*On Spectral Permanence**

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Communicated by E. R. LORCH

1. Preliminaries. Let D be a directed set and let $\{E_\alpha : \alpha \in D\}$ be a family of complex, separated, barreled and quasi-complete locally convex spaces, all being subspaces of a vector space E . For $\alpha \leq \beta$, $\alpha, \beta \in D$ we assume that $E_\alpha \subset E_\beta$ and that the topology induced on E_α by E_β is weaker than the topology of E_α . Let $E = \bigcup_{\alpha \in D} E_\alpha$ with the inductive limit topology. We know that E is barreled. We shall also assume that E is quasi-complete.

Let T be a continuous linear operator on E which leaves each E_α invariant. Let $T_\alpha (= T|E_\alpha)$ be a spectral operator on E_α , for every $\alpha \in D$. Then we show that T is a spectral operator on E . For the definitions of spectral operator, spectral measure and other related topics we refer to [4].

Throughout this paper, for any locally convex space X , which we shall always assume to be barreled and quasi-complete, X' will denote the dual of X and $L(X)$, the algebra of all continuous linear operators on X with the topology, τ , of uniform convergence on the bounded subsets of X . Let C denote the complex plane and $\hat{C}$, the one point compactification of C . Let Γ be the class of the Borel subsets of C .

On a locally convex space X , the spectrum of an element T in $L(X)$ is defined in the sense of Waelbroeck [5] as follows.

Definition 1. $\lambda \in \hat{C}$ is said to belong to the *resolvent set*, $\rho(T)$, of T if and only if there is a neighbourhood V_λ of λ in $\hat{C}$ such that there is a function $\mu \rightarrow R_\mu$ on $V_\lambda \cap C$ to $L(X)$ satisfying the conditions

- (a) $R_\mu(\mu I - T) = (\mu I - T)R_\mu = I$, for each $\mu \in V_\lambda \cap C$
- (b) $\{R_\mu : \mu \in V_\lambda \cap C\}$ is bounded in $L(X)$.

The *spectrum*, $\text{sp}(T)$, of T is given by

$$\text{sp}(T) = \hat{C} - \rho(T).$$

It may be remarked that if $\lambda \in \rho(T)$ then in the above definition the neighbour-

* This research was partially supported by the Department of University affairs, Toronto, Ontario.

hood V_λ may be taken arbitrarily so long as it does not intersect the spectrum of T .

The following definition and the lemma which generalise the corresponding definition and the result of Dunford [2] are due to Ionescu Tulcea [3].

Definition 2. Let T be a spectral operator on a locally convex space X . The resolvent, $\rho_T(x)$, of an element $x \in X$ is the set of all $\lambda \in \hat{C}$, having the following property:

There is a neighbourhood V_λ of λ and a holomorphic mapping $\mu \rightarrow (\mu)$ of V_λ into X such that

$$(1) \quad (\mu I - T)\tilde{x}(\mu) = x, \quad \mu \in V_\lambda \quad \text{and} \quad \mu \neq \lambda.$$

The *spectrum*, $\sigma_T(x)$, of x is defined by

$$\sigma_T(x) = \hat{C} - \rho_T(x).$$

Lemma 1. Let $\sigma \subset \hat{C}$ be compact. Let T be a spectral operator on X with $P(\cdot)$ as the corresponding spectral measure. Then

$$P(\sigma)X = \{x \in X : \sigma_T(x) \subset \sigma\}.$$

2. Spectral permanence. As in the preliminaries, let

$$E(= \bigcup_{\alpha \in D} E_\alpha)$$

be the inductive limit of the spaces E_α . Let T be a continuous linear operator on E which leaves each E_α invariant. For each $\alpha \in D$, let $T_\alpha (\equiv T|E_\alpha)$ be a spectral operator on E_α with $P_\alpha(\cdot)$ as the corresponding spectral measure.

