda'_n(s) = \begin{cases} \lambda_n(s) & \text{if } 0 \leq s \leq t - 2\epsilon_n \\ \frac{1}{2\epsilon_n} \{t - \lambda_n(t - 2\epsilon_n)\}(s - t) + t & \text{if } t - 2\epsilon_n \leq s \leq t. \end{cases}$$

Then $\sup_{0 \leq s \leq t} |\lambda'_n(s) - s| \leq \epsilon_n \rightarrow 0$, and

$$\begin{aligned} \sup_{0 \leq s \leq t} |x_n(s) - x(\lambda'_n s)| &\leq \sup_{0 \leq s \leq 1} |x_n(s) - x(\lambda_n s)| \\ &\quad + \sup_{t - 2\epsilon_n \leq s \leq t} \{|x_n(s) - x(\lambda_n s)| + |x(\lambda_n s) - x(\lambda'_n s)|\} \end{aligned}$$

which tends to zero as $n \rightarrow \infty$ since $x(\cdot)$ is continuous at t . So, $T_t x_n \rightarrow T_t x$ in $D[0, t]$, and we may conclude that T_t is continuous at each point of B_t . So, for every measurable subset A of $D[0, t]$, $(T_t^{-1}A) \cap B_t$ is a measurable subset of $D[0, 1]$. Hence, we may define $\mu^t(A) = \mu\{(T_t^{-1}A) \cap B_t\}$.

We now turn to the main result of this paper. Note that it follows from Theorem 2 that $P(|X_n(1) - x_0| \leq \gamma_n/b_n) > 0$ for all n beyond a certain point. So, we may define random elements Y_n on $D[0, 1]$ by requiring that $P(Y_n \in A) = P(X_n \in A \mid |X_n(1) - x_0| \leq \gamma_n/b_n)$ for each measurable subset A of $D[0, 1]$. The invariance principle for the sequence $\{Y_n\}$ is stated in the following theorem.

Theorem 4. *Y_n converges weakly in $D[0, 1]$ to a random element Y , where Y has finite dimensional distributions given by*

$$\begin{aligned} P(Y(t_1) \in dx_1, \dots, Y(t_k) \in dx_k) \\ = \frac{p\left(\frac{x_0 - x_k - \eta(1 - t_k)}{(1 - t_k)^{1/\alpha}}\right)}{(1 - t_k)^{1/\alpha} p(x_0)} P(X(t_1) \in dx_1, \dots, X(t_k) \in dx_k) \end{aligned}$$

for $0 < t_1 < t_2 < \dots < t_k < 1$ and $k = 1, 2, \dots$; $Y(0) \equiv 0$ and $Y(1) \equiv x_0$.

Proof. We will first prove the convergence of the finite dimensional distributions. Let $t_1, \dots, t_k$ be irrational numbers satisfying $0 < t_1 < \dots < t_k < 1$. Then $(n+1)t_k$ is not an integer for any n , so that $n = [(n+1)t_k] + [(n+1) \cdot (1 - t_k)]$. So, $X_n(1) - X_n(t_k)$ has the same distribution as $X_n(1 - t_k)$. Using this fact, and the independence of $X_n(1) - X_n(t_k)$ from $\{X_n(t_1), \dots, X_n(t_k)\}$, we may write

$$P(Y_n(t_1) \leq x_1, \dots, Y_n(t_k) \leq x_k)$$

$$\begin{aligned}
&= P\left(X_n(t_1) \leq x_1, \dots, X_n(t_k) \leq x_k \mid |X_n(1) - x_0| \leq \frac{\gamma_n}{b_n}\right) \\
&= \int_{-\infty}^{x_k} \frac{P\left(X_n(t_1) \leq x_1, \dots, X_n(t_{k-1}) \leq x_{k-1}, X_n(t_k) \leq x_k, |X_n(1) - x_0| \leq \frac{\gamma_n}{b_n}\right)}{P\left(|X_n(1) - x_0| \leq \frac{\gamma_n}{b_n}\right)} dy \\
&= \int_{-\infty}^{x_k} P(X_n(t_1) \leq x_1, \dots, X_n(t_{k-1}) \leq x_{k-1}, X_n(t_k) \leq x_k) \cdot \frac{P\left(|X_n(1 - t_k) + y - x_0| \leq \frac{\gamma_n}{b_n}\right)}{P\left(|X_n(1) - x_0| \leq \frac{\gamma_n}{b_n}\right)} dy.
\end{aligned}$$

