Singular Integral Equations with Shifts

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The techniques and results of [7] and [8] will be used here to discuss a large class of singular integral equations with shifts on compact Riemann surfaces. As will be seen, accurate information about the solution spaces of all but the simplest equations is obtained for the first time.

I will take for granted Sections 1 and 2 of [8], but the Plemelj formulas as stated there must be corrected. First of all, for $n = 2$ define $\sigma_\nu(x)$ to be the additive inverse of the $\sigma_\nu(x)$ defined in [8]. If x_2^* is of order 2, define the numbers $\alpha_1(x_2^*)$, $\alpha_2(x_2^*)$ as follows: let $\alpha_\mu(x_2^*)$ be the angle θ_μ (in radians) between the tangent line to $\mathcal{C}_{1,x_2^*}$ and $\mathcal{C}_{2,x_2^*}$ at x_2^* as viewed from "inside" $U_{\mu+1}$ if $\sigma_\mu(x_2^*) = 1$ and $2\pi - \theta_\mu$ otherwise (see Figure 1).

Now suppose that order $x_2^* > 2$. For $\nu \neq \mu$ (respectively $\nu = \mu$), define $\alpha_\mu(x_2^*, \nu)$ to be the positive angle between $\sigma_\mu(x_2^*) \cdot t_\mu(x_2^*)$ and $\sigma_\nu(x_2^*) \cdot t_\nu(x_2^*)$ as viewed from inside $U_{\mu+1,x_2^*} \cup \dots \cup U_{n,x_2^*} \cup U_{1,x_2^*} \cup \dots \cup U_{\nu,x_2^*}$ (respectively $= 0$), where $t_\mu(x_2^*)$ is the oriented tangent to $\mathcal{C}_{\mu,x_2^*}$ at x_2^* (see Figure 2). Form $\bar{X}$ relative to S and define as in [8] for $\bar{x} \in \bar{X}_\beta$, $\sigma(\bar{x}) = \sigma_\nu(\text{pr } \bar{x})$ and $\tilde{\alpha}(\bar{x}) = \alpha_\nu(\text{pr } \bar{x})$ provided that $\text{pr } \bar{U}_x^- \subseteq \bar{U}_{\nu, \text{pr } \bar{x}}^-$ and $\alpha(\bar{x}, \bar{y}) = \alpha_\mu(\text{pr } \bar{x}, \nu)$ provided that $\text{pr } \bar{y} = \text{pr } \bar{x}$, $\text{pr } \bar{U}_x^- \subseteq \bar{U}_{\nu, \text{pr } \bar{x}}^-$ and $\text{pr } \bar{U}_y^- \subseteq \bar{U}_{\nu, \text{pr } \bar{y}}^-$. The local Plemelj formulas are then (see [6], Appendix 2)

$$(\Omega_{\mu+1}f)(x_2) - (\Omega_\mu f)(x_2) = \sigma_\mu(x_2)f(x_2)$$

$$(\Omega_{\mu+1}f)(x_2) + (\Omega_\mu f)(x_2) = \sigma_\mu(x_2)(1 - \pi^{-1}\alpha_\mu(x_2))f(x_2) + (\pi i)^{-1} \int_{(S, \Sigma)} f(x_1)\Omega(x_1, x_2).$$

Now by computing essentially as in [8], Theorem (2.2) of that paper becomes

(1) Theorem (Plemelj Formulas). *Assuming that the hypothesis of [8], Proposition (2.1) is fulfilled, the function $\bar{\Omega}f$ can be extended continuously to $\bar{X}$. Moreover, for every point $\bar{x} \in \bar{X}_\beta$,*

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FIG. 1

$$(\bar{\Omega}f)(\beta\bar{x}) - (\bar{\Omega}f)(\bar{x}) = \sigma(\bar{x})f(\text{pr } \bar{x})$$

$$(\bar{\Omega}f)(\beta\bar{x}) + (\bar{\Omega}f)(\bar{x}) = -\sigma(\bar{x})\alpha(\bar{x})f(\text{pr } \bar{x}) + (\pi i)^{-1} \int_{(S, \Sigma)} f(x_1) \Omega(x_1, \text{pr } \bar{x})$$

where

$$\alpha(\bar{x}) = 1 + \pi^{-1} \Sigma \{ \sigma(\bar{x}) \sigma(\bar{y}) \alpha(\bar{x}, \bar{y}) : \bar{y} \in \bar{X}_\beta, \text{pr } \bar{y} = \text{pr } \bar{x} \}.$$

Since $\alpha(\bar{x}) = 0$ whenever $\text{pr } \bar{x}$ is not a vertex of the minimal simplicial structure (see [8], Proposition (1.4)), these formulas agree with those in [8] except at those vertices. Now it is not difficult to show that for every such vertex x , $\alpha(\bar{x})f(\bar{x}) = 0$ for all $\bar{x} \in \text{pr}^{-1}(x)$ if and only if $f(x) = 0$. Consequently the formulas as given in [8], Theorem (2.2) are correct except at those vertices x of the minimal simplicial structure on S where $f(x) \neq 0$.

FIG. 2

(2) Corollary. *Under the same assumptions as in Theorem (1), the Plemelj formulas have the form*

$$(\bar{\Omega}f)(\beta\bar{x}) - (\bar{\Omega}f)(\bar{x}) = \sigma(\bar{x})f(\text{pr } \bar{x})$$

$$(\bar{\Omega}f)(\beta\bar{x}) + (\bar{\Omega}f)(\bar{x}) = (\pi i)^{-1} \int_{(S, \Sigma)} f(x_1) \Omega(x_1, \text{pr } \bar{x})$$

if and only if $f(x) = 0$ at every vertex x of the minimal simplicial structure on S .

For the minimal simplicial structure $\{\mathcal{C}^{(\rho)} : \rho = 1, \dots, k\}$ on S and the star of smooth arcs $\mathcal{C}_{1, x_2^*}, \dots, \mathcal{C}_{n, x_2^*}$, define $\rho(x_2^*, \nu)$ to be that unique ρ for which $\mathcal{C}^{(\rho)} \cap \{\mathcal{C}_{\nu, x_2^*} \setminus \{x_2^*\}\} \neq \emptyset$. One has the following generalization of [8], Proposition (2.1).

(3) Proposition. *Let X be a compact Riemann surface, Ω the Cauchy kernel of X associated with D and $*$, S a smooth 1-complex on X which satisfies $S \cap \{*\} \cup \text{supp } D = \emptyset$, and Σ a coherent set of signatures on S . Let f_ρ be a Hölder continuous function on $\mathcal{C}^{(\rho)}$ for each $\rho = 1, \dots, k$ such that*

$$\sum_{\nu=1}^{n(x)} \sigma_x(\nu) f_{\rho(\nu, x)}(x) = 0 \quad \text{for every } x \in S.$$

Then the singular integral

$$\sum_{\rho=1}^k (2\pi i)^{-1} \int_{\mathcal{C}^{(\rho)}} f_\rho(x_1) \Omega(x_1, x_2)$$

defined as in [8], (1') exists for every $x_2 \in S$.

Denote the integral

$$\sum_{\rho=1}^k (2\pi i)^{-1} \int_{\mathcal{C}^{(\rho)}} f_\rho(x_1) \Omega(x_1, x_2), \quad x_2 \in X \setminus S,$$

by $(\Omega f)(x_2)$ and for $\bar{x} \in \bar{X}_\beta$, let $\rho(\bar{x}) = \rho(\nu, \text{pr } x)$ provided that $\bar{U}_x^- \subseteq \bar{U}_{\nu, \text{pr } \bar{x}}$. A straightforward calculation yields

(4) Theorem. *Assuming that the hypothesis of Proposition (3) is fulfilled, the function $\bar{\Omega}f$ defined as in Theorem (1) can be extended continuously to $\bar{X}$. Moreover, for every point $\bar{x} \in \bar{X}_\beta$,*

$$(\bar{\Omega}f)(\beta\bar{x}) - (\bar{\Omega}f)(\bar{x}) = \sigma(\bar{x})f_{\rho(\bar{x})}(\text{pr } \bar{x}),$$

$$(\bar{\Omega}f)(\beta\bar{x}) + (\bar{\Omega}f)(\bar{x}) = -\sigma(\bar{x})f_{\rho(\bar{x})}(\text{pr } \bar{x}) -$$

$$\Sigma\{\sigma(\bar{y})\pi^{-1}\alpha(\bar{x}, \bar{y})f_{\rho(\bar{y})}(\text{pr } \bar{y}) : \text{pr } \bar{y} = \text{pr } \bar{x}\} + \sum_{\rho=1}^k (\pi i)^{-1} \int_{\mathcal{C}^{(\rho)}} f_\rho(x_1) \Omega(x_1, \text{pr } \bar{x}).$$

The advantage of having these generalized formulas is that the f_ρ may be chosen so that all $f_\rho(x)$ are not zero at a vertex $x \in S$, but the Plemelj formulas above have the form of those in Corollary (2) at all $\bar{x} \in \text{pr}^{-1}(x)$.

§1. Ω -degenerate singular integral equations with shifts. Let X be a compact Riemann surface and S a smooth 1-complex on X ([7]). Let $\{\alpha^\nu : \nu = 0, \dots, p-1\}$ be a cyclic group of homeomorphisms of S onto itself under composition of maps; the elements are indexed so that $\alpha^\nu = \alpha^1 \circ \dots \circ \alpha^1$ ν -times. We assume that α^1 preserves the minimal simplicial structure on S ([7], [8]), i.e. maps simplices onto simplices and vertices onto vertices. Let Ω be the Cauchy kernel on X associated with the normalization divisor D and $*$, where $S \cap \{D \cup \{*\}\} = \emptyset$, and Σ a coherent set of signatures on S . Given Hölder continuous $q \times q$ matrices A^ν on S and K^ν on $S \times S$, and a Hölder continuous $q \times 1$ matrix g on S , find Hölder continuous $q \times 1$ matrices f on S satisfying

$$(1.1) \quad \sum_{\nu=0}^{p-1} A^\nu(\alpha^\nu(x_1))f(\alpha^\nu(x_1)) + \frac{1}{\pi i} \int_{(S, \Sigma)} K^\nu(\alpha^\nu(x_1), x_2)f(x_2)\Omega(x_2, \alpha^\nu(x_1)) = g(x_1)$$

for all $x_1 \in S$; (1.1) is called a *singular integral equation with a shift of period p* .

The integer q is called the *rank* of (1.1). The kernel $K^\nu(x_1, x_2)$ is called *degenerate* if there are Hölder continuous $q \times q$ matrices $B'_1, C'_1, \dots, B'_{n_\nu}, C'_{n_\nu}$ on S such that

$$(1.2) \quad K^\nu(x_1, x_2) = \sum_{\mu=1}^{n_\nu} B'_\mu(x_1)C'_\mu(x_2), \quad x_1, x_2 \in S.$$

The integer n_ν is called the *height* of the kernel K^ν in the representation (1.2). (1.1) is called an *Ω -degenerate singular integral equation with a shift of period p* if each kernel K^ν is degenerate. It is clearly no loss in generality to assume that $n = n_0 = n_1 = \dots = n_{p-1}$.

Because of [8], Proposition (2.1) and Corollary (2) the integral equation (1.1) is in general subjected to the requirements

1) $K^\nu(x_1, x_2) = 0$ at each vertex x_2 of the minimal simplicial structure on S for all $\nu = 0, 1, \dots, p-1$.

If (1.1) is an Ω -degenerate equation, 1) can be replaced with

1)' $C'_\mu(x_2) = 0$ at each vertex x_2 of the minimal simplicial structure on S for all $\nu = 0, 1, \dots, p-1$ and $\mu = 1, \dots, n$.

An *admissible simplicial structure on S relative to the Ω -degenerate integral equation* (1.1) is one which refines the minimal simplicial structure on S and satisfies

2) for every vertex x , either $C'_\mu \circ \alpha^\rho(x) = 0$ for all $\mu = 1, \dots, n$ and all $\nu, \rho = 0, \dots, p-1$ or $B'_\mu \circ \alpha^\rho(x) = 0$ for all $\mu = 1, \dots, n$ and all $\nu, \rho = 0, \dots, p-1$.

Any simplicial structure $\{\sigma_\lambda^1 : \lambda \in \Lambda\}$ considered here is assumed to be admissible relative to (1.1). In view of this hypothesis and the fact that α^ρ is a homeomorphism of S onto itself, it may be assumed that whenever $x \in S$ is a vertex so is $\alpha^\rho(x)$ for all $\rho = 1, \dots, p-1$.

