An Operator Valued Function Space Integral and a Related Integral Equation

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§0. Introduction. Let us begin by considering for positive values of λ ,

$$(0.1) \quad (I_\lambda(F)\psi)(\xi) \equiv \int_{C_0[a,b]} F(\lambda^{-1/2}x + \xi)\psi(\lambda^{-1/2}x(b) + \xi) dx,$$

where $C_0[a, b] = \{x(\cdot) \mid x(a) = 0, x(t) \text{ continuous on } [a, b]\}$, and F is a real or complex valued functional defined for all continuous functions on $[a, b]$ and ψ is a real or complex valued function defined almost everywhere on $(-\infty, \infty)$ and ξ is a real number; and the λ, F, ψ, ξ are so chosen that the above Wiener integral exists. Now suppose that for a certain choice of the λ, F, ψ the integral (0.1) exists for all (or almost all) values of ξ in a set S of real numbers. Then

$$(0.2) \quad I_\lambda(F)\psi$$

denotes the function which maps ξ into $(I_\lambda(F)\psi)(\xi)$ for almost all values of ξ in S . Again, suppose that for a certain choice of λ, F , the function (0.2) exists for ψ in a class of functions $\mathfrak{D}$ and belongs to a class of functions $\mathfrak{E}$. Then the operator

$$(0.3) \quad I_\lambda(F)$$

maps $\mathfrak{D}$ into $\mathfrak{E}$. In what follows, $\mathfrak{D}$ and $\mathfrak{E}$ will usually be $L_2(-\infty, \infty)$, so $I_\lambda(F)$ will usually be an operator that maps the Hilbert space L_2 into L_2 . It can therefore be considered to be an operator valued function space integral. We shall be particularly interested in extending the definitions of (0.1), (0.2), (0.3) to the case where λ is not real, and in particular, to the case where λ is a pure imaginary number. Before doing so, however, we mention that if

$$(0.4) \quad F(x) = \exp \left\{ \int_a^t \theta(s, x(s)) ds \right\}$$

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and certain smoothness and order of growth conditions are placed on θ and ψ , the expression (0.1) (which now depends on t and ξ) turns out to be a solution of the generalized heat flow equation (see [2]). In this case ψ is not necessarily in L_2 .

One way of extending these operator valued function space integrals to include complex values of the parameter λ is simply to make a definition based on analytic continuation. We shall make such a definition and also make a sequential definition and use the latter to show the existence of the former in a large class of cases. Thus, we define

$$(0.5) \quad I_{\lambda}^{\text{an}}(F)$$

to be the operator valued analytic function of λ [10] which agrees with $I_{\lambda}(F)$ for real λ and is analytic throughout $\text{Re } \lambda > 0$. It is clear that if (0.5) exists, it is unique.

In order to obtain existence theorems and specific formulas for $I_{\lambda}^{\text{an}}(F)$, we also make a sequential definition which in a large class of cases coincides with $I_{\lambda}^{\text{an}}(F)$. In addition to the space $C_0[a, b]$ which is used in the Wiener integral above, it will be necessary to use the following function spaces:

$$C[a, b] = \{x(\cdot) \mid x(t) \text{ continuous on } [a, b]\}, \text{ and}$$

$$B[a, b] = \{x(\cdot) \mid x(t) \text{ defined and continuous on } [a, b] \text{ except for a finite number of finite jump discontinuities}\}.$$

Let σ be any subdivision of $[a, b]$, $\sigma: a = t_0 < t_1 < t_2 < \cdots < t_n = b$, and let $\text{norm } \sigma = \max (t_i - t_{i-1})$. For any $x \in B[a, b]$, let

$$(0.6) \quad x_{\sigma}(t) = \begin{cases} x(t_{i-1}) & \text{if } t_{i-1} \leq t < t_i \quad \text{for } j = 1, \dots, n \\ x(b) & \text{if } t = b. \end{cases}$$

Let

$$(0.7) \quad F_{\sigma}(x) = F(x_{\sigma})$$

for $x \in B[a, b]$.

It is easy to see (and is shown below) that for a very large class of functionals F , $I_{\lambda}^{\text{an}}(F_{\sigma})$ exists and is given as a finite dimensional integral by the formula (4.7) below (in which f_{σ} is given in terms of F and z by (4.2) and z is given by (4.1)). We shall denote this operator by $I_{\lambda}^{\sigma}(F)$, and therefore have the specific formula (under weak restrictions on F)

$$(0.8) \quad \begin{aligned} (I_{\lambda}^{\sigma}(F)\psi)(\xi) &\equiv (I_{\lambda}^{\text{an}}(F_{\sigma})\psi)(\xi) \\ &= \lambda^{n/2} [(2\pi)^n (t_1 - a) \cdots (t_n - t_{n-1})]^{-1/2} \int_{-\infty}^{\infty} \cdots (n) \cdots \\ &\quad \cdot \int_{-\infty}^{\infty} f_{\sigma}(\xi, v_1, \dots, v_n) \psi(v_n) \exp \left\{ - \sum_{i=1}^n \frac{\lambda(v_i - v_{i-1})^2}{2(t_i - t_{i-1})} \right\} dv_1 \cdots dv_n, \end{aligned}$$

where $\psi \in L_2(-\infty, \infty)$ and $v_0 \equiv \xi$ and f_σ is given by (4.1) and (4.2) below. We now define

$$(0.9) \quad I_\lambda^{\text{seq}}(F) = w \lim_{\sigma \rightarrow 0} I_\lambda^\sigma(F)$$

where $w \lim$ denotes weak limit.

In Theorem 1 below we shall show that for real positive λ and with rather weak restrictions on F , $I_\lambda^{\text{seq}}(F)\psi = I_\lambda(F)\psi$. Under more restrictive conditions on F we shall show in Theorems 4 and 6 that for $\text{Re } \lambda > 0$, $I_\lambda^{\text{seq}}(F)$ exists and is an analytic operator valued function of λ , and hence that $I_\lambda^{\text{an}}(F)$ exists and for $\text{Re } \lambda > 0$

$$(0.10) \quad I_\lambda^{\text{an}}(F) = I_\lambda^{\text{seq}}(F).$$

At this point we note some relationships to earlier papers. In the first place, if we set $\psi \equiv 1$, the complex sequential integral defined by one of the authors in [3] and denoted by

$$(0.11) \quad \int_{C[a,b]}^{swp} F(x) dx$$

would be given by

$$\lim_{\text{norm } \sigma \rightarrow 0} (I_\sigma^\sigma(F)\psi)(0)$$

if $x_\sigma(t)$ as given by (0.6) were replaced by the corresponding continuous polygonal function in the definition of $F_\sigma(x)$ in (0.7). Thus our $I_\lambda^{\text{seq}}(F)$ differs from (0.11) in that it is operator valued and the limit is a weak limit for operators rather than a limit of a numerical sequence, and in the fact that our present x_σ is the step function (0.6) rather than a polygonal function. (It turns out that the necessary computations are much easier with step functions.) Since our integrals are now operator valued, they are in this respect similar to those of Babbitt, Feldman, Nelson, Ito, [1, 7, 11, 12]. However, the functionals with which we deal are not limited to those which are likely to be useful in quantum theory; *i.e.*, to exponential functions of integrals of functions of $x(s)$. In this respect, our integral is more like the Wiener integral than the original integral of Feynman [8].

Finally in order to obtain these operator valued integrals for pure imaginary λ , we define

$$(0.12) \quad J_p(F) = w \lim_{\eta \rightarrow 0^+} I_{\eta - ip}^{\text{seq}}(F)$$

and give existence theorems for $J_p(F)$. In the special case where $F(x)$ is of the form (0.4) so that F depends on t , the quantity $G(t, \xi, p) \equiv (J_p(F)\psi)(\xi)$ is a function of t and ξ as well as p , and if the θ which appears in (0.4) is bounded and almost everywhere continuous in the strip $[0, t_0] \times (-\infty, \infty)$, it is shown that G satisfies an integral equation, see [13], which is formally related to the Schrodinger equation. The operator valued integral $J_p(F)$ differs from the "Feynman

integrals" defined in [3] and [4] in that it is operator valued and in other ways. It differs from the "Feynman integrals" of Babbitt, Nelson, etc. in that the functionals which may be used as integrands need not be exponentials of integrals of functions of $x(s)$. In particular existence theorems are given in which the exponentials are replaced by other analytic functions (not necessarily entire functions).

Let us begin by considering for each positive λ

$$(I_\lambda(F)\psi)(\xi) \equiv \int_{C_0[a, b]} F(\lambda^{-1/2}x + \xi) \psi(\lambda^{-1/2}x(b) + \xi) dx,$$

where $C_0[a, b] = \{x(t) \mid x(a) = 0, x(t) \text{ continuous on } [a, b]\}$, $\psi(t)$ is any function in $L_2(-\infty, \infty)$, and $F(y)$ is any bounded complex valued functional such that $F(\lambda^{-1/2}x + \xi)$ is Wiener $\times$ Lebesgue measurable in (x, ξ) on $C_0[a, b] \times (-\infty, \infty)$. We present two examples.

Example 1. Let $F(y) \equiv 1$; then

$$\begin{aligned} (I_\lambda(F)\psi)(\xi) &= \int_{C_0[a, b]} \psi(\lambda^{-1/2}x(b) + \xi) dx \\ &= [2\pi(b-a)]^{-1/2} \int_{-\infty}^{\infty} \psi(\lambda^{-1/2}u + \xi) e^{-u^2/(b-a)} du. \end{aligned}$$

Letting $\lambda^{-1/2}u + \xi = v$, $u = (v - \xi)\lambda^{1/2}$ and $du = \lambda^{1/2} dv$ so that

$$(I_\lambda(F)\psi)(\xi) = \lambda^{1/2} (2\pi(b-a))^{-1/2} \int_{-\infty}^{\infty} \psi(v) e^{-\lambda^{1/2}((v-\xi)^2/(b-a))^{1/2}} dv,$$

we have $(I_\lambda(F)\psi)(\xi) \in L_2(-\infty, \infty)$ if $F \equiv 1$. We also note that $I_\lambda(F)\psi$ is analytic in λ for $\text{Re } \lambda > 0$.

Example 2. Let $F(y) \equiv \exp \{i \int_a^b y(s) ds\}$. Then

$$\begin{aligned} (I_\lambda(F)\psi)(\xi) &\equiv I_\lambda \left[\exp \left\{ i \int_a^b x(s) ds \right\} \psi(\xi) \right] \\ &\equiv \int_{C_0[a, b]} \exp \left\{ i \int_a^b \left[\lambda^{-1/2}x(s) + \xi \right] ds \right\} \psi[\lambda^{-1/2}x(b) + \xi] dx \\ &= e^{i\xi(b-a)} \int_{C_0[a, b]} \exp \left\{ i\lambda^{-1/2} \int_a^b (b-s) dx(s) \right\} \psi \left[\lambda^{-1/2} \int_a^b 1 dx(s) + \xi \right] dx. \end{aligned}$$

Let $\varphi_1(t) \equiv (b-a)^{-1/2}$ and

$$\varphi_2(t) \equiv 3^{1/2} \frac{(a+b-2t)}{(b-a)^{3/2}}.$$

It is easy to see that φ_1 and φ_2 are orthonormal on $[a, b]$. Moreover

$$1 = (b-a)^{1/2} \varphi_1(t)$$

and

$$b - t = \left[\frac{1}{2(3)^{1/2}} \varphi_2(t) + \frac{1}{2} \varphi_1(t) \right] (b - a)^{3/2}.$$

Thus

$$\begin{aligned} I &= e^{i\xi(b-a)} \int_{C_0[a,b]} \exp \left\{ i\lambda^{-1/2} \frac{(b-a)^{3/2}}{2} \left[3^{-1/2} \int_a^b \varphi_2(s) dx(s) + \int_a^b \varphi_1(s) dx(s) \right] \right\} \\ &\quad \cdot \psi \left[\lambda^{-1/2} (b-a)^{1/2} \int_a^b \varphi_1(s) dx(s) + \xi \right] dx \\ &= \frac{e^{i\xi(b-a)}}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left\{ i\lambda^{-1/2} \frac{(b-a)^{3/2}}{2} [3^{-1/2}v + u] \right\} \\ &\quad \cdot \psi[\lambda^{-1/2}(b-a)^{1/2}u + \xi] e^{-(u^2+v^2)/2} du dv \\ &= \frac{e^{i\xi(b-a)}}{2\pi} \left\{ \int_{-\infty}^{\infty} \exp \left[i(3\lambda)^{-1/2}(b-a)^{3/2} \frac{v}{2} - \frac{v^2}{2} \right] dv \right\} \\ &\quad \cdot \left\{ \int_{-\infty}^{\infty} \exp \left\{ i(\lambda)^{-1/2}(b-a)^{3/2} \frac{u}{2} - \frac{u^2}{2} \right\} \psi[\lambda^{-1/2}(b-a)^{1/2}u + \xi] du \right\} \\ &= \frac{e^{i\xi(b-a) - (b-a)^2/24\lambda}}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{i(b-a)(\rho-\xi)/2 - \lambda(\rho-\xi)^2/2(b-a)} \psi(\rho) \frac{\lambda^{1/2} d\rho}{(b-a)^{1/2}} \\ &= \left(\frac{\lambda}{2\pi(b-a)} \right)^{1/2} e^{i\xi(b-a)/2 - (b-a)^2/24\lambda} \int_{-\infty}^{\infty} e^{i(b-a)\rho/2} \psi(\rho) e^{-\lambda(\rho-\xi)^2/2(b-a)} d\rho. \end{aligned}$$

By Lemma 1 below it will be seen that this is of class L_2 as a function of ξ whenever $\psi \in L_2$. Moreover, it is easy to see that for $\text{Re } \lambda > 0$,

$$I_\lambda \left[\exp \left\{ i \int_a^b x(s) ds \right\} \right] \psi$$

is analytic in λ (as a Hilbert space valued function of λ).

We shall need a notation for limits in the mean as $A \rightarrow \infty$ of integrals over $(-A, A)$ such as occur in Plancherel's Theorem. We shall think of them as a generalization of the integral over $(-\infty, \infty)$ and shall therefore use a notation similar to the usual notation. To indicate that the integral is to be taken in this sense, and also to specify clearly which variable is involved in the limit in the mean, we introduce the following notation:

$$(0.13) \quad \int_{-\infty}^{\infty} f(u, \xi) du \equiv \text{l.i.m.}_{\substack{A \rightarrow \infty \\ (\xi)}} \int_{-A}^A f(u, \xi) du$$

$$i.e. \quad \int_{-\infty}^{\infty} f(u, \xi) du = \varphi(\xi) \quad \text{means}$$

$$\lim_{A \rightarrow \infty} \int_{-\infty}^{\infty} \left| \varphi(\xi) - \int_{-A}^A f(u, \xi) du \right|^2 d\xi = 0.$$

The (ξ) in the notation will be helpful in cases where there are more than two variables in the integrand.