By Theorem 2,

$$\frac{P\left(|X_n(1 - t_k) + y - x_0| \leq \frac{\gamma_n}{b_n}\right)}{P\left(|X_n(1) - x_0| \leq \frac{\gamma_n}{b_n}\right)} \rightarrow \frac{p\left(\frac{x_0 - y - \eta(1 - t_k)}{(1 - t_k)^{1/\alpha}}\right)}{(1 - t_k)^{1/\alpha} p(x_0)}$$

uniformly for y in the support of $P(X_n(t_k) \leq x_k)$. By Theorem 1, the measure $P(X_n(t_1) \leq x_1, \dots, X_n(t_{k-1}) \leq x_{k-1}, X_n(t_k) \leq x_k) dy$ converges weakly to the measure $P(X(t_1) \leq x_1, \dots, X(t_{k-1}) \leq x_{k-1}, X(t_k) \leq x_k) dy$. So, $P(Y_n(t_1) \leq x_1, \dots, Y_n(t_k) \leq x_k)$ converges to

$$\int_{-\infty}^{x_k} P(X(t_1) \leq x_1, \dots, X(t_{k-1}) \leq x_{k-1}, X(t_k) \leq x_k) dy \frac{p\left(\frac{x_0 - y - \eta(1 - t_k)}{(1 - t_k)^{1/\alpha}}\right)}{(1 - t_k)^{1/\alpha} p(x_0)},$$

which proves the convergence of the finite dimensional distributions of Y_n , at least for irrational t 's. Since $Y_n(0) \equiv Y(0) \equiv 0$, $Y(1) \equiv x_0$ and $|Y_n(1) - x_0| \leq \gamma_n/b_n \rightarrow 0$ with probability one, the convergence of the finite dimensional distributions is established for t 's which are irrational, zero or one.

We turn now to consider the question of tightness for the sequence $\{Y_n\}$. For this purpose, we fix an irrational t in $(0, 1)$. Theorem 2 implies the existence of numbers $M(t)$ and $n(t)$ which depend only on t such that

$$(2) \quad \frac{P\left(|X_n(1 - t) + x - x_0| \leq \frac{\gamma_n}{b_n}\right)}{P\left(|X_n(1) - x_0| \leq \frac{\gamma_n}{b_n}\right)} \leq M(t)$$

for $n \geq n(t)$ and x in the support of $P(X_n(t) \leq x)$. For each n , let X'_n and Y'_n be random elements of $D[0, t]$ which are constructed in the manner described in

Theorem 3 from X_n and Y_n respectively. Then, for any measurable subset A of $D[0, t]$,

$$\begin{aligned} P(Y_n^t \in A) &= P\left(X_n \in (T_t^{-1}A) \cap B_t \mid |X_n(1) - x_0| \leq \frac{\gamma_n}{b_n}\right) \\ &= \int_{-\infty}^{\infty} \frac{P\left(X_n \in (T_t^{-1}A) \cap B_t, X_n(t) \in dx, |X_n(1) - x_0| \leq \frac{\gamma_n}{b_n}\right)}{P\left(|X_n(1) - x_0| \leq \frac{\gamma_n}{b_n}\right)} \\ &= \int_{-\infty}^{\infty} P(X_n \in (T_t^{-1}A) \cap B_t, X_n(t) \in dx) \\ &\quad \cdot \frac{P\left(|X_n(1) - X_n(t) + x - x_0| \leq \frac{\gamma_n}{b_n}\right)}{P\left(|X_n(1) - x_0| \leq \frac{\gamma_n}{b_n}\right)}, \end{aligned}$$

where we have used the independence of $X_n(1) - X_n(t)$ from $\{X_n(s), s \leq t\}$. It now follows from (2) that, for $n \geq n(t)$,

$$\begin{aligned} (3) \quad P(Y_n^t \in A) &\leq M(t) \int_{-\infty}^{\infty} P(X_n \in (T_t^{-1}A) \cap B_t, X_n(t) \in dx) \\ &= M(t)P(X_n^t \in A). \end{aligned}$$

The proof of tightness for the sequence $\{Y_n\}$ will now be reduced to consideration of the behavior of Y_n in the intervals $[0, t]$ and $[t, 1]$ separately. To accomplish the first, we will use the inequality just derived. For the second, it suffices to appeal to the fact that the behavior of the process Y_n near $t = 1$ is very similar to its behavior near $t = 0$.