These requirements enable one to apply the Plemelj formulas as given in Corollary (2) to an Ω -degenerate singular integral equation with a shift of period p in order to associate with it a Riemann–Privalov problem or a boundary value problem of the type studied in [7], whichever may be appropriate. The more general Plemelj formulas given in Theorem (4) can also be used for this purpose, but in order to keep the notation as simple as possible we will use only those from Corollary (2). An example of the application of the general Plemelj formulas is given at the end of this section.

In order that the transmission function of the associated transmission problem mentioned above takes values in $GA(m, \mathbf{C})$ for some positive integer m , (1.1) is also subjected to what is known as regularity conditions. Unlike the case of Ω -degenerate singular integral equations without shifts ((1.1) with $p = 1$) there is no one natural transmission problem to associate with (1.1). Therefore the regularity conditions must be stated relative to the choice of this transmission problem. These conditions will be stated at the appropriate place in this section.

That many transmission problems can arise is due to the fact that for each σ_λ^1 on $\bar{X}_\beta$, α^ρ induces two maps $\alpha_{\lambda'}^\rho : \sigma_\lambda^1 \rightarrow \sigma_{\lambda'}^1$, and $\alpha_{\lambda''}^\rho : \sigma_\lambda^1 \rightarrow \sigma_{\lambda''}^1$, where $\text{pr } \sigma_{\lambda'}^1 = \text{pr } \sigma_{\lambda''}^1 = \alpha^\rho(\text{pr } \sigma_\lambda^1)$. If for each $\lambda \in \Lambda$ one of these two maps is chosen (denote it by α_λ^ρ) so that no two maps $\alpha_{\lambda_1}^\rho$, $\alpha_{\lambda_2}^\rho$ have the same 1-simplex for their ranges, then a transmission problem can be defined, under specific conditions, relative to the system $\{\alpha_\lambda^\rho : \lambda \in \Lambda\}$. The advantage of having several choices for the transmission problem associated with (1.1) will become apparent later in this section.

The unit (resp. zero) $m \times m$ matrix will be denoted by 1 (resp. 0), or when the dimension m of the matrix is not obvious from the context, 1_m (resp. 0_m).

In order to associate with an Ω -degenerate singular integral equation with a shift of period p a holomorphic affine bundle, the techniques of [8] are used to transform the integral equation into a boundary value problem of the type studied in [7] or [8]. For that purpose, choose a set of maps $\{\alpha_\lambda^\rho : 0 \leq \rho \leq p-1, \lambda \in \Lambda\}$ as defined above, assume that f is a solution of the integral equation and form for every pair μ, ν ($\mu = 1, \dots, n; \nu = 0, \dots, p-1$)

$$(1.3) \quad (\Omega F_\mu^\nu)(x_2) = (2\pi i)^{-1} \int_{(s, z)} C_\mu^\nu(x_1) f(x_1) \Omega(x_1, x_2), \quad x_2 \in X \setminus S.$$

ΩF_μ^ν is a meromorphic $q \times 1$ matrix on $X \setminus S$ and has divisor (i.e., each entry of ΩF_μ^ν has divisor) $\geq * - D$. Furthermore, hypothesis 1)' implies that Corollary (2) is applicable, and so for $\bar{x} \in \bar{X}_\beta$ one has

$$(1.4) \quad (\bar{\Omega} F_\mu^\nu)(\beta \bar{x}) - (\bar{\Omega} F_\mu^\nu)(\bar{x}) = \sigma(\bar{x}) C_\mu^\nu(\text{pr } \bar{x}) f(\text{pr } \bar{x})$$

$$(\bar{\Omega} F_\mu^\nu)(\beta \bar{x}) + (\bar{\Omega} F_\mu^\nu)(\bar{x}) = (\pi i)^{-1} \int_{(s, z)} C_\mu^\nu(x_2) f(x_2) \Omega(x_2, \text{pr } \bar{x}).$$

The integral equation (1.1) can be rewritten as

(1.5)₀

$$\sum_{\nu=0}^{p-1} \left\{ \left\{ A^{\nu}(\alpha^{\nu}(\text{pr } \bar{x})) + \sum_{\mu=1}^n B_{\mu}^{\nu}(\alpha^{\nu}(\text{pr } \bar{x})) C_{\mu}^{\nu}(\alpha^{\nu}(\text{pr } \bar{x})) \right\} f(\alpha^{\nu}(\text{pr } \bar{x})) + \sum_{\mu=1}^n B_{\mu}^{\nu}(\alpha^{\nu}(\text{pr } \bar{x})) \right. \\ \left. \cdot \left\{ -C_{\mu}^{\nu}(\alpha^{\nu}(\text{pr } \bar{x})) f(\alpha^{\nu}(\text{pr } \bar{x})) + (\pi i)^{-1} \int_{(S, \Sigma)} C_{\mu}^{\nu}(x_2) f(x_2) \Omega(x_2, \alpha^{\nu}(\text{pr } \bar{x})) \right\} \right\} = g(\text{pr } \bar{x})$$

or equivalently, putting $K_+^{\nu}(x) = A^{\nu}(x) + \sum_{\mu=1}^n B_{\mu}^{\nu}(x) C_{\mu}^{\nu}(x)$ and substituting the Plemelj formulas (1.4) into (1.5)₀,

$$(1.6)_0 \quad \sum_{\nu=0}^{p-1} \left\{ K_+^{\nu}(\alpha^{\nu}(\text{pr } \bar{x})) f(\alpha^{\nu}(\text{pr } \bar{x})) + \sum_{\mu=1}^n B_{\mu}^{\nu}(\alpha^{\nu}(\text{pr } \bar{x})) \right. \\ \left. \cdot \{ (1 - \sigma(\alpha_{\lambda}^{\nu}(\bar{x}))) (\bar{\Omega} F_{\mu}^{\nu})(\beta \alpha_{\lambda}^{\nu}(\bar{x})) + (1 + \sigma(\alpha_{\lambda}^{\nu}(\bar{x}))) (\bar{\Omega} F_{\mu}^{\nu})(\alpha_{\lambda}^{\nu}(\bar{x})) \} \right\} = g(\text{pr } \bar{x}).$$

Thus for every $\rho = 0, 1, \dots, p-1$

$$(1.6)_{\rho} \quad \sum_{\nu=0}^{p-1} \left\{ K_+^{\nu}(\alpha^{\nu+\rho}(\text{pr } \bar{x})) f(\alpha^{\nu+\rho}(\text{pr } \bar{x})) + \sum_{\mu=1}^n B_{\mu}^{\nu}(\alpha^{\nu+\rho}(\text{pr } \bar{x})) \right. \\ \left. \cdot \{ (1 - \sigma(\alpha_{\lambda}^{\nu+\rho}(\bar{x}))) (\bar{\Omega} F_{\mu}^{\nu})(\beta \alpha_{\lambda}^{\nu+\rho}(\bar{x})) \right. \\ \left. + (1 + \sigma(\alpha_{\lambda}^{\nu+\rho}(\bar{x}))) (\bar{\Omega} F_{\mu}^{\nu})(\alpha_{\lambda}^{\nu+\rho}(\bar{x})) \} \right\} = g(\alpha^{\rho}(\text{pr } \bar{x})).$$

Introduce the following notation.

a) $K_+(x)$ is the $pq \times pq$ matrix on S given by

$$(K_+(x))_{qi+\nu, qi+\mu} = (K_+^{p-i+j}(\alpha^i(x)))_{\nu, \mu},$$

where $0 \leq i, j \leq p-1$ and $1 \leq \nu, \mu \leq q$; it is understood that $p-i+j$ is to be taken modulo p . This will be true throughout this paper for any upper indices which appear.

b) For $1 \leq \mu_0 \leq n$ and $0 \leq \rho \leq p-1$, define $\tilde{B}_{\mu_0}^{\rho}(x)$ to be the $q \times pq$ matrix on S given by

$$(\tilde{B}_{\mu_0}^{\rho}(x))_{\nu, \rho i+\mu} = (B_{\mu_0}^i(\alpha^{\rho+i}(x)))_{\nu, \mu}$$

and $\tilde{B}_{\mu_0}^{\rho}(x)$ to be the $pq \times pq$ matrix on S given by

$$(\tilde{B}_{\mu_0}^{\rho}(x))_{\rho i+\nu, \rho i+\mu} = \begin{cases} 0, & i \neq \rho \\ (\tilde{B}_{\mu_0}^{\rho}(x))_{\nu, \rho i+\mu}, & i = \rho \end{cases}$$

where $0 \leq i, j \leq p-1$ and $1 \leq \mu, \nu \leq q$. Moreover, let $B(x)$ be the $nq \times np^2q$ matrix

$$B(x) = (\tilde{B}_1^0(x), \dots, \tilde{B}_n^0(x), \tilde{B}_1^1(x), \dots, \tilde{B}_n^1(x), \dots, \tilde{B}_n^{p-1}(x))$$

c) Denote by P the $pq \times pq$ permutation matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where 1 and 0 denote 1_q and 0_q respectively, and let

$$\hat{C}_\mu^\rho(x) = \left\{ \bigoplus_{\nu=0}^{p-1} C_\mu^\nu(\alpha^{\nu+\rho}(x)) \right\} \cdot P^\rho.$$

d) Let $\sigma^\rho(\bar{x}) = \bigoplus_{\nu=0}^{p-1} 1_q \otimes \sigma(\alpha_\lambda^{\nu+\rho}(\bar{x}))$ and $\tilde{\sigma}(\bar{x}) = \bigoplus_{\nu=0}^{p-1} 1_n \otimes \sigma^\rho(\bar{x})$.

e) Let $\tilde{g}(x)$ be the $pq \times 1$ matrix defined by

$$(\tilde{g}(x))_{qi+\nu} = (g(\alpha^i(x)))_\nu, \quad 0 \leq i \leq p-1 \quad \text{and} \quad 1 \leq \nu \leq q.$$

Any $q \times 1$ matrix v can be considered as a $q \times q$ diagonal matrix whose (μ, μ) th entry is the μ th entry in v and vice versa. Whenever a $q \times 1$ matrix v appears in a direct sum, it will be assumed that v has been replaced with this $q \times q$ diagonal matrix.

f) With this convention in mind, define $(\bar{\Omega}\tilde{F}_\mu^\rho)(\bar{x})$ to be the $pq \times 1$ matrix

$$((\bar{\Omega}\tilde{F}_\mu^\rho)(\bar{x}))_\nu = \left\{ \bigoplus_{\delta=0}^{p-1} (\bar{\Omega}F_\mu^\delta)(\alpha_\lambda^{i+\rho}(\bar{x})) \right\}_{\nu,\nu} \quad 1 \leq \nu \leq pq$$

and define $(\bar{\Omega}F)(\bar{x})$ to be the $np^2q \times 1$ matrix

$$((\bar{\Omega}F)(\bar{x}))_\nu = \left\{ \bigoplus_{\rho=0}^{p-1} \bigoplus_{\mu=1}^n (\bar{\Omega}\tilde{F}_\mu^\rho)(\bar{x}) \right\}_\nu, \quad 1 \leq \nu \leq np^2q.$$

The p equations (1.6) $_\rho$, $\rho = 0, \dots, p-1$ can now be written in the matrix form

$$(1.6) \quad K_+(\text{pr } \bar{x}) \begin{pmatrix} f(\text{pr } \bar{x}) \\ f(\alpha^1(\text{pr } \bar{x})) \\ \vdots \\ f(\alpha^{p-1}(\text{pr } \bar{x})) \end{pmatrix} + B(\text{pr } \bar{x}) \{ (1 - \tilde{\sigma}(\bar{x}))(\bar{\Omega}F)(\beta\bar{x}) + (1 + \tilde{\sigma}(\bar{x}))(\bar{\Omega}F)(\bar{x}) \} = \tilde{g}(\text{pr } \bar{x});$$

it is understood that $(\bar{\Omega}\tilde{F}_\mu^\rho)(\beta\bar{x})$ denotes the matrix transpose of $((\bar{\Omega}\tilde{F}_\mu^0)(\beta\alpha_\lambda^0(\bar{x})), \dots, (\bar{\Omega}\tilde{F}_\mu^{p-1})(\beta\alpha_\lambda^{p-1+\rho}(\bar{x})))$.