Proposition 1. For $\alpha \leq \beta$, $P_\beta(\cdot)$ leaves E_α invariant and for each $\sigma \in \Gamma$,

$$P_\alpha(\sigma) = P_\beta(\sigma)|E_\alpha.$$

Proof. For any $x \in E_\alpha$, let $\lambda \rightarrow \tilde{x}(\lambda)$ be the holomorphic function defined on $\rho_{T_\alpha}(x)$ with values in E_α such that

$$(\lambda I - T)\tilde{x}(\lambda) = x, \quad \lambda \in \rho_{T_\alpha}(x).$$

Since, the topology of E_α is stronger than the topology induced on E_α by E_β , it follows that $\lambda \rightarrow \tilde{x}(\lambda)$ is also holomorphic in E_β . Hence $\rho_{T_\alpha}(x) \subset \rho_{T_\beta}(x)$, so that

$$(1) \quad \sigma_{T_\beta}(x) \subset \sigma_{T_\alpha}(x).$$

Let σ be any compact set in $\hat{C}$. By Lemma 1, $\sigma_{T_\alpha}(P_\alpha(\sigma)x) \subset \sigma$ and hence, from (1)

$$\sigma_{T_\beta}(P_\alpha(\sigma)x) \subset \sigma.$$

Another application of Lemma 1 shows that $P_\alpha(\sigma)x \in P_\beta(\sigma)E_\beta$. Hence,

$$P_\beta(\sigma)P_\alpha(\sigma)x = P_\alpha(\sigma)x.$$

Let δ be any compact set disjoint from σ . By multiplying

$$P_\alpha(\delta)x = P_\beta(\delta)P_\alpha(\delta)x$$

with $P_\beta(\sigma)$ we obtain

$$P_\beta(\sigma)P_\alpha(\delta)x = 0.$$

Since $P_\alpha(\cdot)x$ is countably additive in E_α , it is also countably additive in E_β . From the continuity of $P_\beta(\sigma)$ on E_β , we have

$$P_\beta(\sigma)P_\alpha(\sigma') = 0$$

where σ' is the complement of σ . Furthermore,

$$\begin{aligned} P_\beta(\sigma)x &= P_\beta(\sigma)(P_\alpha(\sigma) + P_\alpha(\sigma'))x \\ &= P_\beta(\sigma)P_\alpha(\sigma)x \\ &= P_\alpha(\sigma)x. \end{aligned}$$

Since $P_\alpha(\cdot)x$ and $P_\beta(\cdot)x$ are both countably additive in E_β , we have

$$P_\beta(\sigma)x = P_\alpha(\sigma)x, \quad \text{for all } \sigma \in \Gamma.$$

This proves the proposition.

We now define a set function $P(\cdot)$ on Γ with values in $L(E)$. Let $x \in E$. Since $E = \bigcup_{\alpha \in D} E_\alpha$, $x \in E_\alpha$ for some $\alpha \in D$. We then define

$$(*) \quad P(\sigma)x = P_\alpha(\sigma)x, \quad \sigma \in \Gamma.$$

By Proposition 1, this is well defined. Clearly, the set $\{P(\sigma) : \sigma \in \Gamma\}$ consists of commuting projections on E .

Proposition 2. $\{P(\sigma) : \sigma \in \Gamma\}$ is a spectral measure on E .

Proof. E being the inductive limit of the spaces $\{E_\alpha : \alpha \in D\}$, it follows that for each $\sigma \in \Gamma$, $P(\sigma)$ is continuous on E . Also, it is immediate that for each $x \in E$, $P(\cdot)x$ is countably additive in E and that $P(C) = I$.

To show that $\{P(\sigma) : \sigma \in \Gamma\}$ is an equicontinuous part of $L(E)$, it is enough to prove that for each $x \in E$, $\{P(\sigma)x : \sigma \in \Gamma\}$ is bounded in E . Let $x \in E_\alpha$. Then $\{P(\sigma)x : \sigma \in \Gamma\}$ is bounded in E_α and hence also in E . This proves the proposition.