Let $\epsilon, \eta > 0$ be given. Since $X(0) \equiv 0$ and $X(s)$ is continuous at $s = 0$, $\sup_{0 \leq r \leq s} |X(r)| \rightarrow 0$ as $s \downarrow 0$ with probability one. So, there is an irrational t , $0 < t < 1/2^{1/2}$, such that $P(\sup_{s \leq t} |X(s)| \geq \epsilon/4) < \eta/4M(1/2^{1/2})$. The functional $\sup_{s \leq t} |x(s)|$ is continuous at each $x(\cdot) \in D[0, 1]$ which is continuous at t . Since X is continuous at t with probability one, and X_n converges weakly to X , $\sup_{s \leq t} |X_n(s)|$ converges in distribution to $\sup_{s \leq t} |X(s)|$. So, there is an n_0 such that for $n \geq n_0$,

$$P\left(\sup_{s \leq t} |X_n(s)| \geq \frac{\epsilon}{4}\right) < \frac{\eta}{2M(1/2^{1/2})}.$$

From (3) it now follows that

$$P\left(\sup_{s \leq t} |Y_n(s)| \geq \frac{\epsilon}{4}\right) \leq \frac{\eta}{2} \quad \text{for } n \geq \max\{n_0, n(1/2^{1/2})\}.$$

Now, for $0 = s_1 < s_2 < \dots < s_k = 1$, where each s_i ($2 \leq i \leq k-1$) is irrational, the random vector $(X_n(1 - s_1), X_n(1 - s_2), \dots, X_n(1 - s_k))$ has

the same distribution as the random vector $(X_n(1) - X_n(s_1), \dots, X_n(1) - X_n(s_k))$. So, the random vectors $(Y_n(1 - s_1), \dots, Y_n(1 - s_k))$ and $(Y_n(1) - Y_n(s_1), \dots, Y_n(1) - Y_n(s_k))$ also have the same distribution. Note that this follows because $X_n(0) \equiv 0$, which implies that $X_n(1)$ is invariant under the transformation $X_n(1 - s) \rightarrow X_n(1) - X_n(s)$. Using this result, we obtain that for $n \geq \max \{n_0, n(1/2^{1/2})\}$,

$$P\left(\sup_{s_1, s_2 \geq 1-t} |Y_n(s_1) - Y_n(s_2)| \geq \frac{\epsilon}{2}\right) \leq P\left(\sup_{s \geq 1-t} |Y_n(1) - Y_n(s)| \geq \frac{\epsilon}{4}\right) = P\left(\sup_{s \leq t} |Y_n(s)| \geq \frac{\epsilon}{4}\right) \leq \frac{\eta}{2}.$$

Following Billingsley [1], we define functionals on $D[0, 1]$ for each $\delta > 0$ and $r \in (0, 1]$ by letting

$$\omega_x[0, \delta] = \sup \{|x(s_1) - x(s_2)|; 0 \leq s_1, s_2 < \delta\}$$

$$\omega_x[1 - \delta, 1] = \sup \{|x(s_1) - x(s_2)|; 1 - \delta \leq s_1, s_2 < 1\}, \text{ and}$$

$$\omega_x''(\delta, r) = \sup \{\min(|x(s) - x(s_1)|, |x(s_2) - x(s)|);$$

$$0 \leq s_1 \leq s \leq s_2 \leq r, s_2 - s_1 \leq \delta\}.$$

According to Theorem 15.3 of [1], the tightness of $\{X_n\}$ implies the existence of numbers a, δ , and n_1 such that $a > \epsilon/2$, $0 < \delta < t$,

$$P\left(\sup_{s \leq 1} |X_n(s)| > a - \frac{\epsilon}{2}\right) \leq \eta/2M(1 - t) \quad \text{for } n \geq 1,$$

$$P(\omega_{X_n}[0, \delta] \geq \epsilon) \leq \eta/M(1 - t) \quad \text{for } n \geq n_1, \text{ and}$$

$$P\left(\omega_{X_n}''(\delta, 1) \geq \frac{\epsilon}{2}\right) \leq \eta/2M\left(1 - \frac{t}{2}\right) \quad \text{for } n \geq n_1.$$