If $\det K_+(x) \neq 0$ for all $x \in S$, (1.6) becomes

$$(1.6)' \quad \begin{pmatrix} f(\text{pr } \bar{x}) \\ f(\alpha^1(\text{pr } \bar{x})) \\ \vdots \\ f(\alpha^{p-1}(\text{pr } \bar{x})) \end{pmatrix} + K_+^{-1}(\text{pr } \bar{x}) B(\text{pr } \bar{x}) \{ (1 - \tilde{\sigma}(\bar{x}))(\bar{\Omega}F)(\beta\bar{x}) + (1 + \tilde{\sigma}(\bar{x}))(\bar{\Omega}F)(\bar{x}) \} = K_+^{-1}(\text{pr } \bar{x}) \tilde{g}(\text{pr } \bar{x}).$$

Multiply (1.6)' through on the left by $\sigma^o(\bar{x})\hat{C}_\mu^o(\text{pr } \bar{x})$; substituting the Plemelj formulas (1.4) into the resulting left-hand side yields

$$(1.7)_\mu^o \quad (\bar{\Omega}\tilde{F}_\mu^o)(\beta\bar{x}) - (\bar{\Omega}\tilde{F}_\mu^o)(\bar{x}) + \sigma^o(\bar{x})\hat{C}_\mu^o(\text{pr } \bar{x})K_+^{-1}(\text{pr } \bar{x})B(\text{pr } \bar{x}) \\ \cdot \{(1 - \tilde{\sigma}(\bar{x}))(\bar{\Omega}F)(\beta\bar{x}) + (1 + \tilde{\sigma}(\bar{x}))(\bar{\Omega}F)(\bar{x})\} \\ = \sigma^o(\bar{x})\hat{C}_\mu^o(\text{pr } \bar{x})K_+^{-1}(\text{pr } \bar{x})\tilde{g}(\text{pr } \bar{x}).$$

Combining these equations into matrix form yields

$$(1.7) \quad (1 - T^-(\bar{x}))(\bar{\Omega}F)(\beta\bar{x}) = (1 - T^+(\bar{x}))(\bar{\Omega}F)(\bar{x}) + \tilde{T}(\bar{x}),$$

where

$$T^\pm(\bar{x}) = \left[\bigoplus_{\rho=0}^{p-1} \bigoplus_{\mu=1}^n \sigma^o(\bar{x})\hat{C}_\mu^o(\text{pr } \bar{x}) \right] \cdot \left[\pm \tilde{I}_{np} \otimes K_+^{-1}(\text{pr } \bar{x}) \otimes \tilde{I}_{np} \right] \\ \cdot \left[\bigoplus_{i=0}^{p-1} \bigoplus_{\nu=1}^n \tilde{B}_\nu^i(\text{pr } \bar{x})(\sigma^i(\bar{x}) \pm 1) \right]$$

and $\tilde{T}(\bar{x})$ is the $np^2q \times 1$ matrix given by

$$(\tilde{T}(\bar{x}))_{npqi+pqj+\nu} = (\sigma^i(\bar{x})\hat{C}_i^o(\text{pr } \bar{x})K_+^{-1}(\text{pr } \bar{x})\tilde{g}(\text{pr } \bar{x}))_\nu,$$

where $0 \leq i \leq p-1$, $0 \leq j \leq n-1$, $1 \leq \nu \leq pq$; $\tilde{I}_{np}$ is the $1 \times np$ matrix $(1, 1, \dots, 1)$.

(1.8) Lemma. Let $B_1, \dots, B_n, C_1, \dots, C_n$ be $q \times q$ matrices over the field of complex numbers. Then

$$\det \begin{bmatrix} 1 - C_1B_1, & -C_1B_2, & \dots, & -C_1B_n \\ -C_2B_1, & 1 - C_2B_2, & \dots, & -C_2B_n \\ \vdots & \vdots & & \vdots \\ -C_nB_1, & -C_nB_2, & \dots, & 1 - C_nB_n \end{bmatrix} = \det \left(1 - \sum_{\nu=1}^n B_\nu C_\nu \right).$$

Proof. See [8], Lemma 1.5.

It follows that

$$(1.9) \quad \det(1 - T^\pm(\bar{x})) = \det \left(1 - \sum_{\mu, \rho} K_+^{-1}(\text{pr } \bar{x})\tilde{B}_\mu^o(\text{pr } \bar{x})(1 \pm \sigma^o(\bar{x}))\hat{C}_\mu^o(\text{pr } \bar{x}) \right) \\ = \det K_+^{-1}(\text{pr } \bar{x}) \cdot \det(K_+(\text{pr } \bar{x}) - \sum_{\mu, \rho} \tilde{B}_\mu^o(\text{pr } \bar{x})(1 \pm \sigma^o(\bar{x}))\hat{C}_\mu^o(\text{pr } \bar{x})).$$

In case $\sigma(\bar{x}) = \sigma(\alpha_\lambda^1(\bar{x})) = \dots = \sigma(\alpha_\lambda^{p-1}(\bar{x}))$, one of the matrices $1 - T^\pm(x)$ is equal to 1 and the value of the determinant of the other is

$$(1.9)' \quad \det K_+^{-1}(\text{pr } \bar{x}) \cdot \det K_-(\text{pr } \bar{x}),$$

where $K_-(\text{pr } \bar{x})$ is that $pq \times pq$ matrix obtained from $K_+(\text{pr } \bar{x})$ by replacing

the $q \times q$ matrices $K_+^* = A^* + \sum_{\mu=1}^n B_\mu^* C_\mu^*$ with $K_-^* = A^* - \sum_{\mu=1}^n B_\mu^* C_\mu^*$. On the other hand, if $\sigma(\bar{x}) = -\sigma(\alpha_\lambda^1(\bar{x})) = \sigma(\alpha_\lambda^2(\bar{x})) = \dots = -\sigma(\alpha_\lambda^{p-1}(\bar{x}))$, then one of the $\det(1 - T^\pm(\bar{x}))$ is equal to $\det K_+^{-1}(\text{pr } \bar{x}) \cdot \det K_1(\text{pr } \bar{x})$ and the other equals $\det K_+^{-1}(\text{pr } \bar{x}) \cdot \det K_2(\text{pr } \bar{x})$, where

$$\begin{aligned} (K_\tau(x))_{\alpha i + \nu, \alpha j + \mu} &= (\tfrac{1}{2}(1 + (-1)^{\tau+i})K_+^{p-i+i}(\alpha^i(x)) \\ &\quad + (\tfrac{1}{2}(1 - (-1)^{\tau+i})K_-^{p-i+i}(\alpha^i(x)))_{\nu, \mu}, \end{aligned}$$

$$0 \leq i, j \leq p-1, 1 \leq \mu, \nu \leq n \quad \text{and} \quad \tau = 1, 2.$$

The Ω -degenerate integral equation (1.1) is said to be *regularizable relative to the maps* $\{\alpha_\lambda^\rho : 0 \leq \rho \leq p-1, \lambda \in \Lambda\}$ if $\det K_+(\text{pr } \bar{x})$, $\det(1 - T^+(\bar{x}))$ and $\det(1 - T^-(\bar{x}))$ vanish nowhere on $\bar{X}_\beta$. Suppose that this is the case for the choice of the maps made above and consider any 1-simplex σ_λ^1 on $\bar{X}_\beta$; each matrix $\sigma^\rho(\bar{x})$ has constant value on σ_λ^1 , i.e. $\sigma^\rho(\bar{x}) = \sigma^\lambda D_\lambda^\rho$ for all $x \in \sigma_\lambda^1$, where σ^λ is the constant value of $\sigma(\bar{x})$ on σ_λ^1 and D_λ^ρ is a diagonal matrix with entries $+1$ or -1 . Therefore the restrictions of T^+ , T^- , and T to σ_λ^1 satisfy

$$(T^\pm | \sigma_\lambda^1)(\bar{x}) = \left[\bigoplus_{\rho=0}^{p-1} \bigoplus_{\mu=1}^n D_\lambda^\rho \hat{C}_\mu^*(\text{pr } \bar{x}) \right] \cdot \left[\bigoplus_{\delta=0}^{p-1} \bigoplus_{\nu=1}^n K_+^{-1}(\text{pr } \bar{x}) \tilde{B}_\nu^\delta(\text{pr } \bar{x}) \{D_\lambda^\delta \pm \sigma^\lambda 1\} \right],$$

and $(\tilde{T} | \sigma_\lambda^1)(\bar{x})$ is the $np^2q \times 1$ matrix

$$[(\tilde{T} | \sigma_\lambda^1)(\bar{x})]_{npqi + pqj + \nu} = [\sigma^\lambda D_\lambda^i C_j^*(\text{pr } \bar{x}) K_+^{-1}(\text{pr } \bar{x}) (g(\text{pr } \bar{x}))]_\nu,$$

where $0 \leq i \leq p-1, 0 \leq j \leq n-1$ and $1 \leq \nu \leq pq$.

Let $\sigma_{\beta(\lambda)}^1$ be the unique 1-simplex for which $\bar{x} \in \sigma_\lambda^1$ if and only if $\beta\bar{x} \in \sigma_{\beta(\lambda)}^1$. Clearly $\lambda = \beta(\beta(\lambda))$. Since $\sigma(\beta\bar{x}) = -\sigma(\bar{x})$ for all $\bar{x} \in \sigma_\lambda^1$, it follows that

$$(1.10) \quad (T^\pm | \sigma_\lambda^1)(\bar{x}) = (T^\mp | \sigma_{\beta(\lambda)}^1)(\beta\bar{x}); \quad (\tilde{T} | \sigma_\lambda^1)(\bar{x}) = -(\tilde{T} | \sigma_{\beta(\lambda)}^1)(\beta\bar{x}).$$

Define $T_\lambda^*(\text{pr } \bar{x}) := (T^\pm | \sigma_\lambda^1)(\bar{x})$ and $\tilde{T}_\lambda(\text{pr } \bar{x}) := (\tilde{T} | \sigma_\lambda^1)(\bar{x})$, and let $T_\lambda(\text{pr } \bar{x})$ be the affine transformation on $\mathbf{C}^{np^2q}$ given by

$$\begin{aligned} (1.11) \quad T_\lambda(\text{pr } \bar{x})G &= (1 - T_\lambda^-(\text{pr } \bar{x}))^{-1}(1 - T_\lambda^+(\text{pr } \bar{x}))G \\ &\quad + (1 - T_\lambda^-(\text{pr } \bar{x}))^{-1}\tilde{T}_\lambda(\text{pr } \bar{x}), \end{aligned}$$

where $\bar{x} \in \sigma_\lambda^1$ and $G \in \mathbf{C}^{np^2q}$.

Because of (1.10) the problem (1.7) then has the form

$$(1.12)_\lambda \quad (\bar{\Omega}F)(\beta\bar{x}) = \hat{T}_\lambda(\text{pr } \bar{x})^{\sigma(\bar{x})}(\bar{\Omega}F)(\bar{x})$$

for all $\bar{x} \in \sigma_\lambda^1 \cup \sigma_{\beta(\lambda)}^1$, where $\hat{T}_\lambda$ is one of the two transformations T_λ and $T_{\beta(\lambda)}$ and $\hat{T}_\lambda^{-1}$ is the other.

In view of the general hypothesis 1)' and 2), $T^+(\bar{x}) = T^-(\bar{x}) = 0$ and $\tilde{T}(\bar{x}) = 0$ at every vertex $\bar{x}$ of the simplicial structure $\{\sigma_\lambda^1 : \lambda \in \Lambda\}$ on $\bar{X}_\beta$. Therefore, whenever $\sigma_{\lambda_1}^1 \cap \sigma_{\lambda_2}^1 \neq \emptyset$, say $\{\bar{x}\} = \sigma_{\lambda_1}^1 \cap \sigma_{\lambda_2}^1$, $\hat{T}_{\lambda_1}$ and $\hat{T}_{\lambda_2}$ can be extended continuously to one another across $\bar{x}$.