§1. **Lemma 1.** Let $\operatorname{Re} \lambda \geq 0$, $\lambda \neq 0$, and $f(x)$ be a complex valued function such that $f(x) \in L_2(-\infty, \infty)$. Let

$$g(y) = \left(\frac{\lambda}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\lambda(y-x)^2/2} f(x) dx.$$

Then $g(x) \in L_2(-\infty, \infty)$ and $\|g\| \leq \|f\|$. (If $\operatorname{Re} \lambda = 0$, we shall interpret the integral as a limit in the mean, i.e.

$$g(y) = \left(\frac{\lambda}{2\pi}\right)^{1/2} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} e^{-\lambda(y-x)^2/2} f(x) dx.$$

Proof. Case I. Let $\lambda = ip$, p real, and without loss of generality assume $p > 0$, hence (if we interpret the integrals below in the sense indicated above)

$$g(y) = \left(\frac{ip}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-ipx^2/2} e^{ipxy} e^{-ipx^2/2} f(x) dx.$$

Let $py = z$, $y = z/p$ and so we obtain

$$\frac{g\left(\frac{z}{p}\right) e^{iz^2/2p}}{(ip)^{1/2}} = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{izx} e^{-ipx^2/2} f(x) dx.$$

Now $f(x) \in L_2(-\infty, \infty)$ implies that $e^{-ipx^2/2} f(x) \in L_2(-\infty, \infty)$ and therefore its Fourier Transform $\in L_2(-\infty, \infty)$ and thus $h(z) = (ip)^{-1/2} g(z/p) e^{iz^2/2p} \in L_2(-\infty, \infty)$. Hence

$$\|h(x)\| = \|e^{-ipx^2/2} f(x)\| = \|f(x)\|,$$

and

$$\|f\|^2 = \|h\|^2 = \int_{-\infty}^{\infty} \left| (ip)^{-1/2} g\left(\frac{z}{p}\right) e^{iz^2/2p} \right|^2 dz = \int_{-\infty}^{\infty} |g(y)|^2 dy = \|g\|^2.$$

Thus when λ is purely imaginary, $\|g\| = \|f\|$.

Case II. Let $\lambda = p$, p real, and by hypothesis, $p > 0$.

Here the integral exists; actually $e^{-\lambda(y-x)^2/2} \in L_1(-\infty, \infty) \cap L_2(-\infty, \infty)$. Now the integral becomes

$$|g(y)| \leq \left(\frac{\lambda}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\lambda(y-x)^2/2} |f(x)| dx$$

and thus

$$\begin{aligned} \|g\|^2 &= \int_{-\infty}^{\infty} |g(y)|^2 dy \leq \int_{-\infty}^{\infty} \frac{\lambda}{2\pi} \left[\int_{-\infty}^{\infty} e^{-\lambda(y-x)^2/2} |f(x)| dx \right]^2 dy \\ &= \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-\lambda(y-x)^2/2} |f(x)| dx \right) \left(\int_{-\infty}^{\infty} e^{-\lambda(y-u)^2/2} |f(u)| du \right) dy \\ &= \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\lambda[(y-x)^2 + (y-u)^2]/2} |f(x)| |f(u)| dx du dy. \end{aligned}$$

Let us make a change of variables,

$$\begin{cases} x = x \\ y - x = z \\ y - u = v \end{cases} \quad \text{with Jacobian,} \quad \frac{\partial(x, v, z)}{\partial(x, y, u)} \equiv 1.$$

The above integral exists as a triple integral as well as a three fold iterated integral and so

$$\begin{aligned} \|g\|^2 &\leq \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\lambda(z^2+v^2)/2} |f(x)| |f(x+z-v)| dx dz dv \\ &= \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\lambda(z^2+v^2)/2} \left[\int_{-\infty}^{\infty} |f(x)| |f(x+z-v)| dx \right] dz dv. \end{aligned}$$

By the Schwarz Inequality we have

$$\left[\int_{-\infty}^{\infty} |f(x)| |f(x+z-v)| dx \right]^2 \leq \int_{-\infty}^{\infty} |f(x)|^2 dx \int_{-\infty}^{\infty} |f(x+z-v)|^2 dx = \|f\|^4,$$

so

$$\begin{aligned} \|g\|^2 &\leq \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\lambda(z^2+v^2)/2} \|f\|^2 dz dv \\ &= \|f\|^2 \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} e^{-\lambda z^2/2} dz \int_{-\infty}^{\infty} e^{-\lambda v^2/2} dv = \|f\|^2, \end{aligned}$$

because of the standard integral $\int_{-\infty}^{\infty} e^{-\lambda t^2/2} dt = (2\pi/\lambda)^{1/2}$. Thus we obtain $\|g\| \leq \|f\|$.

Case III. $\operatorname{Re} \lambda > 0$, and let $\lambda = p + iq$, $p > 0$; then

$$|g(y)| \leq \left(\frac{|\lambda|}{2\pi} \right)^{1/2} \int_{-\infty}^{\infty} e^{-p(y-x)^2/2} |f(x)| dx.$$

Now $g \in L_2(-\infty, \infty)$, because we may consider

$$h(y) = \left(\frac{p}{2\pi} \right)^{1/2} \int_{-\infty}^{\infty} e^{-p(y-x)^2/2} |f(x)| dx$$

and apply Case II to $|f(x)|$ with λ replaced by p and because $f(x)$ is in $L_2(-\infty, \infty)$, therefore $\|h\| \leq \| |f| \| = \|f\|$.

Thus by Case II, $h \in L_2(-\infty, \infty)$. But $|g(y)| \leq (|\lambda|/p)^{1/2} h(y)$ and therefore $g(y) \in L_2(-\infty, \infty)$.

(We now observe that this implies that

$$\|g\| \leq (|\lambda|/p)^{1/2} \|f\|, \quad \text{but} \quad |\lambda| > p$$

and so this isn't sufficient to conclude directly that $\|g\| \leq \|f\|$).

Since the function $g(y) \in L_2(-\infty, \infty)$,

$$\begin{aligned} ||g||^2 &= \frac{|\lambda|}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{-\lambda(y-x)^2/2} f(x) dx \right] \left[\int_{-\infty}^{\infty} e^{-\bar{\lambda}(y-u)^2/2} \overline{f(u)} du \right] dy \\ &= \frac{|\lambda|}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\lambda(y-x)^2/2 - \bar{\lambda}(y-u)^2/2} f(x) \overline{f(u)} dx du dy. \end{aligned}$$

Applying the Fubini Theorem we write

$$||g|| = \frac{|\lambda|}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) \overline{f(u)} \left(\int_{-\infty}^{\infty} e^{-\lambda(y-x)^2/2 - \bar{\lambda}(y-u)^2/2} dy \right) du dx.$$

Now recalling that $\lambda = p + iq$ we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-\lambda(y-x)^2/2 - \bar{\lambda}(y-u)^2/2} dy &= \int_{-\infty}^{\infty} e^{-py^2 + (\lambda x + \bar{\lambda}u)y - (\lambda x^2/2 + \bar{\lambda}u^2/2)} dy \\ &= e^{-(\lambda x^2 + \bar{\lambda}u^2)/2} \int_{-\infty}^{\infty} e^{-py^2 + (\lambda x + \bar{\lambda}u)y} dy. \end{aligned}$$

If we complete the square in the last exponent we observe that the expression above equals

$$\begin{aligned} &\exp \left\{ -\frac{1}{2}(\lambda x^2 + \bar{\lambda}u^2) + \frac{(\lambda x + \bar{\lambda}u)^2}{4p} \right\} \\ &\quad \cdot \int_{-\infty}^{\infty} \exp \left\{ -py^2 + (\lambda x + \bar{\lambda}u)y - \frac{(\lambda x + \bar{\lambda}u)^2}{4p} \right\} dy \\ &= \exp \left\{ \frac{x^2(\lambda^2 - 2p\lambda) + 2\lambda\bar{\lambda}xu + u^2(\bar{\lambda}^2 - 2p\bar{\lambda})}{4p} \right\} \\ &\quad \cdot \int_{-\infty}^{\infty} \exp \left\{ -p \left(y - \frac{(\lambda x + \bar{\lambda}u)}{2p} \right)^2 \right\} dy. \end{aligned}$$

If we evaluate

$$\lambda^2 - 2p\lambda = \lambda(\lambda - 2p) = \lambda(-p + iq) = -\lambda\bar{\lambda} = -|\lambda|^2$$

and

$$\bar{\lambda}^2 - 2p\bar{\lambda} = \bar{\lambda}(\bar{\lambda} - 2p) = \bar{\lambda}(-p - iq) = \bar{\lambda}(-\lambda) = -|\lambda|^2,$$

then we obtain that the expression above is equal to

$$e^{-|\lambda|^2[(x-u)^2/4p]} \int_{-\infty}^{\infty} e^{-p(y - (\lambda x + \bar{\lambda}u)/2p)^2} dy.$$

We now change variables, letting

$$y - \frac{\lambda x + \bar{\lambda}u}{2p} = z,$$

so our expression above becomes (after we move the line of integration back to the real line),

$$e^{-|\lambda|^{-2}(x-u)^2/4p} \int_{-\infty}^{\infty} e^{-pz^2} dz = \left(\frac{\pi}{p}\right)^{1/2} e^{-|\lambda|^{-2}(x-u)^2/4p}.$$

In this fashion, we have evaluated

$$\int_{-\infty}^{\infty} e^{-\lambda(y-x)^2/2 - \bar{\lambda}(y-u)^2/2} dy = \left(\frac{\pi}{p}\right)^{1/2} e^{-|\lambda|^{-2}(x-u)^2/4p}.$$

We now can estimate $\|g\|^2$ in this fashion

$$\|g\|^2 = \frac{|\lambda|}{2\pi} \left(\frac{\pi}{p}\right)^{1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) \overline{f(u)} e^{-|\lambda|^{-2}(x-u)^2/4p} du dx.$$

Again let us change variables,

$$\begin{cases} u - x = t \\ x = x \end{cases}$$

$$\begin{aligned} \|g\|^2 &= \frac{|\lambda|}{2(\pi p)^{1/2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{\infty} f(x) \overline{f(t+x)} e^{-|\lambda|^{-2}t^2/4p} dx dt \\ &= \frac{|\lambda|}{2(\pi p)^{1/2}} \int_{-\infty}^{\infty} e^{-|\lambda|^{-2}t^2/4p} \left(\int_{-\infty}^{\infty} f(x) \overline{f(t+x)} dx \right) dt. \end{aligned}$$

Applying the Schwarz inequality, we find

$$\left| \int_{-\infty}^{\infty} f(x) \overline{f(t+x)} dx \right| \leq \left(\int_{-\infty}^{\infty} |f|^2 dx \int_{-\infty}^{\infty} |f(x+t)|^2 dx \right)^{1/2},$$

and hence

$$\|g\|^2 \leq \frac{|\lambda|}{2(\pi p)^{1/2}} \cdot \int_{-\infty}^{\infty} e^{-|\lambda|^{-2}t^2/4p} dt \cdot \|f\|^2 = \|f\|^2$$

and therefore Lemma 1 is proved.

§2. Lemma 2. If $|F(x)| \leq B$ for all x and λ is positive and if $F(\lambda^{-1/2}x + \xi)$ is Wiener $\times$ Lebesgue measurable in (x, ξ) on $C_0[a, b] \times (-\infty, \infty)$ then $(I_\lambda(F)\psi)(\xi) \in L_2(-\infty, \infty)$ and $\|I_\lambda(F)\psi\| \leq B \|\psi\|$.

Proof.

$$(I_\lambda(F)\psi)(\xi) = \int_{C_0[a, b]} F(\lambda^{-1/2}x + \xi) \psi(\lambda^{-1/2}x(b) + \xi) dx,$$

and so

$$\begin{aligned} |(I_\lambda(F)\psi)(\xi)| &\leq B \int_{C_0[a, b]} |\psi(\lambda^{-1/2}x(b) + \xi)| dx \\ &= B(2\pi(b-a))^{-1/2} \int_{-\infty}^{\infty} |\psi(\lambda^{-1/2}u + \xi)| e^{-u^2/2(b-a)} du. \end{aligned}$$

Let $v = \lambda^{-1/2}u + \xi$, so $u^2 = \lambda(v - \xi)^2$, and let $\Lambda = \lambda/(b - a)$; we obtain

$$|(I_\lambda(F)\psi)(\xi)| \leq B(\Lambda/2\pi)^{1/2} \int_{-\infty}^{\infty} |\psi(v)| e^{-\Lambda(v-\xi)^2/2} dv.$$

From Lemma 1, it follows then that

$$\|I_\lambda(F)\psi\| \leq B \|\psi\|.$$

Lemma 3. *Let $\lambda > 0$ and let $\{F_n(x)\}$ be a sequence of functionals such that $|F_n(x)| \leq B$ and $F_n(\lambda^{-1/2}x + \xi)$ is Wiener $\times$ Lebesgue measurable in (x, ξ) on $C_0[a, b] \times (-\infty, \infty)$ for $n=1, 2, \dots$ and let $\lim_{n \rightarrow \infty} F_n(\lambda^{-1/2}x + \xi) \equiv F(\lambda^{-1/2}x + \xi)$ exist for almost all (x, ξ) in $C_0[a, b] \times (-\infty, \infty)$. Then $\lim_{n \rightarrow \infty} (I_\lambda(F_n)\psi, \varphi) = (I_\lambda(F)\psi, \varphi)$ for all $\psi, \varphi \in L_2(-\infty, \infty)$, where the parentheses denote the inner product.*

Proof. By the previous lemma, $I_\lambda(F_n)\psi \in L_2(-\infty, \infty)$ and $\|I_\lambda(F_n)\psi\| \leq B \|\psi\|$. Clearly F is bounded by B and measurable and thus

$$\begin{aligned} (I_\lambda(F_n)\psi)(\xi) - (I_\lambda(F)\psi)(\xi) \\ = \int_{C_0[a, b]} [F_n(\lambda^{-1/2}x + \xi) - F(\lambda^{-1/2}x + \xi)] \psi(\lambda^{-1/2}x(b) + \xi) dx. \end{aligned}$$

Since

$$\begin{aligned} \int_{C_0[a, b]} |\psi(\lambda^{-1/2}x(b) + \xi)| dx &= (2\pi(b-a))^{-1/2} \\ &\cdot \int_{-\infty}^{+\infty} |\psi(\lambda^{-1/2}u + \xi)| e^{-u^2/2(b-a)} du \\ &= \lambda^{1/2} (2\pi(b-a))^{-1/2} \int_{-\infty}^{\infty} |\psi(v)| e^{-\lambda(v-\xi)^2/2(b-a)} dv < \infty, \end{aligned}$$

we obtain

$$\lim_{n \rightarrow \infty} |(I_\lambda(F_n)\psi)(\xi) - (I_\lambda(F)\psi)(\xi)| = 0$$

for almost all ξ by the Lebesgue dominated convergence theorem for Wiener integrals. Thus, by Theorem 13.44, Page 207, Hewitt and Stromberg [9] we have the desired weak convergence.