Lemma 2. For each $x \in E$ and $x' \in E'$, $\langle P(\cdot)x, x' \rangle$ has a compact support.

Proof. Let $x \in E$ and $x' \in E'$ be fixed. There is an $\alpha \in D$ such that $x \in E_\alpha$. x' is also in E'_α . Hence $\langle P(\cdot)x, x' \rangle = \langle P_\alpha(\cdot)x, x' \rangle$ has a compact support.

We now state and prove the main theorem of this paper.

Theorem 1. *Let E be such that a set B in E is bounded if and only if it is already contained in some E_α and is bounded in the same. Let T be a continuous linear operator on E which leaves each E_α invariant. For each $\alpha \in D$, let $T_\alpha (\equiv T|E_\alpha)$ be a spectral operator on E_α with $P_\alpha(\cdot)$ as the corresponding spectral measure. Then T is a spectral operator on E with $P(\cdot)$, as defined by (*), as the corresponding spectral measure.*

Proof. In virtue of Proposition 2 and Lemma 2, we need only to verify the relation

$$sp(T|P(\sigma)E) \subset \bar{\sigma}, \quad \sigma \in \Gamma.$$

Let $\sigma \in \Gamma$ be fixed and let $\lambda \notin \bar{\sigma}$. We have to show the existence of a neighbourhood V_λ of λ in $\hat{C}$ such that there is a function $\mu \rightarrow R_\mu$ on $V_\lambda \cap C$ to $L(E)$ satisfying the conditions

- (a) $R_\mu(\mu I - T) = (\mu I - T)R_\mu = I, \quad \mu \in V_\lambda \cap C$
- (b) $\{R_\mu : \mu \in V_\lambda \cap C\}$ is bounded in $L(E)$.

Fix any neighbourhood V of λ that does not intersect $\bar{\sigma}$. Since, for each $\alpha \in D$, $sp(T_\alpha|P_\alpha(\sigma)E_\alpha) \subset \bar{\sigma}$, there is a function $\mu \rightarrow R_\mu^\alpha$ on $V \cap C$ to $L(P_\alpha(\sigma)E_\alpha)$ satisfying the conditions

$$\begin{aligned} (a_1) \quad R_\mu^\alpha(\mu I - T) | P_\alpha(\sigma)E_\alpha &= ((\mu I - T) | P_\alpha(\sigma)E_\alpha)R_\mu^\alpha \\ &= I | P_\alpha(\sigma)E_\alpha. \end{aligned}$$

Let x lie in $P_\alpha(\sigma)E_\alpha$ as also in $P_\beta(\sigma)E_\beta$. For definiteness, let $\alpha \leq \beta$, so that $P_\alpha(\sigma)E_\alpha \subset P_\beta(\sigma)E_\beta$. We shall show that for all μ in V , $R_\mu^\alpha x = R_\mu^\beta x$.

Let for some $\mu \in V$, $R_\mu^\alpha x = x_1$ and $R_\mu^\beta x = x_2$ where $x_1 \neq x_2$. Now, x_1 and x_2 are in $P_\beta(\sigma)E_\beta$ and

$$(\mu I - T)x_1 = (\mu I - T)x_2$$

i.e.,

$$(\mu I - T)(x_1 - x_2) = 0.$$

This shows that $\mu \in sp(T|P_\beta(\sigma)E_\beta)$ which is a contradiction. Hence, for all $\mu \in V$, $R_\mu^\alpha x = R_\mu^\beta x$, $x \in P_\alpha(\sigma)E_\alpha$.