(1.13) Theorem. *If the Ω -degenerate singular integral equation (1.1) is regularizable relative to the system $\{\alpha_\lambda^\rho : 0 \leq \rho \leq p-1, \lambda \in \Lambda\}$, then*

- 1) *there is a Hölder continuous map $T : S \rightarrow GA(np^2q, \mathbb{C})$ for which $(T|_{L_\lambda^1}) \circ \text{pr} = \hat{T}_\lambda$ on $\hat{\mathcal{S}}_\lambda^1$ for all $\lambda \in \Lambda$;*
- 2) *$(\bar{\Omega}F)(\bar{x})$ is a continuous map from $\bar{X}_\beta$ into $\mathbb{C}^{np^2q}$ for every Hölder continuous solution f of (1.1);*
- 3) *$(\bar{\Omega}F)(\bar{x})$ satisfies the transmission problem*

$$(1.12) \quad (\bar{\Omega}F)(\beta\bar{x}) = T(\text{pr } \bar{x})^{\sigma(\bar{x})}(\bar{\Omega}F)(\bar{x}).$$

Let $G(\bar{x})$ be any solution of the transmission problem (1.12) of the form

$$(G(\bar{x}))_\nu = \left(\bigoplus_{\rho=0}^{p-1} \bigoplus_{\mu=1}^n \bigoplus_{\delta=0}^{p-1} G_{\mu,\rho}^\delta(\bar{x}) \right)_{\nu,\nu}, \quad 1 \leq \nu \leq np^2q,$$

where $G_{\mu,\rho}^\delta$ is a Hölder continuous $q \times 1$ matrix on $\bar{X}_\beta$ for all $\delta, \rho = 0, \dots, p-1$ and $\mu = 1, \dots, n$ such that whenever $\delta + \rho = 0$ or p there is a meromorphic $q \times 1$ matrix $\hat{G}_{\mu,\rho}^\delta$ on $X \setminus S$ with divisor $\geq * - D$ for which the matrix $\bar{G}_{\mu,\rho}^\delta$ on $\text{pr}^{-1}(X \setminus S)$ defined by $\bar{G}_{\mu,\rho}^\delta(\bar{x}) = \hat{G}_{\mu,\rho}^\delta(\text{pr } \bar{x})$ can be extended continuously to $\bar{X}$ so that $\bar{G}_{\mu,\rho}^\delta(\bar{x}) = G_{\mu,\rho}^\delta(\bar{x})$ for all $\bar{x} \in \bar{X}_\beta$. Keeping (1.6)' in mind, consider for $x \in S$ and $\bar{x} \in \text{pr}^{-1}(x)$ the expression

$$(1.14) \quad K_+^{-1}(x) \{ \tilde{g}(x) - B(x)[(1 - \tilde{\sigma}(\bar{x}))G(\beta\bar{x}) + (1 + \tilde{\sigma}(\bar{x}))G(\bar{x})] \}.$$

Because of the general hypothesis 1)' and 2) it is easy to show that this $pq \times 1$ matrix depends only on $x = \text{pr } \bar{x}$. Using the techniques in [8], it can be shown that the $q \times 1$ matrix f whose ν th entry is the ν th entry in (1.14), $1 \leq \nu \leq q$, is a Hölder continuous solution of (1.1). Moreover, if one forms $(\bar{\Omega}F)(\bar{x})$ for this f , $\bar{\Omega}F_{\mu,\rho}^\delta = G_{\mu,\rho}^\delta$ for all μ, ρ and δ for which $\delta + \rho = 0$ or p .

Denote this space of solutions G of (1.12) by L .

(1.15) Proposition. *The expression (1.14) defines a surjection ϵ from the space L of solutions of the transmission problem (1.12) as described above to the space of Hölder continuous solutions of the Ω -degenerate integral equation (1.1). For the homogeneous equation (1.1), the kernel of the corresponding map ϵ_0 is the space of all solutions of (1.12) (with $\hat{T} \equiv 0$) for which $G_{\mu,\rho}^\delta \equiv 0$ whenever $\delta + \rho = 0$ or p . If $\dim_{\mathbb{C}} \ker \epsilon_0 = 0$, then for every solution of (1.12) as described above one has*

$$(1.16) \quad G_{\mu,\rho}^\delta(\bar{x}) = G_{\mu,p-\rho}^\delta(\alpha_\lambda^{\delta+\rho}(\bar{x}))$$

for all $0 \leq \delta, \rho \leq p-1$ and $1 \leq \mu \leq n$. In general, (1.14) defines a bijection from the subspace of L whose elements satisfy (1.16) to the space of solutions of (1.1).

Proof. See [8].

A solution of the transmission problem (1.12) as described above will be called a meromorphic solution if for every $G_{\mu,\rho}^\delta$ (not just those for which $\rho + \delta = 0$ or p) there is a meromorphic $q \times 1$ matrix $\hat{G}_{\mu,\rho}^\delta$ on $X \setminus S$ for

which the matrix $\bar{G}_{\mu,\rho}^\delta$ on $\text{pr}^{-1}(X \setminus S)$ defined by $\bar{G}_{\mu,\rho}^\delta(\bar{x}) = \hat{G}_{\mu,\rho}^\delta(\text{pr } \bar{x})$ can be extended continuously to $\bar{X}$ so that $\bar{G}_{\mu,\rho}^\delta(\bar{x}) = G_{\mu,\rho}^\delta(\bar{x})$ for all $\bar{x} \in \bar{X}_\beta$.

Suppose now that $\alpha = \alpha^1$ is the restriction to S of a holomorphic automorphism of X . This is then true for each α^ρ . α^ρ induces in the obvious way a homeomorphism $\bar{\alpha}^\rho$ of $\bar{X}$ onto itself which is biholomorphic on $\text{pr}^{-1}(X \setminus S)$. Suppose also that the integral equation (1.1) is regularizable relative to the system $\{\alpha_\lambda^\rho = \bar{\alpha}^\rho \mid \sigma_\lambda^1 : 0 \leq \rho \leq p-1, \lambda \in \Lambda\}$. As in [9] there is a holomorphic affine bundle $W \rightarrow X$ whose global holomorphic sections are in bijective correspondence with the space of holomorphic solutions of the transmission problem (1.12) corresponding to the above system $\{\alpha_\lambda^\rho : 0 \leq \rho \leq p-1, \lambda \in \Lambda\}$. W is given by a holomorphic $\mathfrak{U} = \mathfrak{U}_S \cup \mathfrak{U}_D \cup \{X \setminus (S \cup \bigcup_{\rho=0}^{p-1} \alpha^\rho(* \cup D))$ -cocycle $g_{ij}(x) = (A_{ij}(x), a_{ij}(x))$, where A_{ij} (resp. a_{ij}) takes values in $GL(np^2q, \mathbb{C})$ (resp. $\mathbb{C}^{np^2q}$); $\mathfrak{U}_S$ is an open cover of S as in [9], while $\mathfrak{U}_D$ is an open cover of $\bigcup_{\rho=0}^{p-1} \alpha^\rho(* \cup D)$ whose elements are pairwise disjoint and have empty intersection with every element of $\mathfrak{U}_S$. Moreover, $A_{ij} \equiv 1$ and $a_{ij} \equiv 0$ on $U \cap X$ for every $U \in \mathfrak{U}_D$. Using the techniques given in [7] and [9], a new $\mathfrak{U}$ -cocycle $\tilde{g}_{ij}$ may be defined which coincides with g_{ij} on $\mathfrak{U}_S$ such that the global holomorphic sections in the affine bundle $\tilde{W} \rightarrow X$ defined by $\tilde{g}_{ij}$ are in bijective correspondence with the meromorphic solutions G of (1.12) for which the divisor of $\hat{G}_{\mu,\rho}^\delta$ is $\geq \alpha^{\delta+\rho}(* - D)$ for all $0 \leq \rho, \delta \leq p-1$ and $1 \leq \mu \leq n$. $\tilde{g}_{ij}$ is constructed as follows. Let $\{d_i^\nu : i \in I\}$ be meromorphic functions on U_i defining the divisor $\alpha^\nu(* - D)$. These functions may be chosen so that $d_i^\nu \equiv 1$ on U_i whenever $U_i \in \mathfrak{U}_S$. Define

$$d_i(x) = \bigoplus_{\rho=0}^{p-1} \bigoplus_{\mu=1}^n \bigoplus_{\delta=0}^{p-1} 1_q \otimes d_i^{\delta+\rho}(x); \quad \tilde{g}_{ij}(x) = (d_i^{-1}(x) A_{ij}(x) d_j(x), d_i^{-1}(x) a_{ij}(x)).$$

(1.17) Corollary. *Suppose that α is the restriction to S of a biholomorphic automorphism of X and the Ω -degenerate singular integral equation (1.1) is regularizable relative to the system $\{\alpha_\lambda^\rho = \bar{\alpha}^\rho \mid \sigma_\lambda^1 : 0 \leq \rho \leq p-1, \lambda \in \Lambda\}$. Then there is a surjection η from the space $H^0(X, \tilde{W})$ of holomorphic sections in the affine bundle $\tilde{W} \rightarrow X$ to the space of Hölder continuous solutions of (1.1).*

One result concerning the kernel of η_0 will be given here. For each $\lambda \in \Lambda$, index the columns (resp. rows) of $T_\lambda^\pm$ from top to bottom (resp. from left to right) in the same way as the vector G is indexed. Replace the $\delta_{\mu,\rho}$ th column matrix (resp. row matrix) in $T_\lambda^\pm$ with the $np^2q \times q$ (resp. $q \times np^2q$) 0-matrix whenever $\delta + \rho = 0$ or p . Denote the resulting matrix by $T_{\lambda,k}^\pm$. Determine $\hat{T}_{\lambda,k}$ in the same way as $\hat{T}_\lambda$ was in (1.12) using $T_{\lambda,k}^\pm$ and $\hat{T}_k \equiv 0$ on S , and do the same to determine T_k in place of T in (1.12). Then every element in the kernel of η_0 is a solution of the transmission problem

$$(1.12)_k \quad H(\beta \bar{x}) = T_k(\text{pr } \bar{x})^{\rho(\bar{x})} H(\bar{x}).$$

Let $\tilde{W}_k \rightarrow X$ be the holomorphic vectorbundle over X associated with the transmission problem $(1.12)_k$.

(1.18) *Proposition.* Let η_0 and $\tilde{W}_k \rightarrow X$ be defined as above. Then

$$\dim_{\mathbb{C}} \ker \eta_0 \leq \dim_{\mathbb{C}} H^0(X, \tilde{W}_k).$$

A similar result holds for ϵ_0 in the case $p = 2$. Let $\tilde{W}_{k,\alpha} \rightarrow X$ be the holomorphic vectorbundle associated with the transmission problem

$$H(\beta\mathcal{E}) = T_k(\alpha(\text{pr } \mathcal{E}))^{\sigma(\tilde{x})} H(\mathcal{E}).$$

(1.19) *Proposition.* Let ϵ_0 and $\tilde{W}_{k,\alpha} \rightarrow X$ be as above and $p = 2$. Then

$$\dim_{\mathbb{C}} \ker \epsilon_0 \leq \dim_{\mathbb{C}} H^0(X, \tilde{W}_{k,\alpha}).$$

The Ω -degenerate integral equation (1.1) can be solved completely if its coefficients are subjected to conditions so as to reduce the corresponding transmission problem (1.12) to the Riemann–Privalov problem (see [9]) or to one with a shift (see [7]). There are many such conditions which do this. Only one case will be discussed here; it generalizes known results in [4] and [5]. For simplicity it will be assumed that $g(x) = 0$ for all $x \in S$, i.e. only the homogeneous equation (1.1) will be considered.