Theorem 1. *Let $\lambda > 0$ and $F(x)$ be bounded and continuous in the uniform topology on $B[a, b]$. Let σ be any subdivision of $[a, b]$, $\sigma: a = t_0 < t_1 < \dots < t_n = b$, and for any $x(t) \in B[a, b]$, let $x_\sigma(t) = x(t_{i-1})$ if $t_{i-1} \leq t < t_i$ for $j = 1, \dots, n$ and $x_\sigma(b) = x(b)$. Let $F_\sigma(x) \equiv F(x_\sigma)$ for any $x(t) \in B[a, b]$. Then it follows that for every $\psi, \varphi \in L_2(-\infty, \infty)$.*

$$\lim_{\text{norm } \sigma \rightarrow 0} (I_\lambda(F_\sigma)\psi, \varphi) = (I_\lambda(F)\psi, \varphi),$$

where the parentheses denote the inner product.

Proof. (This proof is worded in such a way as to apply not only to the theorem but also to the corollary below.)

Since $F_\sigma(x) \equiv F(x_\sigma)$ for $x \in B[a, b]$, it follows that $F_\sigma(\lambda^{-1/2}x + \xi) = F(\lambda^{-1/2}x_\sigma + \xi)$. Let $|F(x)| \leq M$ on $B[a, b]$. Let $\sigma_1, \sigma_2, \dots$ be any sequence of subdivisions such that $\text{norm } \sigma_n \rightarrow 0$ as $n \rightarrow \infty$, and consider $F_{\sigma_n}(x)$. The functions $F_{\sigma_n}(\lambda^{-1/2}x + \xi)$ are Wiener $\times$ Lebesgue measurable on $C_0[a, b] \times (-\infty, \infty)$ and satisfy the inequality $|F_{\sigma_n}(x)| \leq M$ for $n = 1, 2, \dots$. Let x_0 be a point of $C_0[a, b]$, ξ_0 a real number where $F(\lambda^{-1/2}x + \xi)$ is continuous at (x_0, ξ_0) . It follows then that

$$\lim_{n \rightarrow \infty} F_{\sigma_n}(\lambda^{-1/2}x_0 + \xi_0) = \lim_{n \rightarrow \infty} F(\lambda^{-1/2}(x_{0, \sigma_n}) + \xi_0) = F(\lambda^{-1/2}x_0 + \xi_0),$$

and hence for almost all $(x, \xi) \in C_0[a, b] \times (-\infty, \infty)$, $\lim_{n \rightarrow \infty} F_{\sigma_n}(\lambda^{-1/2}x + \xi) = F(\lambda^{-1/2}x + \xi)$. An application of Lemma 3 yields

$$\lim_{n \rightarrow \infty} (I_\lambda(F_{\sigma_n})\psi, \varphi) = (I_\lambda(F)\psi, \varphi)$$

for every $\psi, \varphi \in L_2(-\infty, \infty)$, and the theorem is proved.

Corollary 1. *The theorem remains valid if the hypothesis of continuity of $F(x)$ over $B[a, b]$ is replaced by these two hypotheses:*

- i) $F(\lambda^{-1/2}x + \xi)$ is continuous in the uniform topology at almost every (x, ξ) in $C_0[a, b] \times (-\infty, \infty)$
- ii) for each subdivision σ and each positive λ , $F(\lambda^{-1/2}x_\sigma + \xi)$ is Wiener $\times$ Lebesgue measurable in (x, ξ) over $C_0[a, b] \times (-\infty, \infty)$.

§3. In preparation for Theorem 2 we first prove two lemmas.

Lemma 4. *If $\theta(t, u)$ is almost everywhere continuous on $R = [a, b] \times (-\infty, \infty)$, then for almost every $x \in C_0[a, b]$ it follows that $\theta(s, x(s))$ is continuous for almost all $s \in [a, b]$.*

Proof. Let S be a Borel measurable null set in the plane containing all points where $\theta(t, u)$ is discontinuous and let $\chi_s(t, u)$ be its characteristic function. Consider the integral

$$I = \int_{C_0[a, b]} \int_a^b \chi_s(s, x(s)) \, ds \, dx = \int_a^b \int_{C_0[a, b]} \chi_s(s, x(s)) \, dx \, ds.$$

If we hold s fixed and apply Wiener's formula, we obtain

$$I = \int_a^b (2\pi(s - a))^{-1/2} \int_{-\infty}^{\infty} \chi_s(s, u) e^{-u^2/(s-a)} \, du \, ds = 0,$$

because this is a double integral over an infinite strip and $\chi_s(s, u) = 0$ almost everywhere. Because $\chi_s(s, x(s)) \geq 0$ and $I = 0$, $\int_a^b \chi_s(s, x(s)) \, ds = 0$ for almost all $x(s)$ in $C_0[a, b]$. Therefore for such x , $\chi_s(s, x(s)) = 0$ for almost all s and therefore $\theta(s, x(s))$ is continuous for almost all s . Thus Lemma 4 is proved.

Next, if we consider the integral

$$\begin{aligned} J &= \int_{-\infty}^{\infty} \int_{C_0[a, b]} \int_a^b \chi_s(s, \lambda^{-1/2} x(s) + \xi) ds dx d\xi \\ &= \int_a^b \left(\int_{-\infty}^{\infty} \int_{C_0[a, b]} \chi_s(s, \lambda^{-1/2} x(s) + \xi) dx d\xi \right) ds \\ &= \int_a^b \int_{-\infty}^{\infty} (2\pi(s-a))^{-1/2} \int_{-\infty}^{\infty} \chi_s(s, \lambda^{-1/2} u + \xi) e^{-u^2/2(s-a)} du d\xi ds \\ &= \int_a^b (2\pi(s-a))^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_s(s, v) e^{-[\lambda(v-\xi)^2/2(s-a)]} \lambda^{1/2} dv d\xi ds, \end{aligned}$$

it follows that $J = 0$. Hence we have proved the following lemma.

Lemma 5. *If $\theta(t, u)$ is almost everywhere continuous on $R \equiv [a, b] \times (-\infty, \infty)$, $\lambda > 0$, then for almost every $(x, \xi) \in C_0[a, b] \times (-\infty, \infty)$ it follows that $\theta(s, \lambda^{-1/2} x(s) + \xi)$ is continuous for almost all $s \in [a, b]$ and $\theta(s, \lambda^{-1/2} u + v)$ is continuous at $u = x(s)$, $v = \xi$ for almost all $s \in [a, b]$.*

Theorem 2. *Let $\theta(t, u)$ be continuous almost everywhere on $R \equiv [a, b] \times (-\infty, \infty)$, let $\operatorname{Re} \theta(t, u) \leq B$ on R , and let $\theta(t, u)$ be bounded on every compact subset of R . Then if $F(x) = \exp \left\{ \int_a^b \theta(s, x(s)) ds \right\}$ and $\sigma, x_\sigma, f_\sigma$ are defined as in Theorem 1, we have for $\lambda > 0$*

$$\lim_{\operatorname{norm} \sigma \rightarrow 0} (I_\lambda(F_\sigma)\psi, \varphi) = (I_\lambda(F)\psi, \varphi)$$

for all ψ, φ in L_2 .

Proof. Let us observe that if $\theta(t, u)$ is continuous everywhere, Theorem 2 follows directly from Theorem 1.

Our proof will consist of verifying that the hypotheses of Theorem 2 are sufficiently strong to permit application of Corollary 1 of Theorem 1.

Indeed, we have

$$|F(x)| = \left| \exp \left\{ \int_a^b \theta(s, x(s)) ds \right\} \right| \leq e^{B(b-a)}.$$

The fact that the function

$$F(\lambda^{-1/2} x + \xi) = \exp \left\{ \int_a^b \theta(s, \lambda^{-1/2} x(s) + \xi) ds \right\}$$

is continuous in the uniform topology for almost all (x, ξ) in $C_0[a, b] \times (-\infty, \infty)$ follows from Lemmas 4 and 5.

We now verify that hypothesis ii) of the corollary is satisfied:

ii) for each subdivision σ and each positive λ , $F(\lambda^{-1/2} x_\sigma + \xi)$ is Wiener $\times$ Lebesgue measurable in (x, ξ) over $C_0[a, b] \times (-\infty, \infty)$.

Now let $\sigma: a = t_0 < t_1 < \dots < t_n = b$ be any subdivision and define the function $x_\sigma(t)$ by $x_\sigma(t) = x(t_{j-1})$ if $t_{j-1} \leq t < t_j$, $j = 1, \dots, n$, $x_\sigma(b) = x(b)$. Then

$$\begin{aligned} F(\lambda^{-1/2}x_\sigma + \xi) &= \exp \left\{ \int_a^b \theta(s, \lambda^{-1/2}x_\sigma(s) + \xi) ds \right\} \\ &= \exp \left\{ \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \theta(s, \lambda^{-1/2}x(t_{j-1}) + \xi) ds \right\}. \end{aligned}$$

Since $\theta(t, u)$ is continuous almost everywhere and hence measurable and is bounded on every compact subset of R ,

$$\int_{t_{j-1}}^{t_j} \theta(s, \lambda^{-1/2}u + \xi) ds$$

is measurable in (u, ξ) and hence

$$\int_{t_{j-1}}^{t_j} \theta(s, \lambda^{-1/2}x(t_{j-1}) + \xi) ds$$

is Wiener $\times$ Lebesgue measurable in (x, ξ) , and so $F(\lambda^{-1/2}x_\sigma + \xi)$ is Wiener $\times$ Lebesgue measurable in (x, ξ) .

Thus hypothesis ii) is satisfied and the proof of Theorem 2 is complete.

Corollary 1. *If $\lambda > 0$ and $|\theta(t, u)| \leq M$ on $R \equiv [a, b] \times (-\infty, \infty)$ and $\theta(t, u)$ satisfies the continuity condition of Theorem 2, then we can replace the exponential function by any continuous function g whose domain of continuity contains the disk $|z| \leq (b - a)M$, i.e. $F(x) = g(\int_a^b \theta(s, x(s)) ds)$.*

Corollary 2. *Both Theorem 2 and Corollary 1 continue to hold if $\int_a^b \theta(s, x(s)) ds$ is replaced by $\int_a^b \theta(s, x(s)) d\mu(s)$ where μ is a left-continuous function of $B.V.$ on $[a, b]$, provided that the clause " $\theta(t, u)$ is continuous almost everywhere" in the hypothesis is interpreted to mean " $\theta(t, u)$ is continuous except on a set of measure zero in the $\mu \times$ Lebesgue product measure" and in the hypothesis of Corollary 1, $|z| \leq (b - a)M$ is replaced by $|z| \leq M \text{Var}_{[a, b]}(\mu)$.*

§4. In order to proceed to the discussion of the definition of $I_\lambda(F)$ for complex values of λ , it is necessary to introduce some notation. Let σ be any partition of $[a, b]$, $\sigma: a = t_0 < t_1 < t_2 < \dots < t_n = b$, and the norm of σ equals $\max_{j=1, \dots, n} (t_j - t_{j-1})$. If x is a continuous function on $[a, b]$ we define

$$x_\sigma(t) = \begin{cases} x(t_{j-1}) & \text{if } t_{j-1} \leq t < t_j, \quad j = 1, \dots, n \\ x(b) & \text{if } t = b. \end{cases}$$

If σ is given and $\{v_0, \dots, v_n\}$ is any set of $(n + 1)$ real numbers we define the function

$$(4.1) \quad z(\sigma, v_0, v_1, \dots, v_n, t) = \begin{cases} v_k & \text{if } t_k \leq t < t_{k+1} \text{ for } k = 0, 1, \dots, n-1 \\ v_n & \text{if } t = b. \end{cases}$$

We note that $z(\sigma, \xi, x(t_1) + \xi, x(t_2) + \xi, \dots, x(t_n) + \xi, t) = x_\sigma(t) + \xi$. Given a function $F(y)$ we introduce a function $f_\sigma(v_0, v_1, \dots, v_n)$ by means of the equation

$$(4.2) \quad f_\sigma(v_0, v_1, \dots, v_n) = F[z(\sigma, v_0, \dots, v_n, \cdot)].$$

If x is a given function on $[a, b]$, let us substitute $\lambda^{-1/2}x(t_i) + \xi$ for v_i to obtain

$$(4.3) \quad f_\sigma(\xi, \lambda^{-1/2}x(t_1) + \xi, \dots, \lambda^{-1/2}x(t_n) + \xi) = F[z(\sigma, \xi, \lambda^{-1/2}x(t_1) + \xi, \dots, \lambda^{-1/2}x(t_n) + \xi, \cdot)].$$

Hence we write

$$f_\sigma(\xi, \lambda^{-1/2}x(t_1) + \xi, \dots, \lambda^{-1/2}x(t_n) + \xi) = F(\lambda^{-1/2}x_\sigma + \xi).$$

We now return to $I_\lambda(F)\psi$ and write an analogous formula for x_σ ,

$$(4.4) \quad (I_\lambda^\sigma(F)\psi)(\xi) = \int_{C_\sigma[a, b]} F(\lambda^{-1/2}x_\sigma + \xi) \psi(\lambda^{-1/2}x(b) + \xi) dx$$

so that for real positive λ ,

$$(4.5) \quad (I_\lambda^\sigma(F)\psi)(\xi) = (I_\lambda(F_\sigma)\psi)(\xi)$$

and if we apply Wiener's formula, we obtain

$$(4.6) \quad (I_\lambda^\sigma(F)\psi)(\xi) = ((2\pi)^n(t_1 - a) \cdots (t_n - t_{n-1}))^{-1/2} \\ \int_{-\infty}^{\infty} \cdot^{(n)} \cdot \int_{-\infty}^{\infty} f_\sigma(\xi, \lambda^{-1/2}u_1 + \xi, \dots, \lambda^{-1/2}u_n + \xi) \psi(\lambda^{-1/2}u_n + \xi) \\ \cdot \exp \left\{ \frac{-u_1^2}{2(t_1 - a)} - \frac{(u_2 - u_1)^2}{2(t_2 - t_1)} - \dots - \frac{(u_n - u_{n-1})^2}{2(t_n - t_{n-1})} \right\} du_1, \dots, du_n.$$

If we now consider $v_i = \lambda^{-1/2}u_i + \xi$, we have presented the motivation for the following definition.