So, for $n \geq \max \{n_0, n(1/2^{1/2}), n_0(1 - t)\}$, we have

$$\begin{aligned} (4) \quad & P\left(\sup_{s \leq 1} |Y_n(s)| > a\right) \\ & \leq P\left(\sup_{s \leq 1-t} |Y_n(s)| > a - \frac{\epsilon}{2}\right) + P\left(\sup_{s \geq 1-t} |Y_n(s) - Y_n(1 - t)| > \frac{\epsilon}{2}\right) \\ & \leq M(1 - t)P\left(\sup_{s \leq 1} |X_n(s)| > a - \frac{\epsilon}{2}\right) + \frac{\eta}{2} \leq \eta. \end{aligned}$$

Now, for each $x(\cdot) \in D[0, 1]$, $\omega_x''(\delta, 1) \leq \omega_x''(\delta, 1 - t/2) + \sup_{s_1, s_2 \geq 1-t} |x(s_1) - x(s_2)|$. Therefore, for $n \geq \max \{n_0, n_1, n(1/2^{1/2}), n(1 - t), n(1 - t/2)\}$, the following three inequalities hold:

$$\begin{aligned} (5) \quad & P(\omega_{Y_n}''(\delta, 1) \geq \epsilon) \leq M\left(1 - \frac{t}{2}\right)P\left(\omega_{X_n}''(\delta, 1) \geq \frac{\epsilon}{2}\right) \\ & + P\left(\sup_{s_1, s_2 \geq 1-t} |Y_n(s_1) - Y_n(s_2)| \geq \frac{\epsilon}{2}\right) \leq \eta \end{aligned}$$

$$(6) \quad P(\omega_{Y_n}[0, \delta] \geq \epsilon) \leq M(1 - t)P(\omega_{X_n}[0, \delta] \geq \epsilon) \leq \eta,$$

and

$$(7) \quad P(\omega_{Y_n}[1 - \delta, 1] \geq \epsilon) \leq \eta.$$

By Theorem 15.3 of [1], it now follows from (4), (5), (6) and (7) that the sequence $\{Y_n\}$ is tight in $D[0, 1]$. A slight modification of Theorem 15.1 of [1] now allows us to conclude that Y_n converges weakly to Y . Note that the existence of a random element Y in $D[0, 1]$, with the finite dimensional distributions given in the statement of the theorem, follows from the tightness of $\{Y_n\}$ and the convergence of its finite dimensional distributions to the appropriate limits. There is no need to prove its existence separately.

Corollary. *Suppose that the random variables ξ_i have a finite variance $\sigma^2 > 0$, that the normalizing constants are given by $a_n = nE(\xi_i)$ and $b_n = \sigma n^{1/2}$, and that x_0 is chosen to be zero. Then Y_n converges weakly to the Brownian bridge.*

Proof. In this case, $X(1)$ has the standard normal distribution, so X is standard Brownian motion on $[0, 1]$. It is then easy to check that the finite dimensional distributions of Y given in the statement of Theorem 4 are the same as those of the Brownian bridge, which is defined by $X(t) - tX(1)$.

It should be noted that this corollary can also be derived from a theorem due to Billingsley (Theorem 24.2 of [1]) concerning the weak convergence of sums of exchangeable random variables to the Brownian bridge. However, this approach does not extend to the more general situation of Theorem 4. Other related results may be found in Wichura [8].

3. Application to empirical distribution functions. It is well known that the normalized empirical distribution functions obtained from independent, identically distributed random variables with common continuous distribution function $F(t)$, where $F(0) = 0$ and $F(1) = 1$, converge weakly in $D[0, 1]$ to the random element $W^0(F(t))$, where $W^0(t)$ represents the Brownian bridge (see, for example, Chapter 3 of [1]).

Since it is easy to reduce the proof of the above statement to the special case in which the underlying random variables are uniformly distributed on $[0, 1]$, we restrict our consideration to that case. We will deduce the result in that case from the preceding invariance principle for conditioned sums of independent random variables.