Suppose that $p = 2$ and $n = q = 1$. If $\sigma(\mathcal{E}) = \sigma(\alpha_\lambda(\mathcal{E}))$ on $\mathfrak{s}_\lambda^1$, then one of T^+ , T^- is equal to 1 on $\text{pr } \sigma_\lambda^1$ while the other is equal to

$$(1.20) \quad 1 - \frac{2}{\det K_+} \begin{pmatrix} C_1^0 K_+^0 \circ \alpha B_1^0, & C_1^0 K_+^0 \circ \alpha B_1^1 \circ \alpha, & -C_1^0 K_+^1 \circ \alpha B_1^0 \circ \alpha, & -C_1^0 K_+^1 \circ \alpha B_1^1 \\ -C_1^1 \circ \alpha K_+^1 B_1^0, & -C_1^1 \circ \alpha K_+^1 B_1^1 \circ \alpha, & C_1^1 \circ \alpha K_+^0 B_1^0 \circ \alpha, & C_1^1 \circ \alpha K_+^0 B_1^1 \\ -C_1^0 \circ \alpha K_+^1 B_1^0, & -C_1^0 \circ \alpha K_+^1 B_1^1 \circ \alpha, & C_1^0 \circ \alpha K_+^0 B_1^0 \circ \alpha, & C_1^0 \circ \alpha K_+^0 B_1^1 \\ C_1^1 K_+^0 \circ \alpha B_1^0, & C_1^1 K_+^0 \circ \alpha B_1^1 \circ \alpha, & -C_1^1 K_+^1 \circ \alpha B_1^0 \circ \alpha, & -C_1^1 K_+^1 \circ \alpha B_1^1 \end{pmatrix}.$$

If $\sigma(\mathcal{E}) = -\sigma(\alpha_\lambda(\mathcal{E}))$ on σ_λ^1 , then one of T^+ , T^- is equal to

$$(1.21) \quad 1 - \frac{2}{\det K_+} \begin{pmatrix} C_1^0 K_+^0 \circ \alpha B_1^0, & 0, & 0, & -C_1^0 K_+^1 \circ \alpha B_1^1 \\ -C_1^1 \circ \alpha K_+^1 B_1^0, & 0, & 0, & C_1^1 \circ \alpha K_+^0 B_1^1 \\ -C_1^0 \circ \alpha K_+^1 B_1^0, & 0, & 0, & C_1^0 \circ \alpha K_+^0 B_1^1 \\ C_1^1 K_+^0 \circ \alpha B_1^0, & 0, & 0, & -C_1^1 K_+^1 \circ \alpha B_1^1 \end{pmatrix}$$

and the other one equals

$$(1.22) \quad 1 - \frac{2}{\det K_+} \begin{pmatrix} 0, & C_1^0 K_+^0 \circ \alpha B_1^1 \circ \alpha, & -C_1^0 K_+^1 \circ \alpha B_1^0 \circ \alpha, & 0 \\ 0, & -C_1^1 \circ \alpha K_+^1 B_1^1 \circ \alpha, & C_1^1 \circ \alpha K_+^0 B_1^0 \circ \alpha, & 0 \\ 0, & -C_1^0 \circ \alpha K_+^1 B_1^1 \circ \alpha, & C_1^0 \circ \alpha K_+^0 B_1^0 \circ \alpha, & 0 \\ 0, & C_1^1 K_+^0 \circ \alpha B_1^1 \circ \alpha, & -C_1^1 K_+^1 \circ \alpha B_1^0 \circ \alpha, & 0 \end{pmatrix}.$$

If $C_1^0 = C_1^1$ on S , then $(\bar{\Omega}F)(\bar{x})$ is of the form

$$'((\bar{\Omega}F_1)(\bar{x}), (\bar{\Omega}F_1)(\alpha_\lambda(\bar{x})), (\bar{\Omega}F_1)(\alpha_\lambda(\bar{x})), (\bar{\Omega}F_1)(\bar{x}));$$

because of proposition (1.15) it is enough to consider only solutions of (1.12) satisfying (1.16) on ∂_λ^1 , and therefore (1.12) $_\lambda$ is equivalent to

$$(1.23) \quad \begin{bmatrix} (\bar{\Omega}F_1)(\beta\bar{x}) \\ (\bar{\Omega}F_1)(\beta\alpha_\lambda(\bar{x})) \end{bmatrix} = \hat{T}_\lambda(\text{pr } \bar{x})^{\sigma(\bar{x})} \begin{bmatrix} (\bar{\Omega}F_1)(\bar{x}) \\ (\bar{\Omega}F_1)(\alpha_\lambda(\bar{x})) \end{bmatrix},$$

where in the case $\sigma(\bar{x}) = \sigma(\alpha_\lambda(\bar{x}))$ on ∂_λ^1 , $\hat{T}_\lambda$ is equal to

$$(1.24) \quad 1 - \frac{2}{\det K_+} \begin{bmatrix} C_1^0, & 0 \\ 0, & C_1^0 \end{bmatrix} \cdot \begin{bmatrix} K_+^0 \circ \alpha B_1^0 - K_+^1 \circ \alpha B_1^1, & K_+^0 \circ \alpha B_1^1 \circ \alpha - K_+^1 \circ \alpha B_1^0 \circ \alpha \\ K_+^0 B_1^1 - K_+^1 B_1^0, & K_+^0 B_1^0 \circ \alpha - K_+^1 B_1^1 \circ \alpha \end{bmatrix}$$

or its inverse, while if $\sigma(\bar{x}) = -\sigma(\alpha_\lambda(\bar{x}))$ on ∂_λ^1 , $\hat{T}_\lambda$ is then equal to

$$(1.24)' \quad \left\{ 1 - \frac{2}{\det K_+} \begin{bmatrix} C_1^0, & 0 \\ 0, & C_1^0 \end{bmatrix} \cdot \begin{bmatrix} K_+^0 \circ \alpha B_1^0 - K_+^1 \circ \alpha B_1^1, & 0 \\ 0, & K_+^0 B_1^0 \circ \alpha - K_+^1 B_1^1 \circ \alpha \end{bmatrix} \right\}^{-1} \\ \cdot \left\{ 1 - \frac{2}{\det K_+} \begin{bmatrix} C_1^0, & 0 \\ 0, & C_1^0 \end{bmatrix} \cdot \begin{bmatrix} 0, & K_+^0 \circ \alpha B_1^1 \circ \alpha - K_+^1 \circ \alpha B_1^0 \circ \alpha \\ K_+^0 B_1^1 - K_+^1 B_1^0, & 0 \end{bmatrix} \right\}$$

or its inverse.

One set of generalizations of [4] and [5] can be obtained by setting $C_1^0 = C_1^1 \equiv 1$ and letting S be the disjoint union of the closed, smooth, simple curves $\mathcal{C}_1, \dots, \mathcal{C}_*$. In this case (1.24) becomes

$$(1.25) \quad \det K_+^{-1} \begin{bmatrix} K_-^0 K_+^0 \circ \alpha - K_-^1 K_+^1 \circ \alpha, & -2(A^0 \circ \alpha B_1^1 \circ \alpha - A^1 \circ \alpha B_1^0 \circ \alpha) \\ -2(A^0 B_1^1 - A^1 B_1^0), & K_-^0 \circ \alpha K_+^0 - K_-^1 \circ \alpha K_+^1 \end{bmatrix}$$

and (1.24)' becomes

$$(1.25)' \quad \left[K_-^0 K_+^0 \circ \alpha - K_-^1 K_+^1 \circ \alpha, \quad 0 \right]^{-1} \\ \cdot \begin{bmatrix} 0, & K_-^0 \circ \alpha K_+^0 - K_-^1 \circ \alpha K_+^1 \\ \det K_+, & -2(A^0 \circ \alpha B_1^1 \circ \alpha - A^1 \circ \alpha B_1^0 \circ \alpha) \\ -2(A^0 B_1^1 - A^1 B_1^0), & \det K_+ \end{bmatrix}.$$

If $A^1 B_1^0 - A^0 B_1^1 \equiv 0$ and $K_-^0 K_+^0 \circ \alpha - K_-^1 K_+^1 \circ \alpha \neq 0$ on some $\text{pr } \sigma_\lambda^1 \cup \alpha(\text{pr } \sigma_\lambda^1)$, choose α_λ so that $\sigma(\bar{x}) = \sigma(\alpha_\lambda(\bar{x}))$ on ∂_λ^1 ; the transmission problem (1.22) is then equivalent to a Riemann–Privalov problem on $\text{pr } \sigma_\lambda^1 \cup \alpha(\text{pr } \sigma_\lambda^1)$.

On the other hand, if $K_-^0 K_+^0 \circ \alpha - K_-^1 K_+^1 \circ \alpha \equiv 0$ and $(2(A^0 B_1^1 - A^1 B_1^0))^{-1} \equiv 2(A^0 \circ \alpha B_1^1 \circ \alpha - A^1 \circ \alpha B_1^0 \circ \alpha)$ on some $\text{pr } \sigma_\lambda^1 \cup \alpha(\text{pr } \sigma_\lambda^1)$, again choose α_λ so that $\sigma(\bar{x}) = \sigma(\alpha_\lambda(\bar{x}))$ on ∂_λ^1 ; the transmission problem (1.22) is then equivalent to a Haseman problem on $\text{pr } \sigma_\lambda^1 \cup \alpha(\text{pr } \sigma_\lambda^1)$ if $\text{pr } \sigma_\lambda^1$ and $\alpha(\text{pr } \sigma_\lambda^1)$ are not on the same contour $\mathcal{C}_i$ and $\alpha|_{\text{pr } \sigma_\lambda^1}$ is direction preserving, or to a Carleman problem if $\text{pr } \sigma_\lambda^1$ and $\alpha(\text{pr } \sigma_\lambda^1)$ are on the same contour $\mathcal{C}_i$ and $\alpha|_{\text{pr } \sigma_\lambda^1}$ is direction reversing. Similar conditions can be stated using (1.25)' so that one may choose $\sigma(\bar{x}) = -\sigma(\alpha_\lambda(\bar{x}))$ on ∂_λ^1 and get (1.22) equivalent to a Riemann–Privalov, Haseman, or Carleman problem on $\text{pr } \sigma_\lambda^1 \cup \alpha(\text{pr } \sigma_\lambda^1)$. Therefore if the coefficients of the integral equation (1.1) are such that the maps $\{\alpha_\lambda : \lambda \in \Lambda\}$ can be chosen so that (1.22) is equivalent to one of these three boundary value problems on each $\mathcal{C}_i \cup \alpha(\mathcal{C}_i)$, then (1.22) is equivalent to a mixed Riemann–Privalov, Haseman, Carleman problem on S . This problem is completely solvable by the methods discussed in [7] under the conditions given there. These conditions allow one to form the welded Riemann surface X^α ; the solution space of (1.22) is isomorphic to that of a Riemann–Privalov problem on X^α . The usual methods (see [9]) can then be employed in order to find a holomorphic line bundle on X^α whose global holomorphic sections are in bijective correspondence with the meromorphic solutions of this Riemann–Privalov problem on X^α with divisor $\geq \chi(* - D)$.

An example of the application of the general Plemelj formulas in Theorem (4) to singular integral equations with shift is as follows. Let S be the union of a finite number of closed, smooth, simple curves $\{\mathcal{C}_\nu : \nu = 0, \dots, p-1\}$; for simplicity of computations assume that the order of any point in S is ≤ 4 . Let $\{\alpha^0 = id_X, \alpha^1, \dots, \alpha^{p-1}\}$ be a cyclic group of holomorphic automorphisms of X which map S onto itself and such that $\alpha^\nu(\mathcal{C}_\nu) = \mathcal{C}_0$, $\nu = 0, \dots, p-1$. Choose Σ so that each $\mathcal{C}_\nu$ is an oriented curve. Let $n = 1$ in equation (1.1), $A^1(x) = \dots = A^{p-1}(x) = 0$ for all $x \in S$, and let $C_{1,\rho}^\nu$ be Hölder continuous functions on $\mathcal{C}_\rho$ ($\rho = 0, \dots, p-1$) such that $C_{1,\rho}^\nu \equiv 0$ for all $\rho = 1, \dots, p-1$ and all ν . The integral equation (1.1) is then of the form

$$(1.26) \quad A^0(x_1)f(x_1) + (\pi i)^{-1} \sum_{\nu=0}^{p-1} B_1^\nu(\alpha^\nu(x_1)) \cdot \int_{(\mathcal{C}_0, \Sigma)} C_{1,0}^\nu(x_2)f(x_2)\Omega(x_2, \alpha^\nu(x_1)) = g(x_1).$$

If the order of every point in S is 2, this equation is a special case of (1.1) in the sense that the exact same methods can be used to associate with it a transmission problem. If there are points of order 4 in $S \cap \mathcal{C}_0$, the general Plemelj formulas must be used. Define as before the function

$$(\Omega F_1^\nu)(x_2) = (2\pi i)^{-1} \int_{(\mathcal{C}_0, \Sigma)} C_{1,0}^\nu(x_1)f(x_1)\Omega(x_1, x_2).$$