Definition. For $\text{Re } \lambda > 0$, and $\psi \in L_2(-\infty, \infty)$

$$(4.7) \quad (I_\lambda^\sigma(F)\psi)(\xi) = \lambda^{n/2}[(2\pi)^n(t_1 - a) \cdots (t_n - t_{n-1})]^{-1/2} \\ \cdot \int_{-\infty}^{\infty} \cdots (n) \cdots \int_{-\infty}^{\infty} f(\xi, v_1, \dots, v_n) \\ \cdot \psi(v_n) \exp \left[- \sum_{i=1}^n \frac{\lambda(v_i - v_{i-1})^2}{2(t_i - t_{i-1})} \right] dv_1 \cdots dv_n,$$

where $f(v_0, v_1, \dots, v_n) \equiv f_\sigma(v_0, v_1, \dots, v_n) = F[z(\sigma, v_0, \dots, v_n, \cdot)]$, and $z(\sigma, v_0, \dots, v_n, t)$ is as defined above and $v_0 \equiv \xi$.

(It is understood that appropriate hypotheses will be imposed on F to guarantee existence of the integral.) It is important to observe that λ no longer appears in the argument of either f or ψ , it appears only as $\lambda^{n/2}$ and in the argument of the exponential function. We assume $|\arg \lambda| \leq \pi/2$ in determining $\lambda^{n/2}$. Hence λ can be a complex number without great difficulty, and so we extend the definition of $I_\lambda^\sigma(F)\psi$ for complex λ . Let us now define $I_\lambda(F)\psi$ for complex λ by means of the formula

$$(I_\lambda(F)\psi, \varphi) = \lim_{\text{norm } \sigma \rightarrow 0} (I_\lambda^\sigma(F)\psi, \varphi)$$

for every $\varphi \in L_2(-\infty, \infty)$ if such a weak limit exists. We also point out that $I_\lambda^\sigma(F)\psi = I_\lambda(F_\sigma)\psi$ which was previously defined if λ is real and positive. We note that if λ is real and positive and F satisfies the hypotheses of Theorem 2 or one of its corollaries, $I_\lambda(F)$ as defined above agrees with our original definition of $I_\lambda(F)$.

Lemma 6. *Let $\theta(t, u)$ be continuous almost everywhere on $R \equiv [a, b] \times (-\infty, \infty)$, and let $\text{Re } (\theta(t, u)) \leq B$ on R , and let $\theta(t, u)$ be bounded on every compact subset of R . Then if*

$$F(x) = \exp \left\{ \int_a^b \theta(s, x(s)) ds \right\}, \quad \text{Re } \lambda > 0,$$

the operator $I_\lambda^\sigma(F)$ exists and is bounded and

$$||I_\lambda^\sigma(F)\psi|| \leq ||\psi|| e^{B(b-a)}.$$

Proof. We have formally that

$$\begin{aligned} (I_\lambda^\sigma(F)\psi)(\xi) &= \lambda^{n/2} ((2\pi)^n (t_1 - a) \cdots (t_n - t_{n-1}))^{-1/2} \\ &\quad \cdot \int_{-\infty}^{\infty} \cdots (n) \cdots \int_{-\infty}^{\infty} \exp \left\{ \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \theta(t, v_{j-1}) dt \right\} \\ &\quad \cdot \exp \left\{ - \sum_{j=1}^n \frac{\lambda (v_j - v_{j-1})^2}{2(t_j - t_{j-1})} \right\} \psi(v_n) dv_1 \cdots dv_n, \quad \text{and } v_0 = \xi, \end{aligned}$$

and the existence of this integral follows from the Fubini Theorem and the existence of the function g_n defined below. Let

$$g_1(v_{n-1}) = \lambda^{1/2} (2\pi(t_n - t_{n-1}))^{-1/2} \int_{-\infty}^{\infty} \psi(v_n) e^{-\lambda[(v_n - v_{n-1})^2 / (t_n - t_{n-1})]^{1/2}} dv_n$$

and note that by Lemma 1 we obtain

$$||g_1|| \leq ||\psi||.$$

Let $\int_{t_k}^{t_{k+1}} \theta(t, v_k) dt = \varphi_k(v_k)$ for $k = 0, 1, \dots, n-1$, and define

$$g_i(v_{n-i}) = \lambda^{1/2} (2\pi(t_{n-i+1} - t_{n-i}))^{-1/2} \int_{-\infty}^{\infty} g_{i-1}(v_{n-i+1}) \\ \cdot \exp \left\{ \varphi_{n-i+1}(v_{n-i+1}) - \frac{\lambda}{2} \left(\frac{(v_{n-i+1} - v_{n-i})^2}{(t_{n-i+1} - t_{n-i})} \right) \right\} dv_{n-i+1}, \quad \text{for } j = 2, 3, \dots, n.$$

Now

$$|e^{\varphi_{n-i+1}(v_{n-i+1})}| \leq e^{B(t_{n-i+1} - t_{n-i+1})}.$$

Therefore

$$||g_{i-1}e^{\varphi_{n-i+1}}|| \leq ||g_{i-1}|| e^{B(t_{n-i+1} - t_{n-i+1})}.$$

Now apply the lemma again and using mathematical induction we obtain

$$||g_i|| \leq ||g_{i-1}|| e^{B(t_{n-i+1} - t_{n-i+1})} \leq ||\psi|| e^{B[(t_n - t_{n-1}) + \dots + (t_{n-i+1} - t_{n-i+1})]}$$

for $j = 2, 3, \dots, n$;

so

$$||g_n|| \leq ||\psi|| e^{B(t_n - t_1)}.$$

The above inequalities imply that

$$(I_\lambda^\sigma(F)\psi)(\xi) = g_n(\xi)e^{\varphi_n(\xi)} = g_n(\xi) \exp \left\{ \int_{t_0}^{t_1} \theta(t, \xi) dt \right\}$$

and since

$$|(I_\lambda^\sigma(F)\psi)(\xi)| \leq |g_n(\xi)| e^{B(t_1 - t_0)}$$

we obtain

$$||I_\lambda^\sigma(F)\psi|| \leq ||\psi|| e^{B(b-a)}.$$

§5. For the sake of completeness we include here the following theorem which is similar to the classical Vitali Theorem.

Theorem 3. Let $\{x_n(z)\}$ be a sequence of vector-valued holomorphic functions in a domain D of the complex plane whose values lie in a separable Hilbert space H . Let $||x_n(z)|| \leq B$ for $z \in D$. Let $\{z_k\}$ be a sequence of distinct points of D such that $\lim_{k \rightarrow \infty} z_k = z_0 \in D$. Let $\lim_{n \rightarrow \infty} (x_n(z_k), y)$ exist for each $y \in H$ and each k . Then there exists $x(z) \in H$ such that $\lim_{n \rightarrow \infty} (x_n(z), y) = (x(z), y)$ and $x(z)$ is holomorphic in D and $\lim_{n \rightarrow \infty} (x_n(z), y) = (x(z), y)$ uniformly on any compact subset of D for each $y \in H$.

Proof. Let y be any point of H , and let us define $f_n(z) = (x_n(z), y)$ which is then an analytic function in D for each n . We also have $|f_n(z)| \leq ||x_n(z)|| ||y|| \leq B ||y||$ for all n . We now have $\lim_{n \rightarrow \infty} f_n(z_k) = \lim_{n \rightarrow \infty} (x_n(z_k), y)$ for each k . We now apply the classical Vitali theorem to the sequence of analytic functions $\{f_n(z)\}$ to obtain the existence of a function $f(z)$ such that $\lim_{n \rightarrow \infty} f_n(z) = f(z)$

for each $z \in D$ and the convergence is uniform on compact subsets of D . Thus for each $y \in H$, there exists an analytic function $f(z)$ (depending also on y) such that $\lim_{n \rightarrow \infty} (x_n(z), y) = f(z)$.

This is valid for any $y \in H$. Hence for each $z \in D$, there exists a point $x(z) \in H$ such that $\lim_{n \rightarrow \infty} (x_n(z), y) = (x(z), y)$ for each $y \in H$. Thus by the definition of weak limit, we have shown that $(x(z), y)$ is analytic, and consequently $x(z)$ is a holomorphic vector valued function.

We now state two definitions indicated in the introduction.

Definition. Let F be a functional such that $I_\lambda(F)\psi$ exists for λ real and positive. For each $\psi \in L_2(-\infty, \infty)$, let a function $A(\lambda; \psi)$ exist for $\operatorname{Re} \lambda > 0$ and be analytic (as a vector valued function) in λ for $\operatorname{Re} \lambda > 0$ and let $A(\lambda; \psi) = I_\lambda(F)\psi$ when λ is real and positive and $\psi \in L_2$. Then we define $I_\lambda^{\text{an}}(F)\psi = A(\lambda; \psi)$ for $\psi \in L_2$ and $\operatorname{Re} \lambda > 0$.

Definition. Let $\operatorname{Re} \lambda > 0$ and let F be a functional such that $I_\lambda^\sigma(F)\psi$ exists for $\psi \in L_2(-\infty, \infty)$ and every subdivision σ of $[a, b]$. Then if $\lim_{\text{norm } \sigma \rightarrow 0} (I_\lambda^\sigma(F)\psi, \varphi)$ exists for all $\varphi, \psi \in L_2(-\infty, \infty)$, we shall define $I_\lambda^{\text{seq}}(F)$ by the equation

$$\lim_{\text{norm } \sigma \rightarrow 0} (I_\lambda^\sigma(F)\psi, \varphi) = (I_\lambda^{\text{seq}}(F)\psi, \varphi).$$

Theorem 4. Let $\theta(t, u)$ satisfy the hypotheses of Theorem 2. Then it follows that for $\operatorname{Re} \lambda > 0$, $I_\lambda^{\text{seq}}(F)$ exists and is holomorphic for $\operatorname{Re} \lambda > 0$. Thus $I_\lambda^{\text{an}}(F)$ exists and

$$I_\lambda^{\text{an}}(F) = I_\lambda^{\text{seq}}(F)$$

and

$$\|I_\lambda^{\text{seq}}(F)\| = \|I_\lambda^{\text{an}}(F)\| \leq e^{B(b-a)}.$$

Proof. We first show that $I_\lambda^\sigma(F)\psi$ is holomorphic for $\operatorname{Re} \lambda > 0$. Consider the inner product

$$\begin{aligned} (I_\lambda^\sigma(F)\psi, \varphi) &= \lambda^{n/2} ((2\pi)^n (t_1 - a) \cdots (t_n - t_{n-1}))^{-1/2} \int_{-\infty}^{\infty} \binom{n+1}{\cdot} \int_{-\infty}^{\infty} \psi(v_n) \varphi(\xi) \\ &\quad \cdot \exp \left[\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \theta(t, v_{i-1}) dt - \sum_{i=1}^n \frac{\lambda(v_i - v_{i-1})^2}{2(t_i - t_{i-1})} \right] dv_1 \cdots dv_n d\xi. \end{aligned}$$

An application of Morera's theorem to the right hand member of the above equation enables us to conclude that $I_\lambda^\sigma(F)\psi$ is holomorphic for $\operatorname{Re} \lambda > 0$. Because $I_\lambda^\sigma(F)\psi = I_\lambda(F_\sigma)\psi$ for real $\lambda, \lambda > 0$, Theorem 2 implies that $\lim_{\text{norm } \sigma \rightarrow 0} (I_\lambda^\sigma(F)\psi, \varphi)$ exists and equals $(I_\lambda(F)\psi, \varphi)$ for real $\lambda, \lambda > 0$. Because Lemma 6 yields $\|I_\lambda^\sigma(F)\psi\| \leq \|\psi\| e^{B(b-a)}$ we may apply our Vitali theorem (Theorem 3) to complete the proof of Theorem 4.

§6. We now define weak limits of $I_\lambda^{\text{an}}(F)$ and $I_\lambda^{\text{seq}}(F)$ as λ approaches the imaginary axis along a horizontal line, and give an existence theorem for these limits.

Definition. Let q be real and let F be a functional such that $I_\lambda^{\text{an}}(F)\psi$ exists for every $\psi \in L_2(-\infty, \infty)$ and for $\text{Re } \lambda > 0$. Then, if

$$\lim_{p \rightarrow 0+} (I_{p-iq}^{\text{an}}(F)\psi, \varphi)$$

exists for all $\varphi, \psi \in L_2(-\infty, \infty)$, we shall define $J_q^{\text{an}}(F)$ by the equation

$$\lim_{p \rightarrow 0+} (I_{p-iq}^{\text{an}}(F)\psi, \varphi) = (J_q^{\text{an}}(F)\psi, \varphi).$$

Definition. Let q be real and let F be a functional such that $I_\lambda^{\text{seq}}(F)\psi$ exists for every $\psi \in L_2(-\infty, \infty)$ and for $\lambda = p - iq$, $0 < p < p_0$ for some $p_0 > 0$. Then, if

$$\lim_{p \rightarrow 0+} (I_{p-iq}^{\text{seq}}(F)\psi, \varphi)$$

exists for all $\varphi, \psi \in L_2(-\infty, \infty)$ we shall define $J_q^{\text{seq}}(F)$ by the equation

$$\lim_{p \rightarrow 0+} (I_{p-iq}^{\text{seq}}(F)\psi, \varphi) = (J_q^{\text{seq}}(F)\psi, \varphi).$$

Whenever $J_q^{\text{seq}}(F)\psi$ and $J_q^{\text{an}}(F)\psi$ exist and are equal we write

$$J_q(F)\psi = J_q^{\text{an}}(F)\psi = J_q^{\text{seq}}(F)\psi.$$

Theorem 5. Under the hypotheses of Theorem 2, the bounded analytic function $I_\lambda^{\text{seq}}(F)\psi \equiv I_\lambda^{\text{an}}(F)\psi$ possesses a (weak) limit as $\lambda \rightarrow -iq$ along any horizontal line in the right half plane for almost all real q . In other words $J_q(F)\psi$ exists for almost all q .

Proof. The proof follows immediately from the Fatou theorem for vector valued holomorphic functions. The Fatou theorem for vector valued holomorphic functions is easily obtained from the classical Fatou theorem by considering the inner products of the vector valued holomorphic function with the elements of a complete orthonormal sequence.

Corollary. Under the hypotheses to Theorem 5,

$$||J_q(F)\psi|| \leq ||\psi|| e^{B(b-a)}$$

for almost all q .