For $x \in P(\sigma)E = \bigcup_{\alpha \in D} P_\alpha(\sigma)E_\alpha$, define R_μ by

$$R_\mu x = R_\mu^\alpha x \quad \text{where } x \in P_\alpha(\sigma)E_\alpha.$$

We have proved that R_μ is well defined. Also, for all $\mu \in V$, R_μ is continuous on $P(\sigma)E$. From relations (a₁), it is clear that R_μ satisfies the relation

$$R_\mu(\mu I - T) | P(\sigma)E = ((\mu I - T) | P(\sigma)E)R_\mu = I | P(\sigma)E.$$

To show that the set $\Sigma \equiv \{R_\mu : \mu \in V\}$ is bounded in $L(P(\sigma)E)$, let B be any bounded set in $P(\sigma)E$. By hypothesis, there is an $\alpha \in D$ such that $B \subset P_\alpha(\sigma)E_\alpha$ and is bounded therein. For each $x \in B$, $R_\mu x = R_\mu^\alpha x$. Since

$$sp(T_\alpha \mid P_\alpha(\sigma)E_\alpha) \subset \bar{\sigma}$$

it follows that the set $\{R_\mu^\alpha x : \mu \in V, x \in B\}$ is bounded in E_α . Thus, the set $\{R_\mu x : \mu \in V, x \in B\}$ is bounded in E proving, thereby, that Σ is bounded in $L(P(\sigma)E)$. This proves the theorem.

Remark. In [4] the author defined an operator of *type* $[B]$ as an operator for which the definitions of the spectrum in the Waelbroeck sense and in the classical sense are equivalent. The examples of operators of type $[B]$ are the compact operators and the completely bounded operators. If T is an operator of type $[B]$ then it is not hard to see that the above theorem is valid without the assumption that a set is bounded in E if and only if it is already contained in some E_α and is bounded therein.

The proof of the following theorem is immediate.

Theorem 2. *Let $T \in L(E)$ leave each E_α invariant. Let T_α be a scalar operator on E_α with $P_\alpha(\cdot)$ as the corresponding spectral measure. Then T is a scalar operator on E with $P(\cdot)$ as defined by (*) as the corresponding spectral measure.*

We, now, proceed to get a sort of a converse of Theorem 1. We, first, remark that if T is a spectral operator on a locally convex space X and $P(\cdot)$ is the spectral measure corresponding to T , then the integral

$$\int_{sp(T)} \lambda dP(\lambda)$$

may not exist in the topology, τ , of uniform convergence on the bounded subsets of X . However, if T satisfies the *compactness condition* [4; Definition 1], and, in particular, if the spectrum of T is a compact subset of C , then the above integral exists in the topology τ [4; Lemma 2]. In this case for each $\lambda \in \rho(T)$ we have,

$$(\lambda I - T)^{-1} = \sum_{n=0}^{\infty} N^n \int \frac{dP(\xi)}{(\lambda - \xi)^{n+1}}$$

(the series being convergent in the topology τ) where N is the radical of T .

With the help of the Proposition 1, the following theorem can now be established by imitating the proof of [1; Theorem 1].

Theorem 3. *Let T be a spectral operator on X satisfying the compactness condition. Let $P(\cdot)$ be the spectral measure corresponding to T . Let Y be a sequentially complete subspace of X which is invariant under T . Then T_Y is a spectral operator on Y if and only if Y is invariant under $P(\sigma)$, for all $\sigma \in \Gamma$.*

Let, now, E be the strict inductive limit of the spaces E_α , so that, for all $\alpha \in D$, the topology of E_α is identical with the topology of E_α as induced by E . The following corollary of Theorem 3 gives a converse of Theorem 1.

Corollary 1. *Let E be the strict inductive limit of the spaces E_α . Let T be a spectral operator on E , satisfying the compactness condition. Let $P(\cdot)$ be the spectral measure corresponding to T . If T and $P(\cdot)$ leave each E_α invariant, then T_α is a spectral operator on E_α with $P(\cdot) \upharpoonright E_\alpha$ as the corresponding spectral measure.*

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Date communicated: MARCH 30, 1968