Clearly the hypothesis of Theorem (4) is satisfied. It is a straightforward matter to calculate that

$$(\bar{\Omega}F_1^v)(\beta\bar{x}) - (\bar{\Omega}F_1^v)(\bar{x}) = \sigma(\bar{x})C_{1,\rho(\bar{x})}^v(\text{pr } \bar{x})f(\text{pr } \bar{x})$$

for all $\bar{x} \in \bar{X}_\beta$, where $\rho(\bar{x})$ is defined so that $\text{pr } (\bar{U}_x \setminus \{\bar{x}\}) \cap \text{pr } (\bar{U}_{\beta x} \setminus \{\beta\bar{x}\}) \cap \mathcal{C}_{\rho(\bar{x})} \neq \emptyset$; such a $\rho(\bar{x})$ exists and is unique. Moreover,

$$(\bar{\Omega}F_1^v)(\beta\bar{x}) + (\bar{\Omega}F_1^v)(\bar{x}) = (\pi i)^{-1} \int_{(\mathcal{C}_0, \Sigma)} C_{1,0}^v(x_1)f(x_1)\Omega(x_1, \text{pr } \bar{x})$$

whenever $\text{pr } \bar{x} \in S \setminus \mathcal{C}_0$ or $\text{pr } \bar{x} \in \mathcal{C}_0$ and $\text{pr } (\bar{U}_x \setminus \{\bar{x}\}) \cap \text{pr } (\bar{U}_{\beta x} \setminus \{\beta\bar{x}\}) \cap \mathcal{C}_0 \neq \emptyset$; if $\bar{x}$ is such that $\text{pr } \bar{x} \in \mathcal{C}_0$ but $\text{pr } (\bar{U}_x \setminus \{\bar{x}\}) \cap \text{pr } (\bar{U}_{\beta x} \setminus \{\beta\bar{x}\}) \cap \mathcal{C}_0 = \emptyset$, then

$$\begin{aligned} & (\bar{\Omega}F_1^v)(\beta\bar{x}) + (\bar{\Omega}F_1^v)(\bar{x}) \\ &= (\pi i)^{-1} \int_{(\mathcal{C}_0, \Sigma)} C_{1,0}^v(x_1)f(x_1)\Omega(x_1, \text{pr } \bar{x}) - \sigma(\beta\bar{x})C_{1,0}^v(\text{pr } \bar{x})f(\text{pr } \bar{x}). \end{aligned}$$

It is easy to see that, because of the fact that $\sigma(\beta\bar{x}_1) = -\sigma(\beta\bar{x}_2)$ for the two points $\bar{x}_1$ and $\bar{x}_2$ in the latter case, to associate with the singular integral equation (1.26) a transmission problem of the form (1.12) it is necessary that $B_1^v(x)C_{1,0}^v(x) = 0$ for all $v = 0, \dots, p-1$ and $x \in \mathcal{C}_0$ for which order $x = 4$.

If $\det K_+(x) \neq 0$ for all $x \in S$, it is not difficult to calculate that for every $\mathcal{C}_\nu$ and $x \in \mathcal{C}_\nu \setminus \mathcal{C}_0$, $\bar{C}_1^\nu(x)K_+^{-1}(x)\bar{B}_1^\nu(x) = 0$ if $\mu \neq \nu$, while $\bar{C}_1^\nu(x)K_+^{-1}(x)\bar{B}_1^\nu(x) =$

$$C_1^{\nu-\rho}(\alpha^\nu(x))K_+^0(\alpha^\nu(x))^{-1} \begin{pmatrix} 0 & , & 0 & , & \cdots & , & 0 \\ \vdots & & \vdots & & & & \vdots \\ 0 & , & 0 & , & \cdots & , & 0 \\ B_1^0(\alpha^\nu(x)), & B_1^1(\alpha^{\nu+1}(x)), & \cdots & , & B_1^{p-1}(\alpha^{\nu+p-1}(x)) \\ 0 & , & 0 & , & \cdots & , & 0 \\ \vdots & & \vdots & & & & \vdots \\ 0 & , & 0 & , & \cdots & , & 0 \end{pmatrix},$$

the nonzero entries being in the $(\nu - \rho + 1)$ th row. For any choice of the transmission function in (1.12) and any σ_λ^1 for which $\text{pr } \sigma_\lambda^1 \subseteq \mathcal{C}_\nu$, one of $\det T_\lambda^+(\bar{x})$, $\det T_\lambda^-(\bar{x})$ is equal to 1 while the other is equal to

$$\det K_+^0(\alpha^\nu(\text{pr } \bar{x}))^{-1} \det K_-^0(\alpha^\nu(\text{pr } \bar{x})).$$

Thus (1.26) is regularizable relative to any system $\{\alpha_\lambda^0 : 0 \leq \rho \leq p-1, \lambda \in \Lambda\}$ if

$$(1.27) \quad \det K_+^0(x) \neq 0, \quad \det K_-^0(x) \neq 0$$

for all $x \in \mathcal{C}_0$.

The integral equation (1.26) contains many of those studied by F. D. Gahov

and L. I. Cibrikova in [1]. The regularity condition (1.27) agrees with that in [1] in the simple cases and corrects the others. The simplest equation studied in [1] is obtained from (1.26) by setting $q = 1$, $C_{1,0}^0(x) = C_{1,0}^1(x) = \cdots = C_{1,0}^{p-1}(x)$ for all $x \in \mathcal{C}_0$, $B_1^0(\alpha^\nu(x)) = B_1^0(x)$ for all $x \in S$ and all $\nu = 1, \dots, p-1$, and choosing Σ so that $\alpha'(\mathcal{C}_\nu) = \mathcal{C}_0$ as oriented arcs. Assume for simplicity that $g(x) \equiv 0$. Then relative to the system $\{\alpha_\lambda^\rho = \bar{\alpha}^\rho \mid \sigma_\lambda^1: 0 \leq \rho \leq p-1, \lambda \in \Lambda\}$, the transmission problem associated with (1.26) on $\mathcal{C}$, consists of p transmission problems of the form

$$(1.28)_\nu \quad \begin{bmatrix} (\bar{\Omega}F_1)(\alpha_\lambda^\rho(\beta\bar{x})) \\ \vdots \\ (\bar{\Omega}F_1)(\alpha_\lambda^{\rho+p-1}(\beta\bar{x})) \end{bmatrix} = \begin{bmatrix} 0, & \dots, & 0 \\ \vdots & & \vdots \\ 1, & \dots, & 1 \\ \vdots & & \vdots \\ 0, & \dots, & 0 \end{bmatrix}^{\sigma(x)} \cdot \begin{bmatrix} (\bar{\Omega}F_1)(\alpha_\lambda^\rho(\bar{x})) \\ \vdots \\ (\bar{\Omega}F_1)(\alpha_\lambda^{\rho+p-1}(\bar{x})) \end{bmatrix}$$

By adding components in each of these problems, one obtains on each $\mathcal{C}$, a transmission problem of the form

$$(1.28)_\nu \quad \{(\bar{\Omega}F_1)(\beta\bar{x}) + \cdots + (\bar{\Omega}F_1)(\beta\alpha_\lambda^{p-1}(\bar{x}))\} \\ = \{K_+^0(\alpha^\nu(\text{pr } \bar{x}))^{-1}K_-^0(\alpha^\nu(\text{pr } \bar{x}))\}^{\sigma(x)} \cdot \{(\bar{\Omega}F_1)(\bar{x}) + \cdots + (\bar{\Omega}F_1)(\alpha_\lambda^{p-1}(\bar{x}))\}.$$

The integral equations with shift considered in [1] are studied by means of transmission problems of this form. However, difficulties do arise because of the Plemelj formulas (see above), and so the reduction of (1.26) to problems (1.28) is not valid in some of the equations in [1].

§2. The Noether theorems. For singular integral equations without shift (equations (1.1) with $p = 1$), the three Noether theorems have been proved ([6], [8]). These are: the solution space of the homogeneous integral equation and that of its adjoint equation ([8], §5) are finite dimensional vector spaces over $\mathbb{C}$; the singular integral equation is normally solvable; and the difference between the dimension of the solution space of the homogeneous integral equation and the dimension of the solution space of the adjoint equation is a topological invariant.

The first two Noether theorems will be proved in this section for singular integral equations with shifts of period $p > 1$ under the assumptions

a) S is a $C^{1+\alpha}$ complex on X and α is the restriction to S of a biholomorphic map α of X onto itself;

b) α is either orientation preserving (*i.e.* $\sigma(\bar{x}) = \sigma(\alpha(\bar{x})) = \dots = \sigma(\alpha^{p-1}(\bar{x}))$) for all nonvertices $\bar{x}$ of $\bar{X}_\beta$ or orientation reversing ($\sigma(\bar{x}) = -\sigma(\alpha(\bar{x})) = \dots = \sigma(\alpha^{p-2}(\bar{x})) = -\sigma(\alpha^{p-1}(\bar{x}))$) for all nonvertices $\bar{x}$ of $\bar{X}_\beta$; that the integral equation is also regular is understood.

In place of the third Noether theorem, an index theorem will be presented which gives a formula for the difference between the dimensions of the solution spaces under the assumptions a) and

b') α is orientation preserving on S ;

c) The integral equation is Ω -degenerate.

The integral equation adjoint to (1.1) is defined to be

$$(1.1)^* \quad \sum_{\nu=0}^{p-1} \tau^\nu \left\{ {}^t A^{p-\nu}(x_1) \eta(\alpha^\nu(x_1)) \right. \\ \left. + (\pi i)^{-1} \int_{(S, \Sigma)} {}^t K^{p-\nu}(x_2, x_1) \Omega(x_1, x_2) \eta(\alpha^\nu(x_2)) \right\} = 0,$$

where $\eta(x_2)$ is a differential form on S and $\tau = 1$ (resp. -1) whenever α is orientation preserving (resp. reversing).

The most straightforward way to establish the first two Noether theorems for equations (1.1) and (1.1)* under assumptions a) and b) is essentially due to Kordzadze ([2]) (the first Noether theorem for Ω -degenerate equations with shift is contained in proposition (1.8)). To both (1.1) and (1.1)* are associated singular integral equations without shifts which are adjoint to one another as follows. For every $\rho = 0, \dots, p-1$ equation (1.1) yields

$$(2.1)_\rho \quad \sum_{\nu=0}^{p-1} \left\{ A^\nu(\alpha^{\nu+\rho}(x_1)) f(\alpha^{\nu+\rho}(x_1)) \right. \\ \left. + (\pi i)^{-1} \int_{(S, \Sigma)} K^\nu(\alpha^{\nu+\rho}(x_1), x_2) f(x_2) \Omega(x_2, \alpha^{\nu+\rho}(x_1)) \right\} = g(\alpha^\rho(x_1));$$

note that $(2.1)_0$ is the same as (1.1). Rewrite this equation as follows.

$$(2.2) \quad \sum_{\nu=0}^{p-1} \left\{ A^\nu(\alpha^{\nu+\rho}(x_1)) f(\alpha^{\nu+\rho}(x_1)) + \tau^{\nu+\rho} (\pi i)^{-1} \int_{(S, \Sigma)} K^\nu(\alpha^{\nu+\rho}(x_1), \alpha^{\nu+\rho}(x_2)) \cdot \right. \\ \left. f(\alpha^{\nu+\rho}(x_2)) \Omega(\alpha^{\nu+\rho}(x_2), \alpha^{\nu+\rho}(x_1)) \right\} = g(\alpha^\rho(x_1)).$$

In view of condition a), $\Omega(\alpha^\nu(x_2), \alpha^\nu(x_1))$ as well as $\Omega(x_2, x_1)$ is a meromorphic section in the line bundle $K \hat{\otimes} 1$ on $X \times X$, where K is the canonical line bundle on X . In local bundle coordinates $(\Omega(x_2, x_1))^{-1}$ is a meromorphic function.

Clearly the collection of local meromorphic functions $(\Omega(\alpha^\nu(x_2), \alpha^\nu(x_1)))^{-1} \Omega(x_2, x_1)$ defines a global meromorphic function $\omega^\nu(x_2, x_1)$ on $X \times X$. From the general properties of $\Omega(x_2, x_1)$ it follows that $\omega^\nu(x_2, x_1)$ has no poles on $S \times S$. Given the representation of $\Omega(x_2, x_1)$ in local coordinates (see [8]), it is easy to check that $\omega^\nu(x_2, x_2) = 1$ for all $x_2 \in S$ and all $\nu = 0, \dots, p-1$.