Proof. By Theorem 4, we have the inequality

$$||I_{p-iq}^{\text{an}}(F)\psi|| \leq ||\psi|| e^{B(b-a)},$$

and by Theorem 5 we know that $I_{p-iq}^{\text{an}}(F)\psi \rightarrow J_q(F)\psi$ weakly. If a sequence f_n tends weakly to f , $||f|| \leq \liminf_{n \rightarrow \infty} ||f_n||$. Thus in our case,

$$||J_a(F)\psi|| \leq \liminf_{p \rightarrow 0+} ||I_{p-i a}^{\text{an}}(F)\psi|| \leq ||\psi|| e^{B(b-a)}.$$

§7. In order to extend Theorems 4 and 5 to more general analytic functions than exponentials we now prove three lemmas.

Lemma 7. *Let $\text{Re } \lambda > 0$ and σ be a subdivision of $[a, b]$. If $F(x) = \varphi_0(x(t_0)) \cdot \varphi_1(x(t_1)) \cdots \varphi_n(x(t_n))$, where the φ_i are measurable on $(-\infty, \infty)$ and $|\varphi_i| \leq M_i$, then $||I_\lambda^\sigma(F)\psi|| \leq ||\psi|| \prod_{i=0}^n M_i$.*

Proof. Let $g_0(v_n) = \psi(v_n)$ and let

$$g_1(v_{n-1}) = (\lambda/2\pi(t_n - t_{n-1}))^{1/2} \int_{-\infty}^{\infty} g_0(v_n) \varphi_n(v_n) \cdot \exp \{ -\lambda(v_n - v_{n-1})^2/2(t_n - t_{n-1}) \} dv_n.$$

By Lemma 1, $g_1 \in L_2$ and $||g_1|| \leq ||\varphi_n \psi|| \leq M_n ||\psi||$. In general, we let

$$g_{i+1}(v_{n-i-1}) = (\lambda/2\pi(t_{n-i} - t_{n-i-1}))^{1/2} \int_{-\infty}^{\infty} \varphi_{n-i}(v_{n-i}) g_i(v_{n-i}) \cdot \exp \{ -\lambda(v_{n-i} - v_{n-i-1})^2/2(t_{n-i} - t_{n-i-1}) \} dv_{n-i}$$

for $j = 1, 2, \dots, n-1$. Then it follows that

$$||g_{i+1}|| \leq ||\varphi_{n-i} g_i|| \leq M_{n-i} ||g_i|| \leq M_{n-i} M_{n-i+1} \cdots M_n ||\psi||.$$

Setting $j = n-1$, we obtain

$$||g_n|| \leq M_1 M_2 \cdots M_n ||\psi||.$$

We now consider

$$(I_\lambda^\sigma(F)\psi)(\xi) = \lambda^{n/2} ((2\pi)^n (t_1 - a) \cdots (t_n - t_{n-1}))^{-1/2} \int_{-\infty}^{\infty} \cdot^{(n)} \int_{-\infty}^{\infty} f(\xi, v_1, \dots, v_n) \cdot \psi(v_n) \exp \left\{ -\sum_{i=1}^n \lambda(v_i - v_{i-1})^2/2(t_i - t_{i-1}) \right\} dv_1 \cdots dv_n$$

where

$$v_0 = \xi, f(v_0, v_1, \dots, v_n) = F[z(\sigma, v_0, \dots, v_n, \cdot)] = \varphi_0(v_0) \cdot \varphi_1(v_1) \cdots \varphi_n(v_n).$$

Thus we obtain

$$(I_\lambda^\sigma(F)\psi)(\xi) = \lambda^{n/2} ((2\pi)^n (t_1 - a) \cdots (t_n - t_{n-1}))^{-1/2} \cdot \int_{-\infty}^{\infty} \cdot^{(n)} \int_{-\infty}^{\infty} \varphi_0(\xi) \varphi_1(v_1) \cdots \varphi_n(v_n) \psi(v_n) \cdot \exp \left\{ -\sum_{i=1}^n \lambda(v_i - v_{i-1})^2/2(t_i - t_{i-1}) \right\} dv_1 \cdots dv_n = g_n(\xi) \varphi_0(\xi).$$

From this equation the inequality

$$||I_{\lambda}^{\sigma}(F)\psi|| \leq M_0 M_1 \cdots M_n ||\psi||$$

follows and the proof of Lemma 7 is complete.

We also have the following corollary to Lemma 7:

Corollary. *Under the hypotheses of Lemma 7,*

$$||I_{\lambda}^{\sigma}(F)|| \leq M_0 \cdots M_n .$$

Lemma 8. *Let $\operatorname{Re} \lambda > 0$, and let σ be a subdivision of $[a, b]$. If $F(x) = [\int_a^b \theta(t, x(t)) dt]^m$ where $|\theta| \leq M$ and θ is continuous almost everywhere in the strip $a \leq t \leq b, -\infty < u < \infty$; then*

$$||I_{\lambda}^{\sigma}(F)\psi|| \leq ||\psi|| M^m (b - a)^m .$$

Proof. We first observe that

$$F(x) = \left(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \theta(t, x(t)) dt \right)^m ,$$

and hence in the notation of the definition of $(I_{\lambda}^{\sigma}(F)\psi)(\xi)$ given in the equations (4.1)–(4.7) above,

$$\begin{aligned} f(v_0, v_1, \dots, v_n) &= F[z(\sigma, v_0, \dots, v_n, \cdot)] \\ &= \left(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \theta(t, v_{i-1}) dt \right)^m = \sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_m=1}^n \varphi_{i_1}(v_{i_1-1}) \cdots \varphi_{i_m}(v_{i_m-1}), \end{aligned}$$

where

$$\varphi_i(u) = \int_{t_{i-1}}^{t_i} \theta(s, u) ds .$$

we now consider:

$$\begin{aligned} (I_{\lambda}^{\sigma}(F)\psi)(\xi) &= \lambda^{n/2} ((2\pi)^n (t_1 - a) \cdots (t_n - t_{n-1}))^{-1/2} \\ &\quad \cdot \int_{-\infty}^{\infty} \cdots (n) \cdots \int_{-\infty}^{\infty} f(\xi, v_1, \dots, v_n) \psi(v_n) \\ &\quad \cdot \exp \left\{ \sum_{k=1}^n \lambda (v_k - v_{k-1})^2 / 2 (t_k - t_{k-1}) \right\} dv_1 \cdots dv_n \\ &= (\lambda/2\pi)^{n/2} ((t_1 - a) \cdots (t_n - t_{n-1}))^{-1/2} \int_{-\infty}^{\infty} \cdots (n) \cdots \\ &\quad \cdot \int_{-\infty}^{\infty} \sum_{i_1=1}^n \sum_{i_2=1}^n \cdots (m) \cdots \sum_{i_m=1}^n \varphi_{i_1}(v_{i_1-1}) \cdots \varphi_{i_m}(v_{i_m-1}) \cdot \psi(v_n) \\ &\quad \cdot \exp \left\{ - \sum_{k=1}^n \lambda (v_k - v_{k-1})^2 / 2 (t_k - t_{k-1}) \right\} dv_1 \cdots dv_n \end{aligned}$$

$$\begin{aligned}
&= \sum_{i_1=1}^n \sum_{i_2=1}^n \dots \sum_{i_m=1}^n (\lambda/2\pi)^{n/2} ((t_1 - a) \cdots (t_n - t_{n-1}))^{-1/2} \\
&\quad \cdot \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \Phi_0(v_0) \Phi_1(v_1) \cdots \Phi_{n-1}(v_{n-1}) \cdot \psi(v_n) \\
&\quad \cdot \exp \left\{ - \sum_{i=1}^n \lambda (v_i - v_{i-1})^2 / 2(t_i - t_{i-1}) \right\} dv_1 \cdots dv_n,
\end{aligned}$$

where

$$\Phi_k(v_k) = [\varphi_{k+1}(v_k)]^{p_k} \quad \text{and} \quad p_k = \sum_{q=1}^m \delta_{k+1, i_q},$$

where $\delta_{i,j}$ is the Kronecker delta function. Thus we have

$$(I_\lambda^\sigma(F)\psi)(\xi) = \sum_{i_1=1}^n \sum_{i_2=1}^n \cdots (m) \cdots \sum_{i_m=1}^n (I_\lambda^\sigma(F_{i_1, i_2, \dots, i_m})\psi)(\xi),$$

where

$$F_{i_1, i_2, \dots, i_m}(x) = \Phi_0(x(t_0)) \Phi_1(x(t_1)) \cdots \Phi_{n-1}(x(t_{n-1})).$$

Hence by Lemma 7 we have the following inequalities

$$||I_\lambda^\sigma(F)\psi|| \leq \sum_{i_1=1}^n \sum_{i_2=1}^n \cdots (m) \cdots \sum_{i_m=1}^n ||\psi|| \prod_{k=0}^n M'_k$$

where $|\Phi_k| \leq M'_k = [M(t_{k+1} - t_k)]^{p_k}$. Thus we have

$$\begin{aligned}
||I_\lambda^\sigma(F)\psi|| &\leq \sum_{i_1=1}^n \cdots \sum_{i_m=1}^n ||\psi|| \prod_{k=0}^{n-1} [M(t_{k+1} - t_k)]^{\sum_{q=1}^m \delta_{k+1, i_q}} \\
&= \sum_{i_1=1}^n \cdots \sum_{i_m=1}^n ||\psi|| M^{\sum_{k=0}^{n-1} \sum_{q=1}^m \delta_{k+1, i_q}} \prod_{k=0}^{n-1} (t_{k+1} - t_k)^{\sum_{q=1}^m \delta_{k+1, i_q}} \\
&= ||\psi|| M^m \sum_{i_1=1}^n \cdots \sum_{i_m=1}^n \prod_{k=0}^{n-1} (t_{k+1} - t_k)^{\sum_{q=1}^m \delta_{k+1, i_q}} \\
&= ||\psi|| M^m \sum_{i_1=1}^n \cdots \sum_{i_m=1}^n \prod_{q=1}^m (t_{i_q} - t_{i_q-1}) \\
&= ||\psi|| M^m \left(\sum_{j=1}^n (t_j - t_{j-1}) \right)^m = (b - a)^m M^m ||\psi||.
\end{aligned}$$

Lemma 9. Let $\operatorname{Re} \lambda > 0$ and let σ be a subdivision of $[a, b]$. Let $\theta(t, u)$ be continuous almost everywhere on $R \equiv [a, b] \times (-\infty, \infty)$ and let $|\theta(t, u)| \leq M$ on R . Let $G(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function which is analytic in a disk of radius greater than $M(b - a)$ and centered at the origin. Let $F(x) = G\{\int_a^b \theta(s, x) ds\}$; then

$$||I_{\lambda}^{\sigma}(F)\psi|| \leq ||\psi|| \sum_{n=0}^{\infty} |a_n| (M(b-a))^n.$$

Proof. Let $F_n(x) = [\int_a^b \theta(s, x(s)) ds]^n$; then by Lemma 8, we have $||I_{\lambda}^{\sigma}(F_n)\psi|| \leq ||\psi|| M^n(b-a)^n$. Now $F(x) = \sum_{n=0}^{\infty} a_n F_n(x)$, and thus we have

$$\begin{aligned} ||I_{\lambda}^{\sigma}(F)\psi|| &= \left\| \sum_{n=0}^{\infty} a_n I_{\lambda}^{\sigma}(F_n)\psi \right\| \\ &\leq \sum_{n=0}^{\infty} |a_n| ||I_{\lambda}^{\sigma}(F_n)\psi|| \leq ||\psi|| \sum_{n=0}^{\infty} |a_n| M^n(b-a)^n. \end{aligned}$$

§8. In this section, for bounded $\theta(t, u)$ we extend Theorems 4 and 5 by replacing the exponential with more general analytic functions.

Theorem 6. Let $\theta(t, u)$ be continuous almost everywhere on $R \equiv [a, b] \times (-\infty, \infty)$, and let $|\theta(t, u)| \leq M$ on R ; let $G(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function which is analytic in a disk centered at the origin of radius greater than $M(b-a)$. Let $F(x) = G\{\int_a^b \theta(s, x(s)) ds\}$, then the functional $I_{\lambda}^{\text{seq}}(F)\psi$ exists and is analytic in λ , $\text{Re } \lambda > 0$ for all $\psi \in L_2(-\infty, \infty)$, so that $I_{\lambda}^{\text{an}}(F)\psi$ also exists and

$$I_{\lambda}^{\text{seq}}(F) = I_{\lambda}^{\text{an}}(F).$$

Moreover we have

$$||I_{\lambda}^{\text{seq}}(F)|| = ||I_{\lambda}^{\text{an}}(F)|| \leq \sum_{n=0}^{\infty} |a_n| [M(b-a)]^n.$$

Proof. We first show that $I_{\lambda}^{\sigma}(F)\psi$ is holomorphic for $\text{Re } \lambda > 0$. Consider the inner product

$$\begin{aligned} (I_{\lambda}^{\sigma}(F)\psi, \varphi) &= (\lambda/2\pi)^{n/2} ((t_1 - a) \cdots (t_n - t_{n-1}))^{-1/2} \\ &\quad \cdot \int_{-\infty}^{\infty} \cdots (n+1) \cdots \int_{-\infty}^{\infty} \psi(v_n) \varphi(\xi) G\left(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \theta(t, v_{i-1}) dt\right) \\ &\quad \cdot \exp\left\{-\sum_{i=1}^n \lambda(v_i - v_{i-1})^2/2(t_i - t_{i-1})\right\} dv_1 \cdots dv_n d\xi. \end{aligned}$$

An application of Morera's theorem to the right hand member of the above equation enables us to conclude that $I_{\lambda}^{\sigma}(F)\psi$ is holomorphic for $\text{Re } \lambda > 0$. Because $I_{\lambda}^{\sigma}(F)\psi = I_{\lambda}(F_{\sigma})\psi$ for real λ , $\lambda > 0$, Corollary 1 of Theorem 2 above implies that $\lim_{\lambda_{\sigma} \rightarrow \lambda} (I_{\lambda}^{\sigma}(F)\psi, \varphi)$ exists and equals $(I_{\lambda}(F)\psi, \varphi)$ for real λ , $\lambda > 0$. Because Lemma 9 implies that

$$||I_{\lambda}^{\sigma}(F)\psi|| \leq ||\psi|| \sum_{n=0}^{\infty} |a_n| (M(b-a))^n,$$

we may apply our Vitali theorem (Theorem 3) to complete the proof of Theorem 6.

Theorem 7. Under the hypotheses of Theorem 6, the bounded analytic function $I_{\lambda}^{\text{sa}}(F)\psi \equiv I_{\lambda}^{\text{an}}(F)\psi$ possesses a weak limit as $\lambda \rightarrow -iq$ along a horizontal line in the right half plane for almost all real q . In other words, $J_q(F)\psi$ exists for almost all q .