Theorem 5. *Let $\eta_1, \eta_2, \dots$ be independent random variables, each of which is uniformly distributed on $[0, 1]$. The empirical distribution function of $\eta_1, \dots, \eta_n$ is defined by $F_n(t) = (1/n) \sum_{k=1}^n 1_{[0, t]}(\eta_k)$, where $1_{[0, t]}(x)$ is the indicator function of the interval $[0, t]$. Let $Z_n(t) = n^{1/2}(F_n(t) - t)$. Then Z_n converges weakly in $D[0, 1]$ to W^0 , the Brownian bridge.*

Proof. Note that the finite dimensional distributions of $n^{1/2}Z_n(t) + nt$ for

$0 < t_1 < \dots < t_k < 1$ are multinomial with parameters $n, t_1, t_2 - t_1, \dots, t_k - t_{k-1}$, and $1 - t_k$. For each n , define a new random element of $D[0, 1]$ by $\tilde{Z}_n(t) = Z_n([(n+1)t]/n)$ if $t < 1$, and $\tilde{Z}_n(1) = Z_n(1)$.

Now, let $\xi_1, \xi_2, \dots$ be a sequence of independent random variables, each of which is Poisson distributed with parameter 1. Let Y_n be the random element of $D[0, 1]$ which is constructed from the ξ_i 's by conditioning their partial sums in the manner of the corollary to Theorem 4. We take $\gamma_n = 0$ for each n , which we may do since the Poisson distribution is of lattice type, and in this case $a_n = n$. By applying the corollary to Theorem 4, we may conclude that Y_n converges weakly in $D[0, 1]$ to W^0 . It is easy to see that the finite dimensional distributions of $n^{1/2}Y_n(t) + [(n+1)t]$ and of $n^{1/2}\tilde{Z}_n(t) + [(n+1)t]$ for $0 < t_1 < \dots < t_k < 1$ are multinomial with parameters

$$n, \frac{[(n+1)t_1]}{n}, \frac{[(n+1)t_2] - [(n+1)t_1]}{n}, \dots, \frac{[(n+1)t_k] - [(n+1)t_{k-1}]}{n}, \frac{n - [(n+1)t_k]}{n}.$$

So, $Y_n(t)$ and $\tilde{Z}_n(t)$ induce the same measures on $D[0, 1]$ for each n . Hence, $\tilde{Z}_n$ converges weakly in $D[0, 1]$ to W^0 .

It now only remains to show that Z_n and $\tilde{Z}_n$ are sufficiently close to each other. For this purpose, we define a functional ϕ on $D[0, 1]$ by $\phi(x) = \sup \{|x(t) - x(t-)|; t \in (0, 1]\}$. Note that $\phi(x) < \infty$ for each $x \in D[0, 1]$. We will show that ϕ is continuous a.s. with respect to W^0 . Let $x_n(\cdot) \rightarrow x(\cdot)$ in $D[0, 1]$, where $x(\cdot)$ is continuous on $[0, 1]$. Since $x(\cdot)$ is continuous, $x_n(\cdot) \rightarrow x(\cdot)$ uniformly on $[0, 1]$. So,

$$\phi x_n \leq \sup_t |x_n(t) - x(t)| + \sup_t |x_n(t-) - x(t-)| \leq 2 \sup_t |x_n(t) - x(t)| \rightarrow 0.$$

Since $\phi x = 0$, ϕ is continuous at $x(\cdot)$. So, using the fact that the paths of W^0 are a.s. continuous, we see that $\phi \tilde{Z}_n$ converges to zero in probability.

For $(k-1)/n \leq t \leq k/n$,

$$|\tilde{Z}_n(t) - Z_n(t)| \leq \frac{2}{n^{1/2}} + \left| \tilde{Z}_n\left(\frac{k}{n+1}\right) - \tilde{Z}_n\left(\frac{k}{n+1}-\right) \right|.$$

So, $\sup_{0 \leq t \leq 1} |Z_n(t) - \tilde{Z}_n(t)| \leq 2/n^{1/2} + \phi \tilde{Z}_n \rightarrow 0$ in probability. By Theorem 4.1 of [1], we may conclude that Z_n converges weakly to W^0 in $D[0, 1]$.

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