Denote $K^\nu(\alpha^{\nu+\rho}(x_1), \alpha^{\nu+\rho}(x_2)) \omega^{\nu+\rho}(x_2, x_1)$ by $\tilde{K}^\nu(\alpha^{\nu+\rho}(x_1), \alpha^{\nu+\rho}(x_2))$. (2.2) can then be written as

$$(2.3)_\rho \quad \sum_{\nu=0}^{p-1} \left\{ A^\nu(\alpha^{\nu+\rho}(x_1)) f(\alpha^{\nu+\rho}(x_1)) + \tau^{\nu+\rho}(\pi i)^{-1} \int_{(S, \Sigma)} \tilde{K}^\nu(\alpha^{\nu+\rho}(x_1), \alpha^{\nu+\rho}(x_2)) f(\alpha^{\nu+\rho}(x_2)) \Omega(x_2, x_1) \right\} = g(\alpha^\rho(x_1)).$$

Denote by $A(x_1)$ the $pq \times pq$ matrix

$$(A(x_1))_{ai+\nu, ai+\mu} = (A^{p-i+i}(\alpha^i(x_1)))_{\nu, \mu};$$

define the $pq \times pq$ matrices $K(x_1, x_2)$ and $\tilde{K}(x_1, x_2)$ in precisely the same manner using $\tau^i K^{p-i+i}(\alpha^i(x_1), \alpha^i(x_2))$ and $\tau^i \tilde{K}^{p-i+i}(\alpha^i(x_1), \alpha^i(x_2))$ respectively. Let $\tilde{f}$ be the $pq \times 1$ matrix defined by

$$(\tilde{f}(x_1))_{ai+\nu} = (f(\alpha^i(x_1)))_\nu.$$

Define the $pq \times 1$ matrix $\tilde{g}(x)$ in the same manner using $g(x)$ in place of $f(x)$. The p equations (2.3) _{ρ} can now be written in the form

$$(2.3) \quad A(x_1) \tilde{f}(x_1) + (\pi i)^{-1} \int_{(S, \Sigma)} \tilde{K}(x_1, x_2) \tilde{f}(x_2) \Omega(x_2, x_1) = \tilde{g}(x_1).$$

Since $\omega^\nu(x, x) = 1$ for all $x \in S$, $\tilde{K}(x, x) = K(x, x)$. The usual regularity conditions for (2.3) are

$$(2.4) \quad \det(A(x) \pm K(x, x)) \neq 0 \text{ for all } x \in S.$$

If α is direction preserving, then (2.4) is precisely the condition that (1.1) be regularizable relative to the system $\{\alpha_\lambda^\rho = \bar{\alpha}^\rho | \sigma_\lambda^1 : 0 \leq \rho \leq p-1, \lambda \in \Lambda\}$ (see (1.9)); if α is direction reversing, (2.4) together with $\det K_+(x) \neq 0$ is that condition. In either case, $\det(A(x) - K(x, x)) \cdot \det(A(x) + K(x, x))^{-1}$ is equal to one of $\det(1 - T^\pm(x)) \cdot \det(1 - T^\mp(x))^{-1}$, where $\tilde{x} \in \text{pr}^{-1}(x)$.

If one goes through the same construction for (1.1)*, the result is the singular integral equation

$$(2.3)^* \quad {}^i A(x_1) \tilde{\eta}(x_1) + (\pi i)^{-1} \int_{(S, \Sigma)} {}^i \tilde{K}(x_1, x_2) \Omega(x_1, x_2) \tilde{\eta}(x_2) = 0,$$

where $\tilde{\eta}(x_2)$ is the $pq \times 1$ matrix of differential forms defined by

$$(\tilde{\eta}(x_2))_{ai+\nu} = ((-1)^i \alpha(\eta^i(x_2))).$$

The following proposition can be found in [2]. The proof is straight-forward.

(2.5) Proposition. *If f (resp. $\tilde{f}$) is a Hölder continuous solution of the integral equation (1.1) (resp. (1.1)*), then $\tilde{f}$ (resp. $\tilde{\eta}$) is a Hölder continuous solution of (2.3) (resp. (2.3)*). Conversely, if $F(x) = {}^t(F_0(x), \dots, F_{p-1}(x))$ (resp. $H(x) = {}^t(H_0(x), \dots, H_{p-1}(x))$) is a Hölder continuous solution of (2.3) (resp. (2.3)*), where each F_ν and each H_μ are $q \times 1$ matrices, then $f(x) = p^{-1} \sum_{\nu=0}^{p-1} \tau^\nu F_\nu(\alpha^{p-\nu}(x))$ (resp. $\eta(x) = p^{-1} \sum_{\mu=0}^{p-1} \tau^\mu H_\mu(\alpha^{p-\mu}(x))$) is a Hölder continuous solution of (1.1) (resp. (1.1)*).*

These relations define surjections from the spaces of Hölder continuous solutions of equations (2.3) and (2.3)* to the spaces of Hölder continuous solutions of (1.1) and (1.1)* respectively. Therefore, the results in [8] for singular integral equations without shift yield

(2.6) Corollary. *Every integrable solution of (1.1) and (1.1)* is Hölder continuous.*

(2.7) Corollary (Noether theorem 1). *If (2.3) is a regular equation, then the spaces of Hölder continuous solutions of the homogeneous equation (1.1) and of equation (1.1)* are finite dimensional vectorspaces over $\mathbb{C}$.*

(2.8) Corollary (Noether theorem 2). *If (2.3) is a regular equation, then the inhomogeneous equation (1.1) has a solution if and only if for every solution η of the adjoint equation (1.1)*,*

$$(2.9) \quad \int_{(S, \Sigma)} {}^t\eta(x)g(x) = 0$$

holds, i.e. (1.1) is normally solvable.

Proof. If the integral equation (1.1) with right hand side $g(x)$ has a solution, then so does (2.3) with right hand side $\tilde{g}$. Therefore

$$\begin{aligned} 0 &= \int_{(S, \Sigma)} {}^t\tilde{\eta}(x)\tilde{g}(x) = \sum_{\nu=0}^{p-1} \int_{(S, \Sigma)} \tau^\nu \cdot {}^t\eta(\alpha^\nu(x))g(\alpha^\nu(x)) \\ &= \sum_{\nu=0}^{p-1} \int_{(S, \Sigma)} {}^t\eta(x)g(x) = p \cdot \int_{(S, \Sigma)} {}^t\eta(x)g(x) \end{aligned}$$

for every solution η of equation (1.1)*. On the other hand, for any solution $H(x)$ of equation (2.3)*,

$$\int_{(S, \Sigma)} {}^tH(x)\tilde{g}(x) = \sum_{\nu=0}^{p-1} \int_{(S, \Sigma)} {}^tH_\nu(x)g(\alpha^\nu(x)) = \int_{(S, \Sigma)} {}^t\left\{\sum_{\nu=0}^{p-1} \tau^\nu H_\nu(\alpha^{p-\nu}(x))\right\}g(x).$$

Therefore, since the map in proposition (2.5) is a surjection, if (2.9) holds for every solution η of (1.1)*, $\int_{(S, \Sigma)} {}^tH(x)\tilde{g}(x) = 0$ for every solution H of (2.3)*. Thus (2.3) has a solution for the right hand side $\tilde{g}$, and so (1.1) has a solution for the right hand side g .

Now for the third Noether theorem. If the integral equation (1.1) is Ω -degener-

ate, so is the adjoint equation (1.1)*. Therefore the technique in §1 can be applied to equation (1.1)* in order to associate with it a holomorphic vectorbundle over X . For this purpose let η be a solution of the equation (1.1)* and form for every pair ρ, μ

$$(1.3)^* \quad (\Omega \eta_\mu^\rho)(x_2) = (2\pi i)^{-1} \int_{(S, \Sigma)} B_\mu^\rho(x_1) \Omega(x_2, x_1) \eta(\alpha^\rho(x_1)), \quad x_2 \in X \setminus S.$$

$(\Omega \eta_\mu^\rho)(x_2)$ is a $q \times 1$ matrix of meromorphic differential forms on $X \setminus S$ with divisor $\geq D - *$.

Due to assumption a) of this section, there is no loss in generality in assuming that the simplicial structure $\{\sigma_\lambda^1 : \lambda \in \Lambda\}$ coincides with the minimal simplicial structure on $\bar{X}_\beta$. Let

$$\begin{aligned} (K_+^*(x))_{qi+\nu, qi+\mu} &= (\tau^{p-i+i} \cdot {}^t K_+^{i-i}(\alpha^i(x)))_{\nu, \mu} \\ (\tilde{B}_{\mu_0}^{\rho*}(x))_{\nu, \nu i+\mu} &= (\tau^{p-i} \cdot {}^t C_\mu^i(\alpha^\rho(x)))_{\nu, \mu} \\ (\tilde{B}_{\mu_0}^{\rho*}(x))_{\nu i+\nu, \nu i+\mu} &= \begin{cases} 0 & i \neq \rho \\ (\tilde{B}_{\mu_0}^{\rho*}(x))_{\nu, \nu i+\mu}, & i = \rho \end{cases} \\ \hat{C}_\mu^{\rho*}(x) = \left\{ \bigoplus_{\nu=0}^{p-1} {}^t B_\mu^\nu(\alpha^\rho(x)) \right\} P^\rho, \sigma^{\rho*}(\bar{x}) &= - \bigoplus_{\nu=0}^{p-1} 1_a \otimes \sigma(\alpha_\lambda^\rho(\bar{x})) \\ (\bar{\Omega} \eta)(\bar{x}) &= \bigoplus_{\rho=0}^{p-1} \bigoplus_{\mu=1}^n \bigoplus_{\delta=0}^{p-1} (\bar{\Omega} \tilde{\eta}_\mu^\delta)(\alpha_\lambda^\rho(\bar{x})). \end{aligned}$$

Consider the transmission problem

$$(1.7)' \quad (1 - R^-(\bar{x}))(\bar{\Omega} \eta)(\beta \bar{x}) = (1 - R^+(\bar{x}))(\bar{\Omega} \eta)(\bar{x}),$$

where $R^\pm(\bar{x}) =$

$$\left[\bigoplus_{\rho=0}^{p-1} \bigoplus_{\mu=1}^n \sigma^{\rho*}(\bar{x}) \hat{C}^{\rho*}(\text{pr } \bar{x}) \right] [{}^t \tilde{1}_{np} \otimes K_+^{*-1}(\text{pr } \bar{x}) \otimes \tilde{1}_{np}] \cdot \left[\bigoplus_{\delta=0}^{p-1} \bigoplus_{\nu=1}^n \tilde{B}_{\nu}^{\delta*}(\text{pr } \bar{x}) (\sigma^{\delta*}(\bar{x}) \pm 1) \right].$$

Under the general assumption

1)* $B_\mu^\nu(x_2) = 0$ at each vertex x_2 of the minimal simplicial structure on S for all $\nu = 0, \dots, p-1$ and all $\mu = 1, \dots, n$, one has

(2.10) Proposition. *There is a surjection η'_0 from the space of meromorphic solutions of the transmission problem*

$$(2.11) \quad E(\beta \bar{x}) = R(\text{pr } \bar{x})^{-\sigma(\bar{x})} E(\bar{x})$$

(see the definition of T in (1.12) for the definition of R), where

$$(E(\bar{x}))_{\nu} = \left(\bigoplus_{\rho=0}^{p-1} \bigoplus_{\mu=1}^n \bigoplus_{\delta=0}^{p-1} E_{\mu,\rho}^{\delta}(\bar{x}) \right)$$

is a $np^2q \times 1$ matrix of meromorphic differential forms on $\text{pr}^{-1}(X \setminus S)$ with continuous extensions to $\bar{X}$ such that

$$\text{divisor } \hat{E}_{\mu,\rho}^{\delta} \geq \alpha^{\rho}(D - *)$$

(see the definition of $\hat{G}$), onto the space of continuous solutions of the integral equation (1.1)*.