Proof. The proof is identical with the proof of Theorem 5.

§9. Theorem 8. Let $\theta(t, u)$ be continuous almost everywhere in the strip $R: 0 \leq t \leq t_0, -\infty < u < \infty$, and let $|\theta(t, u)| \leq M$ for (t, u) in the strip. Let $\psi \in L_2(-\infty, \infty)$, let λ be a complex number, $\text{Re } \lambda > 0, t > 0$, and ξ be real. Let $G(t, \xi, \lambda)$ be defined by

$$(9.0) \quad G(t, \xi, \lambda) = \int_{C_0[0, t]} \exp \left\{ \int_0^t \theta(t-s, \lambda^{-1/2}x(s) + \xi) ds \right\} \psi[\lambda^{-1/2}x(t) + \xi] dx.$$

Then $G(t, \xi, \lambda)$ has an analytic extension in λ for $\text{Re } \lambda > 0, t \geq 0, \xi$ real, and it satisfies the following integral equation:

$$(9.1) \quad G(t, \xi, \lambda) = \left(\frac{\lambda}{2\pi t} \right)^{1/2} \int_{-\infty}^{\infty} \psi(u) e^{-\lambda(\xi-u)^2/2t} du \\ + \left(\frac{\lambda}{2\pi} \right)^{1/2} \int_0^t (t-s)^{-1/2} ds \int_{-\infty}^{\infty} \theta(s, u) G(s, u, \lambda) e^{-\lambda(\xi-u)^2/2(t-s)} du.$$

Proof. In order to derive the integral equation above (see for instance Darling and Siegert [6]), let us begin by defining the function

$$(9.2) \quad \Theta(t, u) = \int_{C_0[0, t]} \exp \left\{ \int_0^t \gamma(\rho, x(t-\rho) + u) d\rho \right\} \varphi(u + x(t)) dx,$$

where $\gamma(t, u) = \theta(t, u\lambda^{-1/2})$ and $\varphi(u) = \psi(u\lambda^{-1/2})$ for all (t, u) in the strip R . The integral in the exponent exists for almost all x by Lemma 4, and for $0 < t \leq t_0$ and $-\infty < u < \infty$, the function

$$\Theta(t, u) = \int_{C_0[0, t]} \exp \left\{ \int_0^t \gamma(\rho, x(t-\rho) + u) d\rho \right\} \varphi(u + x(t)) dx$$

satisfies the following inequalities:

$$|\Theta(t, u)| \leq e^{Mt} \int_{C_0[0, t]} |\varphi(u + x(t))| dx = e^{Mt} (2\pi t)^{-1/2} \\ \cdot \int_{-\infty}^{\infty} |\varphi(u + v)| e^{-v^2/2t} dv = e^{Mt} (2\pi t)^{-1/2} \int_{-\infty}^{\infty} |\varphi(w)| e^{-(w-u)^2/2t} dw.$$

An application of Lemma 1 to this integral yields the following inequality for the norm of Θ which is valid for $0 \leq t \leq t_0$:

$$(9.3) \quad \|\Theta(t, \cdot)\| \leq e^{Mt} \|\varphi\|.$$

An application of the Schwarz inequality to the same integral yields

$$|\Theta(t, u)| \leq e^{Mt}(2\pi t)^{-1/2} \|\varphi\| \left(\int_{-\infty}^{\infty} e^{-(w-u)^2/t} dw \right)^{1/2},$$

and after using the standard integral

$$\int_{-\infty}^{\infty} e^{-q^2} dq = \pi^{1/2},$$

we also obtain the following inequality:

$$(9.4) \quad |\Theta(t, u)| \leq 4e^{Mt}(4\pi t)^{-1/4} \|\varphi\|.$$

Since $\gamma(t, u)$ is continuous almost everywhere in the strip R and is bounded in R , for almost all (t, u) in the strip by applying the fundamental theorem for Lebesgue integrals, we obtain for almost all x :

$$\begin{aligned} \int_0^t \gamma(s, u + x(t-s)) \exp \left\{ \int_0^s \gamma(\rho, u + x(t-\rho)) d\rho \right\} ds \\ = \exp \left\{ \int_0^t \gamma(\rho, u + x(t-\rho)) d\rho \right\} - 1. \end{aligned}$$

After multiplying this equation by $\varphi(u + x(t))$ and integrating over Wiener space we obtain

$$\begin{aligned} Q(t, u) &\equiv \int_{C_0[0, t]} \varphi(u + x(t)) \\ &\quad \cdot \int_0^t \gamma(s, u + x(t-s)) \exp \left\{ \int_0^s \gamma(\rho, u + x(t-\rho)) d\rho \right\} ds dx \\ &= \int_{C_0[0, t]} \varphi(u + x(t)) \left[\exp \left\{ \int_0^t \gamma(\rho, u + x(t-\rho)) d\rho \right\} - 1 \right] dx, \end{aligned}$$

and consequently

$$(9.5) \quad Q(t, u) = \Theta(t, u) - \int_{C_0[0, t]} \varphi(u + x(t)) dx.$$

Because the integrand of $Q(t, u)$ is the product of an integrable function over C_0 multiplied by a bounded function, we may apply the Fubini theorem to $Q(t, u)$,

$$\begin{aligned} Q(t, u) &= \int_0^t ds \int_{C_0[0, t]} \varphi(u + x(t)) \gamma(s, u + x(t-s)) \\ &\quad \cdot \exp \left\{ \int_0^s \gamma(\rho, u + x(t-\rho)) d\rho \right\} dx. \end{aligned}$$

By using the following property of the Wiener integral [5],

$$\int_{C_0[a, b]} F(x) dx = \int_{C_0[a, c] \times C_0[c, b]} F(y + z) d(y \times z),$$

where for $t \in [a, c]$, $z(t) = 0$ and for $t \in [c, b]$, $y(t) = y(c)$; since $0 \leq s \leq t$, we may write

$$Q(t, u) = \int_0^t ds \int_{C_0[0, t-s] \times C_0[t-s, t]} \varphi(u + y(t) + z(t)) \gamma(s, u + y(t-s) + z(t-s)) \\ \cdot \exp \left\{ \int_0^s \gamma(\rho, u + y(t-\rho) + z(t-\rho)) d\rho \right\} d(y \times z).$$

We observe that $z(t-s)$ equals zero and in the exponential, $y(t-\rho) = y(t-s)$ because $0 \leq \rho \leq s$ and $t-\rho \geq t-s$, so that we may now transform the double integral into these repeated integrals,

$$Q(t, u) = \int_0^t ds \int_{C_0[0, t-s]} \gamma(s, u + y(t-s)) dy \int_{C_0[t-s, t]} \varphi(u + y(t-s) + z(t)) \\ \cdot \exp \left\{ \int_0^s \gamma(\rho, u + y(t-s) + z(t-\rho)) d\rho \right\} dz.$$

We next change variables so that the second Wiener integral is extended over the space $C_0[0, s]$ to obtain:

$$Q(t, u) = \int_0^t ds \int_{C_0[0, t-s]} \gamma(s, u + y(t-s)) dy \int_{C_0[0, s]} \varphi(u + y(t-s) + z(s)) \\ \cdot \exp \left\{ \int_0^s \gamma(\rho, u + y(t-s) + z(s-\rho)) d\rho \right\} dz.$$

We now use the definition of $\Theta(t, u)$, (9.2), to obtain

$$Q(t, u) = \int_0^t ds \int_{C_0[0, t-s]} \gamma(s, u + y(t-s)) [\Theta(s, u + y(t-s))] dy.$$

We integrate this to obtain

$$Q(t, u) = \int_0^t ds (2\pi(t-s))^{-1/2} \int_{-\infty}^{\infty} \gamma(s, u+v) \Theta(s, u+v) e^{-v^2/2(t-s)} dv,$$

and from (9.5) we have

$$Q(t, u) = \Theta(t, u) - (2\pi t)^{-1/2} \int_{-\infty}^{\infty} \varphi(u+v) e^{-v^2/2t} dv.$$

Utilizing the definition of $Q(t, u)$, (9.5), we have derived for $0 < t \leq t_0$ an integral equation for Θ

$$(9.6) \quad \Theta(t, u) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \varphi(w) e^{-(w-u)^2/2t} dw \\ + (2\pi)^{-1/2} \int_0^t (t-s)^{-1/2} ds \int_{-\infty}^{\infty} \gamma(s, w) \Theta(s, w) e^{-(w-u)^2/2(t-s)} dw.$$

Rewriting Θ as defined in (9.2) in terms of θ, ψ we have

$$(9.7) \quad \Theta(t, u) = \int_{c_0 t_0, t_1} \exp \left\{ \int_0^t \theta(\rho, \lambda^{-1/2} x(t - \rho) + \lambda^{-1/2} u) d\rho \right\} \\ \cdot \psi(\lambda^{-1/2} x(t) + \lambda^{-1/2} u) dx.$$

We now observe from (9.0) and (9.7) that $G(t, \xi, \lambda) = \Theta(t, \lambda^{1/2} \xi)$ and rewriting (9.6) in terms of θ and ψ and G we obtain

$$G(t, \xi, \lambda) = \left(\frac{\lambda}{2\pi t} \right)^{1/2} \int_{-\infty}^{\infty} \psi(v) e^{-\lambda(\xi - v)^2/2t} dv \\ + \left(\frac{\lambda}{2\pi} \right)^{1/2} \int_0^t (t - s)^{-1/2} ds \int_{-\infty}^{\infty} \theta(s, v) G(s, v, \lambda) e^{-\lambda(\xi - v)^2/2(t-s)} dv.$$

In order to use Theorem 4, we note that $\theta(t - s, u)$ satisfies in $0 \leq s \leq t$, $-\infty < u < \infty$ for each fixed $t \in [0, t_0]$ the same hypotheses as $\theta(t, u)$ in R . Moreover if we set

$$F(x) \equiv \exp \left\{ \int_0^t \theta(t - s, x(s)) ds \right\}$$

for real positive λ we have

$$G(t, \xi, \lambda) = (I_\lambda(F)\psi)(\xi) \\ = \int_{c_0 t_0, t_1} \exp \left\{ \int_0^t \theta(t - s, \lambda^{-1/2} x(s) + \xi) ds \right\} \psi(\lambda^{-1/2} x(t) + \xi) dx.$$

We have by Theorem 4 that for each $\psi \in L_2$ and $(t, \xi) \in R$, $G(t, \xi, \lambda)$ has an analytic extension in λ to the whole half-plane $\text{Re } \lambda > 0$, thus

$$G(t, \xi, \lambda) = (I_\lambda^{\text{an}}(F)\psi)(\xi) \quad \text{Re } \lambda > 0.$$

By Theorem 4 we have

$$||I_\lambda^{\text{an}}(F)\psi|| \leq ||\psi|| e^{M't_0}$$

and so

$$||G(t, \cdot, \lambda)|| \leq ||\psi|| e^{M't_0}.$$

The right hand side of the last integral equation above is analytic in λ for $\text{Re } \lambda > 0$, $t > 0$, and ξ real by Morera's theorem. Hence the integral equation (9.1) is valid for $0 < t \leq t_0$, λ complex, $\text{Re } \lambda > 0$, and ξ real, and the proof of Theorem 8 is complete.

§10. In order to effect an interchange of order of integration at one point of the proof of Theorem 9, it is convenient to apply this lemma.

Lemma 10. *Let $f \in L_2$ and $\varphi \in L_1 \cap L_2$. Then it follows that*

$$\int_{-\infty}^{\infty} \varphi(t) \int_{-\infty}^{\infty} f(s) e^{i(s-t)^2/2} ds dt = \int_{-\infty}^{\infty} f(s) \int_{-\infty}^{\infty} \varphi(t) e^{i(s-t)^2/2} dt ds.$$

Proof. We begin by observing that the limit in the mean of $\int_{-A}^A f(s)e^{i(s-t)^2/2} ds$ exists because of Plancherel's theorem, since

$$\int_{-A}^A f(s)e^{i(s-t)^2/2} ds = e^{it^2/2} \int_{-A}^A f(s)e^{is^2/2} e^{-ist} ds,$$

and we are simply taking the Fourier transform of $f(s)e^{is^2/2}$ in the usual way for functions of L_2 . We have

$$\begin{aligned} \int_{-\infty}^{\infty} \varphi(t) \int_{-\infty}^{\infty} f(s)e^{i(s-t)^2/2} ds dt &= \int_{-\infty}^{\infty} \varphi(t) \left[\text{l.i.m.}_{A \rightarrow \infty} \int_{-A}^A f(s)e^{i(s-t)^2/2} ds \right] dt \\ &= \lim_{A \rightarrow \infty} \int_{-\infty}^{\infty} \varphi(t) \left[\int_{-A}^A f(s)e^{i(s-t)^2/2} ds \right] dt \\ &= \lim_{A \rightarrow \infty} \int_{-A}^A f(s) \left[\int_{-\infty}^{\infty} \varphi(t)e^{i(s-t)^2/2} dt \right] ds \\ &= \int_{-\infty}^{\infty} f(s) \int_{-\infty}^{\infty} \varphi(t)e^{i(s-t)^2/2} dt ds, \end{aligned}$$

and the lemma is proved.

Corollary. Let $f, \varphi \in L_2$. Then it follows that

$$\int_{-\infty}^{\infty} \varphi(t) \int_{-\infty}^{\infty} f(s)e^{i(s-t)^2/2} ds dt = \int_{-\infty}^{\infty} f(s) \int_{-\infty}^{\infty} \varphi(t)e^{i(s-t)^2/2} dt ds.$$

This follows by applying the lemma to $f(s)$ and $\varphi(t)\chi_{[-B, B]}(t)$ and taking limits as $B \rightarrow \infty$.

§11. Theorem 9. Let $\theta(t, u)$ be continuous almost everywhere in the strip $R: 0 \leq t \leq t_0, -\infty < u < \infty, |\theta(t, u)| \leq M$ for $(t, u) \in R$. Let $F(x) = \exp \{ \int_0^t \theta(t-s, x(s)) ds \}$, and let $\Gamma(t, \xi, q) = (J_q(F)\psi)(\xi)$ for $(t, \xi) \in R$ and almost all real q . Then for each $t \in (0, t_0]$ and almost every real q ,

$$\begin{aligned} \Gamma(t, \xi, q) &= \text{l.i.m.}_{\substack{A \rightarrow \infty \\ \xi}} \left(\frac{q}{2\pi i t} \right)^{1/2} \int_{-A}^A \psi(u) e^{iq(\xi-u)^2/2t} du \\ (10.1) \quad &+ \text{l.i.m.}_{\substack{A \rightarrow \infty \\ \xi}} \left(\frac{q}{2\pi i} \right)^{1/2} \int_0^t (t-s)^{-1/2} ds \int_{-A}^A \theta(s, u) \Gamma(s, u, q) e^{iq(\xi-u)^2/2(t-s)} du, \end{aligned}$$

or in simplified notation

$$\begin{aligned} \Gamma(t, \xi, q) &= \left(\frac{q}{2\pi i t} \right)^{1/2} \int_{-\infty}^{\infty} \psi(u) e^{iq(\xi-u)^2/2t} du \\ (10.2) \quad &+ \left(\frac{q}{2\pi i} \right)^{1/2} \int_0^t (t-s)^{-1/2} ds \int_{-\infty}^{\infty} \theta(s, u) \Gamma(s, u, q) e^{iq(\xi-u)^2/2(t-s)} du. \end{aligned}$$

As the notation indicates, these limits in the mean are to be taken with respect to the variable ξ and these limits on the right hand side exist and the equality holds for each t and almost all real q . Here Γ can be and should be chosen so as to be measurable in (t, ξ, q) .

Proof. Theorem 5 asserts that for each fixed valued of the parameter t appearing in the function $F(x) = \exp \{ \int_0^t \theta(t-s, x(s)) ds \}$, the bounded analytic function $I_\lambda(F) \equiv I_\lambda^{\text{an}}(F) \equiv I_\lambda^{\text{sa}}(F)$ possesses a weak limit as $\lambda \rightarrow iq$ horizontally for almost all real q . We have denoted this weak limit as $J_q(F)$. We will be using the function G defined by

$$G(t, \xi, \lambda) = (I_\lambda(F)\psi)(\xi)$$

for λ with positive real part and its limit $\Gamma(t, \xi, q)$ defined on the imaginary axis. For $\varphi \in L_2(-\infty, \infty)$ we have for each $t \in [0, t_0]$ and almost every q ,

$$\lim_{\eta \rightarrow 0^+} (G(t, \cdot, \eta - iq), \varphi) = (\Gamma(t, \cdot, q), \varphi)$$

or

$$\lim_{\eta \rightarrow 0^+} \int_{-\infty}^{\infty} G(t, \xi, \eta - iq) \varphi(\xi) d\xi = \int_{-\infty}^{\infty} \Gamma(t, \xi, q) \varphi(\xi) d\xi$$

for each t on $[0, t_0]$ and almost all real q . Since we can choose

$$G(t, \xi, \eta - iq) \varphi(\xi)$$

to be measurable in (t, ξ, η, q) , it follows that $\Gamma(t, \xi, q)$ can be so chosen as to be measurable in (t, ξ, q) . But for each t , $\Gamma(t, \xi, q)$ exists for almost all (ξ, q) , so the set of (t, ξ, q) where it does not exist is a null set in the three dimensional product space. Thus by the Fubini theorem there is an exceptional null set Q independent of t and ξ such that if $q \notin Q$, $\Gamma(t, \xi, q)$ exists for almost all (t, ξ) in R and is measurable over R .

The basic idea of the proof of Theorem 9 is to begin with the integral equation of Theorem 8, multiply both sides of that equation by

$$(10.3) \quad \varphi_n(\xi) \equiv H_n(\xi) e^{-\xi^2/2}$$

(where $H_n(\xi)$ is essentially the Hermite polynomial of order n , i.e., $H_n(\xi) = (-1)^n e^{\xi^2/2} (d^n/d\xi^n)(e^{-\xi^2/2})$), integrate with respect to ξ over $(-\infty, \infty)$, and pass to the limit as $\lambda \rightarrow -iq$. By our Lemma 10, this amounts to taking a weak limit and thus we will obtain the integral equation of Theorem 9.

Let us begin by considering the second term on the right hand side of the integral equation (9.1) which we write as follows after multiplying by φ_n and integrating:

$$(10.4) \quad \int_{-\infty}^{\infty} \varphi_n(\xi) \int_0^t g(s, \xi) ds d\xi$$

where

$$(10.5) \quad g(s, \xi) = \left(\frac{\lambda}{2\pi(t-s)} \right)^{1/2} \int_{-\infty}^{\infty} \theta(s, u) G(s, u, \lambda) e^{-\lambda(\xi-u)^2/2(t-s)} du.$$

Then by Lemma 1 and Theorem 4, the following estimates hold

$$(10.6) \quad \begin{aligned} ||g(s, \cdot)|| &\leq ||\theta(s, u)G(s, \cdot, \lambda)|| \\ &\leq M ||G(s, \cdot, \lambda)|| \\ &\leq M ||\psi|| e^{Mt_0}, \end{aligned}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} \left(\int_0^t |g(s, \xi)| ds \right)^2 d\xi &\leq \int_{-\infty}^{\infty} \int_0^t |g(s, \xi)| ds \int_0^t |g(s', \xi)| ds' d\xi \\ &= \int_0^t ds \int_0^t ds' \int_{-\infty}^{\infty} |g(s, \xi)| |g(s', \xi)| d\xi \\ &\leq \int_0^t ds \int_0^t ||g(s, \cdot)|| \cdot ||g(s', \cdot)|| ds' \\ &\leq \int_0^t ds \int_0^t M^2 ||\psi||^2 e^{2Mt_0} ds' \\ &= t^2 M^2 ||\psi||^2 e^{2Mt_0}. \end{aligned}$$

This enables us to conclude that

$$\int_{-\infty}^{\infty} \left(\int_0^t |g(s, \xi)| ds \right)^2 d\xi \leq t^2 M^2 ||\psi||^2 e^{2Mt_0},$$

and so by applying the Schwarz inequality we obtain for an arbitrary function $\varphi \in L_2(-\infty, \infty)$,

$$\int_{-\infty}^{\infty} |\varphi(\xi)| \int_0^t |g(s, \xi)| ds d\xi \leq ||\varphi|| \cdot |t| \cdot M ||\psi|| e^{Mt_0}.$$

We now apply the Fubini Theorem to interchange the order of integration

$$(10.7) \quad \begin{aligned} \int_{-\infty}^{\infty} \varphi(\xi) \int_0^t g(s, \xi) ds d\xi &= \int_0^t \int_{-\infty}^{\infty} \varphi(\xi) g(s, \xi) d\xi ds \\ &= \left(\frac{\lambda}{2\pi} \right)^{1/2} \int_0^t (t-s)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(\xi) \theta(s, u) G(s, u, \lambda) e^{-\lambda(\xi-u)^2/2(t-s)} du d\xi ds. \end{aligned}$$

We now consider the double integral:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(\xi) h(u) e^{-\Lambda(\xi-u)^2/2} du d\xi$$

where

$$h(u) = \theta(s, u) G(s, u, \lambda) \quad \text{and} \quad \Lambda = \lambda/(t-s).$$

By Lemma 1, since $||h|| < \infty$ we may observe that

$$\left| \left(\frac{\operatorname{Re} \Lambda}{2\pi} \right)^{1/2} \int_{-\infty}^{\infty} |h(u)| e^{-\operatorname{Re} \Lambda (\xi-u)^2/2} du \right| \leq ||h||,$$

hence by the Schwarz inequality

$$\left(\frac{\operatorname{Re} \Lambda}{2\pi} \right)^{1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\varphi(\xi)| |h(u)| e^{-\operatorname{Re} \Lambda (\xi-u)^2/2} du d\xi \leq ||\varphi|| \cdot ||h||.$$

Therefore by the Fubini Theorem we may conclude that

$$(10.8) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(\xi) h(u) e^{-\Lambda (\xi-u)^2/2} du d\xi = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(\xi) h(u) e^{-\Lambda (\xi-u)^2/2} d\xi du,$$

and therefore

$$(10.9) \quad \begin{aligned} & \left(\frac{\lambda}{2\pi} \right)^{1/2} \int_0^t (t-s)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(\xi) \theta(s, u) G(s, u, \lambda) e^{-\lambda (\xi-u)^2/2 (t-s)} du d\xi ds \\ &= \left(\frac{\lambda}{2\pi} \right)^{1/2} \int_0^t (t-s)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(\xi) \theta(s, u) G(s, u, \lambda) e^{-\lambda (\xi-u)^2/2 (t-s)} d\xi du ds. \end{aligned}$$

We have established that for an arbitrary function $\varphi \in L_2(-\infty, \infty)$, by (10.5), (10.7), and (10.9)

$$(10.10) \quad \begin{aligned} & \int_{-\infty}^{\infty} \varphi(\xi) \int_0^t (t-s)^{-1/2} \int_{-\infty}^{\infty} \theta(s, u) G(s, u, \lambda) e^{-\lambda (\xi-u)^2/2 (t-s)} du ds d\xi \\ &= \int_0^t (t-s)^{-1/2} \int_{-\infty}^{\infty} \theta(s, u) G(s, u, \lambda) du \int_{-\infty}^{\infty} \varphi(\xi) e^{-\lambda (\xi-u)^2/2 (t-s)} d\xi ds. \end{aligned}$$

We now consider the Hermite polynomials $H_n(\xi)$ and write

$$\begin{aligned} & \int_{-\infty}^{\infty} H_n(\xi) e^{-\xi^2/2} \int_0^t (t-s)^{-1/2} \int_{-\infty}^{\infty} \theta(s, u) G(s, u, \lambda) e^{-\lambda (\xi-u)^2/2 (t-s)} du ds d\xi \\ &= \int_0^t (t-s)^{-1/2} ds \int_{-\infty}^{\infty} \theta(s, u) G(s, u, \lambda) \Phi_n \left(u, \frac{\lambda}{t-s} \right) du, \end{aligned}$$

where

$$\Phi_n(u, \Lambda) = \int_{-\infty}^{\infty} H_n(\xi) e^{-\xi^2/2} e^{-\Lambda (\xi-u)^2/2} d\xi \quad \text{and} \quad \Lambda = \frac{\lambda}{t-s}.$$

Now we consider the Fourier Transform of Φ_n ,

$$\mathfrak{F}(\Phi_n) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \Phi_n(u, \Lambda) e^{i u \rho} du,$$

and observe that

$$\mathfrak{F}(H_n(\xi) e^{-\xi^2/2}) = (i\rho)^n e^{-\rho^2/2}.$$

Because the Fourier Transform of a convolution is the product of the Fourier Transforms we have

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} \Phi_n(u, \Lambda) e^{i u \rho} du = \mathfrak{F}(\Phi_n) = (2\pi)^{1/2} (i\rho)^n e^{-\rho^2/2} \Lambda^{-1/2} e^{-\rho^2/2 \Lambda} \\ = \left(\frac{2\pi(t-s)}{\lambda} \right)^{1/2} (i\rho)^n e^{-(\lambda+t-s)/2 \lambda} \rho^s.$$

Because the Fourier Transform is an isometric transform, we have:

$$||\Phi_n|| = ||\mathfrak{F}(\Phi_n)|| \\ = \left[2 \int_0^{\infty} \frac{2\pi(t-s)}{|\lambda|} \rho^{2n} \exp \{ -(|\lambda|^2 + (t-s) \operatorname{Re} \lambda) / |\lambda|^2 \cdot \rho^2 \} d\rho \right]^{1/2}.$$

If we let $\sigma = (|\lambda|^2 + (t-s) \operatorname{Re} \lambda) / |\lambda|^2 \rho^2$ and $\rho = |\lambda| \sigma^{1/2} (|\lambda|^2 + (t-s) \operatorname{Re} \lambda)^{-1/2}$ then

$$||\Phi_n|| = \left[\frac{2\pi(t-s)}{|\lambda| (|\lambda|^2 + (t-s) \operatorname{Re} \lambda)^{n+1/2}} \int_0^{\infty} c^{n-1/2} e^{-\sigma} d c \right]^{1/2},$$

and using the Gamma function we have

$$||\Phi_n|| = \left[\frac{2\pi(t-s)}{(|\lambda|^2 + (t-s) \operatorname{Re} \lambda)^{n+1/2}} \frac{|\lambda|^{2n}}{\Gamma(n + \frac{1}{2})} \right]^{1/2}.$$

We shall later perform a similar analysis for differences of Φ_n 's.

We shall undertake to establish both the pointwise limit and the limit in the mean for Φ_n . We begin with the pointwise limit. For a fixed value of u ,

$$\lim_{\substack{\Lambda \rightarrow -i/(t-s) \\ \operatorname{Re} \Lambda > 0}} \int_{-\infty}^{\infty} H_n(\xi) e^{-\xi^2/2} e^{-\Lambda (\xi-u)^2/2} d\xi = \int_{-\infty}^{\infty} H_n(\xi) e^{-\xi^2/2} e^{i(\xi-u)^2/2(t-s)} d\xi.$$

This follows from Lebesgue's convergence theorem because $|e^{-\Lambda (\xi-u)^2/2}| = e^{-\operatorname{Re} \Lambda (\xi-u)^2/2} \leq 1$ if $\operatorname{Re} \Lambda > 0$, and we have

$$|H_n(\xi) e^{-\xi^2/2} e^{-\Lambda (\xi-u)^2/2}| \leq |H_n(\xi)| e^{-\xi^2/2}.$$

Thus we have the pointwise limit for $\operatorname{Re} \Lambda > 0$ and for each fixed u ,

$$\lim_{\Lambda \rightarrow -i/(t-s)} \Phi_n(u, \Lambda) = \Phi_n\left(u, \frac{-iq}{t-s}\right).$$

If we now apply Lemma 1 to the function Φ_n ,

$$\Phi_n(u, \Lambda) = \int_{-\infty}^{\infty} H_n(\xi) e^{-\xi^2/2} e^{-\Lambda (\xi-u)^2/2} d\xi,$$

we obtain the inequality

$$||\Phi_n(\cdot, \Lambda)|| \leq \frac{(2\pi)^{1/2}}{|\Lambda|} ||H_n(\cdot) e^{-\cdot^2/2}||.$$

In order to establish the limit in the mean for the Φ_n 's we observe the general property; if a sequence of functions f_n possesses the property that $\text{l.i.m.}_{n \rightarrow \infty} f_n(x) = g(x)$ and if $\lim_{n \rightarrow \infty} f_n(x) = h(x)$ almost everywhere. Then $g(x) = h(x)$ almost everywhere.