(2.12) Corollary. *There is a holomorphic vectorbundle $\tilde{V}$ over X constructed for (2.11) in the same way that $\tilde{W}$ was constructed for (1.12) such that there is a surjection from the space $H^0(X, \tilde{V} \otimes K)$ to the space of Hölder continuous solutions of (1.1)*.*

In this same direction one has from §1

(1.17)* Corollary. *If $g(x) \equiv 0$, then there is a surjection η_0^* from $H^0(X, \tilde{W}^* \otimes K)$ to the space of meromorphic solutions of the transmission problem*

$$(1.12)^* \quad \tilde{E}(\beta\bar{x}) = {}^tT(\text{pr } \bar{x})^{-\sigma(\bar{x})} \tilde{E}(\bar{x}),$$

where $(\tilde{E}(\bar{x}))_{\nu} = (\bigotimes_{\rho=0}^{p-1} \bigotimes_{\mu=1}^n \bigotimes_{\delta=0}^{p-1} \tilde{E}_{\mu,\rho}^{\delta}(\bar{x}))_{\nu}$, is a $np^2q \times 1$ matrix of meromorphic differential forms on $\text{pr}^{-1}(X \setminus S)$ with continuous extension to $\bar{X}$ such that

$$\text{divisor } \hat{\tilde{E}}_{\mu,\rho}^{\delta} \geq \alpha^{\rho+\delta}(D - *).$$

These results imply that whenever there is an isomorphism between the solution space of (2.11) as described in proposition (2.10) and the solution space of (1.12)* as described in Corollary (1.17)*, as well as one from $\ker \eta_0^*$ to $\ker \eta_0^*$, the index can be computed in terms of the Riemann-Roch Theorem for holomorphic vectorbundles over compact Riemann surfaces ([5]). These isomorphisms can be established whenever α is orientation preserving. In this case

$${}^tT(x) = 1 - 2 \left[\bigoplus_{\rho=0}^{p-1} \bigoplus_{\mu=1}^n {}^t\tilde{B}_{\mu}^{\rho}(x) \right] [{}^t\tilde{I}_{np} \otimes {}^tK_+^{-1}(x) \otimes I_{np}] \left[\bigoplus_{\delta=0}^{p-1} \bigoplus_{\nu=1}^n {}^t\tilde{C}_{\nu}^{\delta}(x) \right],$$

while

$$R(x) = 1 - 2 \left[\bigoplus_{\rho=0}^{p-1} \bigoplus_{\mu=1}^n \hat{C}_{\mu}^{\rho*}(x) \right] [{}^t\tilde{I}_{np} \otimes {}^tK_+^{-1}(x) \otimes I_{np}] \left[\bigoplus_{\delta=0}^{p-1} \bigoplus_{\nu=1}^n \hat{B}_{\nu}^{\delta*}(x) \right].$$

Let $\mathcal{P}$ be the $np^2q \times np^2q$ permutation matrix which has the effect, when operating from the right on a matrix of the same dimension, of replacing the $(p(\mu + \rho n - 1) - \delta + 1)$ th column with the $(p(\mu + (\rho - \delta)n - 1) + \delta + 1)$ th column if $\rho \geq \delta$ or with the $(p(\mu + (p - \delta + \rho)n - 1) + \delta + 1)$ th column if $\rho < \delta$ for all $\mu = 1, \dots, n$ and all $\rho, \delta = 0, \dots, p - 1$. It can be shown that

$$(2.13) \quad {}^tT(x) = [\mathcal{P} \otimes I_q]^{-1} R(x) [\mathcal{P} \otimes I_q]$$

for all x . An immediate consequence of this relation is

(2.14) Proposition. *The vectorbundles $\tilde{V}$ and $\tilde{W}^*$, as well as V_k and W_k^* are isomorphic.*

This establishes the above isomorphisms. Computing as in [8], one has

(2.15) Proposition. *Let $\tilde{W}$ be the vectorbundle associated with the homogeneous Ω -degenerate integral equation (1.1) as in Theorem (1.25), where α satisfies a) and b)' above. Then*

$$\deg \tilde{W} = (g - 1)np^2q + (2\pi)^{-1} \sum_{\lambda=1}^k \sigma^\lambda \arg_{e^\lambda} (\det K_- \cdot \det K_+^{-1}),$$

where g is the genus of X .

Let L (resp. L^*) denote the space of all solutions of the homogeneous Ω -degenerate integral equation (1.1) (resp. (1.1)*). Then from propositions (2.14) and (2.15) and the Riemann–Roch Theorem one has

(2.16) Theorem (Index Theorem). *If α satisfies conditions a) and b)', then*

$$\dim L - \dim L^* = \sum_{\lambda=1}^k \sigma^\lambda \arg_{e^\lambda} (\det K_- \cdot \det K_+^{-1}) - \dim \ker \eta_0 + \dim \ker \eta_0^*.$$

In the case $X = \mathbf{P}^1$, S is a simple, closed, oriented, $C^{1+\alpha}$ curve, α is an orientation preserving $C^{1+\alpha}$ homeomorphism of S onto itself, and $q = 1$, A. Kordzadze has published proofs of these theorems ([2], [3]). The index theorem proposed in [2] turned out to be incorrect, and so in [3] the correct, modified formula is presented.

If X , S and q are as above but α is not necessarily orientation preserving, and if $n = 1$, $p = 2$, and $C_1^0(x) = C_1^1(x) \equiv 1$, the proofs of these theorems have been announced by G. S. Litvinčuk ([4]). For integral equations which satisfy the conditions in both [3] and [4], the two definitions of the adjoint equation agree. Also agreeing are the index formulas if one takes Kordzadze's uncorrected version in [2].

The relation between the results in this section and those in [2] and [3] is as follows. For integral equations which satisfy the conditions of all three, the definitions of regularity relative to the system $\{\alpha_\lambda^\rho = \bar{\alpha}^\rho \mid \sigma_\lambda^1 : 0 \leq \rho \leq p - 1, \lambda \in \Lambda\}$ in §1 and of regularity in [2] are the same, as are the definitions of the adjoint equation. The index formulas agree if one takes Kordzadze's modified version in [3].

The relations with [4] are just as those above in the case when α is orientation preserving. When α is orientation reversing, the adjoint equation in [4] has been generalized to obtain the second Noether theorem in this section. However, the index formula in [4] is in general incorrect, even if correction factors of the type in [3] or Theorem (2.16) are included.

A counterexample to the index formula in [4] for orientation reversing shifts is as follows. Let $X = \mathbf{P}^1$ and S be the boundary of the unit disc with the positive orientation, and consider the singular integral equation

$$(2.17) \quad x_1^{-\kappa} f(x_1) + 2f(-x_1^{-1}) - (\pi i)^{-1} x_1^{-\kappa} \int_{(S, \Sigma)} f(x_2)(x_2 - x_1)^{-1} dx_2 = 0,$$

where f is a complex-valued function on S and κ is a positive, even integer.

In the notation of §1, $n = q = 1$ and $p = 2$, $\alpha(x) = -x^{-1}$, $A^0(x) = x^{-\kappa}$, $A^1(x) = 2$, $B_1^0(x) = -x^{-\kappa}$, $C_1^0(x) = 1$, $B_1^1(x) = C_1^1(x) = 0$. It is not difficult to compute that (2.17) is regularizable relative to the system of maps induced by α and that the transmission matrix in (1.12) is given by

$$T(x) = \begin{pmatrix} 0 & 0 & -x^{\kappa} & 0 \\ 0 & 1 & 0 & 0 \\ x^{-\kappa} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x \in \partial D.$$

If U_1 is the unit disc D and $U_{\infty} = \{x : x \in \mathbf{P}^1, x \neq 0\}$, the vectorbundle W in corollary (1.17) is given by $\{U_1, U_{\infty}\}$ -cocycle

$$g_{1\infty}(x) = \begin{pmatrix} 0 & 0 & -x^{\kappa} & 0 \\ 0 & 1 & 0 & 0 \\ x^{-\kappa} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x \in U_1 \cap U_{\infty}.$$

Let

$$\lambda_1(x) = \begin{pmatrix} 1 & 0 & x^{\kappa} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \lambda_{\infty}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -x^{-\kappa} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

λ_1 is holomorphic in U_1 , λ_{∞} is holomorphic in U_{∞} , and $\det \lambda_1(x) = \det \lambda_{\infty}(x) = 1$. Moreover, $\lambda_1(x)g_{1\infty}(x)\lambda_{\infty}(x) = 1$, and so W is the trivial bundle on $\mathbf{P}^1$. Therefore, the only holomorphic section in $\tilde{W}$ is the zero section, and so the only solution of equation (2.17) is the trivial one.

Litvinčuk claims that the index of equation (2.17) is equal to the index of the corresponding singular integral equation (2.3), which in turn is equal to $\dim_{\mathbb{C}} H^0(\mathbf{P}^1, \tilde{W}) - \dim_{\mathbb{C}} H^0(\mathbf{P}^1, \tilde{W}^* \otimes K)$. This means that, since in this case $\dim_{\mathbb{C}} \ker \eta'_0 = \dim_{\mathbb{C}} \ker \eta_0 = 0$ (see Proposition (1.18)), there must be a bijection from $H^0(\mathbf{P}^1, \tilde{V} \otimes K)$ to $H^0(\mathbf{P}^1, \tilde{W}^* \otimes K)$, where $\tilde{V}$ is the vectorbundle in Corollary (2.12).

Now the equation adjoint to (2.17) is

$$(2.17)^* \quad x_1^{-\kappa} \eta(x_1) - 2\eta(-x_1^{-1}) - (\pi i)^{-1} \int_{(S, \Sigma)} x_2^{-\kappa} \eta(x_2) (x_1 - x_2)^{-1} dx_1 = 0.$$

It too is regularizable relative to the system of maps induced by α . The transmission matrix of (2.11) is given by

$$R(x) = \begin{bmatrix} 0 & 0 & x^{-\kappa} & 0 \\ 0 & 1 & 0 & 0 \\ -x^{\kappa} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}, \quad x \in \partial D.$$

The vectorbundle V in Corollary (2.12) is given by the cocycle

$$h_{1\infty}(x) = \begin{bmatrix} 0 & 0 & x^{-\kappa} & 0 \\ 0 & 1 & 0 & 0 \\ -x^{\kappa} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad x \in U_1 \cap U_{\infty}.$$

Let

$$\mu_1(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -x^{\kappa} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mu_{\infty}(x) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$\mu_1(x)$ is holomorphic in U_1 and $\det \mu_1(x) = \det \mu_{\infty}(x) = 1$. Moreover,

$$\mu_1(x)h_{1\infty}(x)\mu_{\infty}(x) = \begin{bmatrix} x^{-\kappa} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x^{\kappa} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad x \in U_1 \cap U_{\infty}.$$

Therefore, $H^0(\mathbf{P}^1, \tilde{W})$ must be isomorphic to $H^0(\mathbf{P}^1, \tilde{V}^*)$. $\tilde{V}^*$ is given by the $\{U_1, U_{\infty}\}$ -cocycle

$$h_{1\infty}^*(x) = \begin{bmatrix} x^{\kappa+1} & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x^{-\kappa-1} & 0 \\ 0 & 0 & 0 & x \end{bmatrix}, \quad x \in U_1 \cap U_{\infty}.$$

But then $\tilde{V}^*$ has $\kappa + 4$ linearly independent global holomorphic sections, and so there can be no such isomorphism.

On the other hand, let D_r be the disc of radius r in $\mathbf{C}$ centered at the origin and replace $S = \partial D$ with $S = \partial D_r \cup \partial D_{1/r}$ ($r \neq 1$) in the above equations, S oriented so that $\alpha(x) = -x^{-1}$ is orientation preserving. Then the equations (2.17) and the adjoint

$$(2.17)^{*} \quad x_1^{-\kappa} \eta(x_1) + 2\eta(-x_1^{-1}) - (\pi i)^{-1} \int_{(S, \Sigma)} x_2^{-\kappa} \eta(x_2) (x_1 - x_2)^{-1} dx_1 = 0$$

have the same number of linearly independent Hölder continuous solutions, *i.e.* the index computed by Litvinchuk's formulas (or Theorem (2.16)) is correct.

Remark. The question naturally arises as to how restrictive are the conditions on the shift maps $\{\alpha^\rho : \rho = 0, \dots, p-1\}$ in §2. If the genus of the compact Riemann surface X is either 0 or 1, then X admits infinitely many holomorphic automorphisms, some of which generate finite groups of automorphisms. If the genus of X is ≥ 2 , the group of holomorphic automorphisms of X is finite. Let $N(g)$ be the order of the largest group of automorphisms that a Riemann surface of genus $g \geq 2$ can admit. R. Accola ([10]) and C. Maclachlan ([11]) have shown that $N(g) \geq 8(g+1)$. If S_0 is a $C^{1+\alpha}$ 1-complex on X and $\{\alpha^\rho : \rho = 0, \dots, p-1\}$ is a finite group of holomorphic automorphisms of X , then $S = \bigcup \{\alpha^\rho(S_0) : \rho = 0, \dots, p-1\}$ can clearly be given the structure of a $C^{1+\alpha}$ 1-complex on X . Moreover, $\alpha^\rho(S) = S$ for every $\rho = 0, \dots, p-1$. Therefore, for every integer $g \geq 0$ there is at least one compact Riemann surface X of genus g , finite group $\{\alpha^\rho : \rho = 0, \dots, p-1\}$ and $C^{1+\alpha}$ 1-complex on X satisfying the conditions in §2.

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