Now, if λ_1, λ_2 are distinct complex numbers with positive real parts,

$$\begin{aligned} \left\| \Phi_n\left(\cdot, \frac{\lambda_1}{t-s}\right) - \Phi_n\left(\cdot, \frac{\lambda_2}{t-s}\right) \right\|^2 &= \left\| \mathfrak{F}\Phi_n\left(\cdot, \frac{\lambda_1}{t-s}\right) - \mathfrak{F}\Phi_n\left(\cdot, \frac{\lambda_2}{t-s}\right) \right\|^2 \\ &= 2\pi(t-s) \int_{-\infty}^{\infty} \left[\frac{(i\rho)^n e^{-((\lambda_1+t-s)/2\lambda_1)\rho^2}}{(\lambda_1)^{1/2}} - \frac{(i\rho)^n e^{-((\lambda_2+t-s)/2\lambda_2)\rho^2}}{(\lambda_2)^{1/2}} \right] \\ &\quad \cdot \left[\frac{(-i\rho)^n e^{-((\bar{\lambda}_1+t-s)/2\bar{\lambda}_1)\rho^2}}{(\lambda_1)^{1/2}} - \frac{(i\rho)^n e^{-((\bar{\lambda}_2+t-s)/2\bar{\lambda}_2)\rho^2}}{(\lambda_2)^{1/2}} \right] d\rho. \end{aligned}$$

Because the integrand tends to zero as $\lambda_1, \lambda_2 \rightarrow -iq$ while their real parts remain positive, we have established that the integral tends to 0 as $\lambda_1, \lambda_2 \rightarrow -iq$, $\text{Re } \lambda > 0$, since its integrand is dominated by a function integrable with respect to ρ and independent of λ_1 and λ_2 , namely, $8e^{-\rho^2/2} \rho^{2n}/|q|$ for $|\lambda_1| > \frac{1}{2}|q|$, $|\lambda_2| > \frac{1}{2}|q|$. Thus by dominated convergence we have established that

$$\left\| \Phi_n\left(\cdot, \frac{\lambda_1}{t-s}\right) - \Phi_n\left(\cdot, \frac{\lambda_2}{t-s}\right) \right\| \rightarrow 0$$

as $\lambda_1, \lambda_2 \rightarrow -iq$, $\text{Re } \lambda > 0$ and by the general property indicated above,

$$\text{l.i.m.}_{\substack{\lambda \rightarrow -iq \\ \text{Re } \lambda > 0}} \Phi_n\left(u, \frac{\lambda}{t-s}\right) = \Phi_n\left(u, \frac{-iq}{t-s}\right).$$

Let us consider the function

$$I_n(\lambda) \equiv \int_0^t (t-s)^{-1/2} ds \int_{-\infty}^{\infty} \theta(s, u) G(s, u, \lambda) \Phi_n\left(u, \frac{\lambda}{t-s}\right) du$$

and let

$$J_n(q) \equiv \int_0^t (t-s)^{-1/2} ds \int_{-\infty}^{\infty} \theta(s, u) \Gamma(s, u, q) \Phi_n\left(u, \frac{-iq}{t-s}\right) du$$

and the difference

$$\begin{aligned} I_n(\lambda) - J_n(q) &= \int_0^t (t-s)^{-1/2} ds \int_{-\infty}^{\infty} \theta(s, u) G(s, u, \lambda) \left[\Phi_n\left(u, \frac{\lambda}{t-s}\right) - \Phi_n\left(u, \frac{-iq}{t-s}\right) \right] du \\ &\quad + \int_0^t (t-s)^{-1/2} ds \int_{-\infty}^{\infty} \theta(s, u) \Phi_n\left(u, \frac{-iq}{t-s}\right) [G(s, u, \lambda) - \Gamma(s, u, q)] du \\ &\equiv \mathcal{G}_1 + \mathcal{G}_2. \end{aligned}$$

We established that $G(s, u, \lambda)$ tends to $\Gamma(s, u, q)$ weakly as $\lambda \rightarrow -iq^+$ for each

fixed s and almost every q , and since $\theta(s, u)$ is bounded and $\Phi_n(u, -iq/(t-s)) \in L_2$ we have for fixed s that

$$\int_{-\infty}^{\infty} \theta(s, u) \Phi_n\left(u, \frac{-iq}{t-s}\right) [G(s, u, \lambda) - \Gamma(s, u, q)] du \rightarrow 0$$

as $\lambda \rightarrow -iq^+$. Moreover:

$$\begin{aligned} \left| \int_{-\infty}^{\infty} \theta(s, u) \Phi_n\left(u, \frac{-iq}{t-s}\right) [G(s, u, \lambda) - G(s, u, -iq)] du \right| \\ \leq 2M \left\| \Phi_n\left(u, \frac{-iq}{t-s}\right) \right\| \|\psi\| e^{M\epsilon}. \end{aligned}$$

and since

$$\|\Phi_n(\cdot, \Lambda)\| \leq \left(\frac{2\pi}{|\Lambda|}\right)^{1/2} \|H_n(\cdot)e^{-\langle \cdot \rangle^2}\|,$$

we have

$$\begin{aligned} \left| \int_{-\infty}^{\infty} \theta(s, u) \Phi_n\left(u, \frac{-iq}{t-s}\right) [G(s, u, \lambda) - \Gamma(s, u, q)] du \right| \\ \leq 2M \left(\frac{2\pi(t-s)}{|q|}\right)^{1/2} \|H_n(\cdot)e^{-\langle \cdot \rangle^2}\| \|\psi\| e^{M\epsilon}. \end{aligned}$$

and by bounded convergence,

$$\lim_{\substack{\lambda \rightarrow -iq \\ \operatorname{Re} \lambda > 0}} g_2 = 0.$$

The integral

$$g_1 = \int_0^t (t-s)^{-1/2} ds \int_{-\infty}^{\infty} \theta(s, u) G(s, u, \lambda) \left[\Phi_n\left(u, \frac{\lambda}{t-s}\right) - \Phi_n\left(u, \frac{-iq}{t-s}\right) \right] du,$$

and by an application of the Schwarz inequality we have

$$\begin{aligned} |g_1| &\leq \int_0^t (t-s)^{-1/2} M \|G(s, \cdot, \lambda)\| \left\| \Phi_n\left(\cdot, \frac{\lambda}{t-s}\right) - \Phi_n\left(\cdot, \frac{-iq}{t-s}\right) \right\| ds \\ &\leq M e^{M\epsilon} \int_0^t \|\psi\| \cdot \left\| \Phi_n\left(\cdot, \frac{\lambda}{t-s}\right) - \Phi_n\left(\cdot, \frac{-iq}{t-s}\right) \right\| (t-s)^{-1/2} ds. \end{aligned}$$

We have already established that for each fixed s , the above integrand tends to zero as $\lambda \rightarrow -iq^+$. We also have

$$\begin{aligned} \left\| \Phi_n\left(\cdot, \frac{\lambda}{t-s}\right) - \Phi_n\left(\cdot, \frac{-iq}{t-s}\right) \right\| \\ \leq \left(\frac{2\pi(t-s)}{|\lambda|}\right)^{1/2} \|H_n(\cdot)e^{-\langle \cdot \rangle^2}\| + \left(\frac{2\pi(t-s)}{|q|}\right)^{1/2} \|H_n(\cdot)e^{-\langle \cdot \rangle^2}\| \end{aligned}$$

where the estimate for $\Phi_n(\cdot, -iq/(t-s))$ is obtained by taking the limit of the estimate of $\Phi_n(\cdot, \lambda/(t-s))$ as $\lambda \rightarrow -iq$. This is justified by the fact that the second Φ_n is the limit in the mean of the first Φ_n . Since we may assume that $|\lambda| \geq |q|/2$, $|\mathcal{I}_1|$ is bounded, and we have $\mathcal{I}_1 \rightarrow 0$. Consequently $I_n(\lambda) - J_n(q) \rightarrow 0$ as $\lambda \rightarrow -iq^+$, *i.e.*

$$\int_{-\infty}^{\infty} H_n(\xi) e^{-\xi^2/2} \int_0^t (t-s)^{-1/2} \int_{-\infty}^{\infty} \theta(s, u) G(s, u, \lambda) e^{-\lambda(\xi-u)^2/2(t-s)} du ds d\xi \rightarrow J_n(q)$$

as $\lambda \rightarrow -iq^+$. By applying Lemma 10 to the inner two integrals of

$$J_n(q) \equiv \int_0^t (t-s)^{-1/2} \int_{-\infty}^{\infty} \theta(s, u) \Gamma(s, u, q) \int_{-\infty}^{\infty} H_n(\xi) e^{-\xi^2/2} e^{-iq(\xi-u)^2/2(t-s)} d\xi du ds$$

and using $\varphi_n(\xi) = H_n(\xi) e^{-\xi^2/2}$ and $f(u) = \theta(s, u) \Gamma(s, u, q)$ we have

$$\begin{aligned} J_n(q) &= \int_0^t (t-s)^{-1/2} \int_{-\infty}^{\infty} \varphi_n(\xi) \int_{-\infty}^{\infty} \theta(s, u) \Gamma(s, u, q) e^{-iq(\xi-u)^2/2(t-s)} du d\xi ds \\ &= \left(\frac{2\pi}{-iq} \right)^{1/2} \int_0^t \int_{-\infty}^{\infty} \varphi_n(\xi) g(s, \xi, q) d\xi ds \end{aligned}$$

where

$$g(s, \xi, q) = \left(\frac{-iq}{2\pi(t-s)} \right)^{1/2} \int_{-\infty}^{\infty} \theta(s, u) \Gamma(s, u, q) e^{-iq(\xi-u)^2/2(t-s)} du$$

and (as in equation 10.6)

$$||g(s, \cdot)|| \leq M ||\psi|| e^{Mt_0}.$$

Thus by the Fubini Theorem

$$\begin{aligned} J_n(q) &= \left(\frac{2\pi}{-iq} \right)^{1/2} \int_{-\infty}^{\infty} \int_0^t \varphi_n(\xi) g(s, \xi, q) ds d\xi \\ &= \int_{-\infty}^{\infty} \varphi_n(\xi) \int_0^t (t-s)^{-1/2} \int_{-\infty}^{\infty} \theta(s, u) \Gamma(s, u, q) e^{-iq(\xi-u)^2/2(t-s)} du ds d\xi. \end{aligned}$$

We now turn our attention to the first integral on the right hand side of the equation. By dominated convergence, since $\varphi \in L_1$,

$$\lim_{\lambda \rightarrow -iq^+} \int_{-\infty}^{\infty} \varphi_n(\xi) e^{-\lambda(\xi-u)^2/2t} d\xi = \int_{-\infty}^{\infty} \varphi_n(\xi) e^{iq(\xi-u)^2/2t} d\xi$$

for all fixed u . By Lemma 1, we see that

$$\left\| \int_{-\infty}^{\infty} \varphi_n(\xi) e^{-\lambda(\xi-u)^2/2t} d\xi \right\| \leq c ||\varphi_n||.$$

Hence the pointwise limit can be replaced by the weak limit

$$\begin{aligned} \lim_{\lambda \rightarrow -iq^+} \left(\frac{\lambda}{2\pi t} \right)^{1/2} \int_{-\infty}^{\infty} \psi(u) \int_{-\infty}^{\infty} \varphi_n(\xi) e^{-\lambda(\xi-u)^2/2t} d\xi du \\ = \left(\frac{-iq}{2\pi t} \right)^{1/2} \int_{-\infty}^{\infty} \psi(u) \int_{-\infty}^{\infty} \varphi_n(\xi) e^{iq(\xi-u)^2/2t} d\xi du. \end{aligned}$$

By Lemma 10, since $\varphi \in L_1 \cap L_2$, we obtain

$$\begin{aligned} \lim_{\lambda \rightarrow -iq} \left(\frac{\lambda}{2\pi t} \right)^{1/2} \int_{-\infty}^{\infty} \varphi_n(\xi) \int_{-\infty}^{\infty} \psi(u) e^{-\lambda(\xi-u)^2/2t} du d\xi \\ = \left(\frac{q}{2\pi it} \right)^{1/2} \int_{-\infty}^{\infty} \varphi_n(\xi) \int_{-\infty}^{\infty} \psi(u) e^{iq(\xi-u)^2/2t} du d\xi. \end{aligned}$$

Let us multiply the following integral equation of Theorem 8, valid for $\text{Re } \lambda > 0$

$$\begin{aligned} G(t, \xi, \lambda) = \left(\frac{\lambda}{2\pi t} \right)^{1/2} \int_{-\infty}^{\infty} \psi(u) e^{-\lambda(\xi-u)^2/2t} du \\ + \left(\frac{\lambda}{2\pi} \right)^{1/2} \int_0^t (t-s)^{-1/2} ds \int_{-\infty}^{\infty} \theta(s, u) G(s, u, \lambda) e^{-\lambda(\xi-u)^2/2(t-s)} du, \end{aligned}$$

by $\varphi_n(\xi) = H_n(\xi)e^{-\xi^2/2}$ and integrate both sides with respect to ξ from $-\infty$ to ∞ and pass to the limit as $\lambda \rightarrow -iq$, $\text{Re } \lambda > 0$. We have

$$\begin{aligned} \lim_{\lambda \rightarrow -iq^+} \int_{-\infty}^{\infty} \varphi_n(\xi) G(t, \xi, \lambda) d\xi = \lim_{\lambda \rightarrow -iq^+} \left(\frac{\lambda}{2\pi t} \right)^{1/2} \int_{-\infty}^{\infty} \varphi_n(\xi) \int_{-\infty}^{\infty} \psi(u) e^{-\lambda(\xi-u)^2/2t} du d\xi \\ + \lim_{\lambda \rightarrow -iq^+} \left(\frac{\lambda}{2\pi} \right)^{1/2} \int_{-\infty}^{\infty} \varphi_n(\xi) \int_0^t (t-s)^{-1/2} \\ \cdot \int_{-\infty}^{\infty} \theta(s, u) G(s, u, \lambda) e^{-\lambda(\xi-u)^2/2(t-s)} du ds d\xi \end{aligned}$$

or

$$\begin{aligned} \int_{-\infty}^{\infty} \varphi_n(\xi) \Gamma(t, \xi, q) d\xi = \left(\frac{q}{2\pi it} \right)^{1/2} \int_{-\infty}^{\infty} \varphi_n(\xi) \int_{-\infty}^{\infty} \psi(u) e^{iq(\xi-u)^2/2t} du d\xi \\ + \left(\frac{q}{2\pi i} \right)^{1/2} \int_{-\infty}^{\infty} \varphi_n(\xi) \int_0^t (t-s)^{-1/2} \int_{-\infty}^{\infty} \theta(s, u) \Gamma(s, u, q) e^{iq(\xi-u)^2/2(t-s)} du ds d\xi. \end{aligned}$$

But the finite linear combinations of the functions $\varphi_n(\xi) = H_n(\xi)e^{-\xi^2/2}$ are dense in L_2 , and consequently the multipliers of $\varphi_n(\xi)$ on the two sides are equal almost everywhere. Thus Theorem 9 is proved.

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