Variational Properties of Nonlinear Spectra

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1. Introduction. It can be quickly shown that the eigenvalues of a self-adjoint linear operator on a real or complex inner product space are the stationary values of the Rayleigh functional of the operator. Minimax characterizations of stationary values of real functionals are known in great generality. This is a product of Morse's theory not often expounded [1].

Briefly and intuitively, the idea is the following. Consider a real-valued function ρ defined on a domain which can fall in a variety of spaces but which we will take as the Euclidean plane with the usual x - y coordinates. Now let us imagine the origin to be an isolated stationary point of ρ and let us credit ρ with having continuous derivatives there. Our attention is now restricted to a circular neighborhood N of the origin in which the origin is the only stationary point. The subsets N^+ on which $\rho > \rho(0, 0)$ and N^- on which $\rho < \rho(0, 0)$ are disjoint subsets of N . If N is small enough, each connected component of N^+ and each connected component of N^- will have $(0, 0)$ as a boundary point and will have boundary points in common with N . Except in those degenerate cases in which the Hessian $\rho_{xx}\rho_{yy} - \rho_{xy}^2$ is zero, N^+ and N^- are composed of at least two disjoint connected components each. A graphic representation of such a situation is given in the diagram.

With the structure described in the preceding paragraph before us, we are in a position to give a minimax property of the stationary point $(0, 0)$ of ρ . Let P^+ (respectively, P^-) be the family of all rectifiable arcs in N whose ends are in distinct connected components of N^+ (respectively, N^-). Then the stationary value

$$\rho(0, 0) = \inf_{A \in P^-} \max_{(x, y) \in A} \rho(x, y) = \sup_{A \in P^+} \min_{(x, y) \in A} \rho(x, y).$$

Such a relationship is not possible for non-stationary points. This development finds its natural setting and generalization in algebraic topology [1] where our arcs become cycles of fixed dimension.

Thus knowing the eigenvalues of self-adjoint operators to be the stationary values of the real-valued Rayleigh functional, we can expect them to possess minimax characterizations. Indeed this is a classical result whose roots lie in work of Rayleigh [2], Poincaré [3], Fischer [4], Weyl [5], and others. However, a

Diagram for "Variational Properties of Nonlinear Spectra" by E. H. Rogers

striking simplification presents itself in this case. This is in the replacement of the cycles of the Morse minimax relation with subspaces of fixed dimension. Moreover, the analysis is not localized to a small neighborhood of the stationary point but is global.

These admirable minimax traits of eigenvalues are possessed by the stationary values of a much wider class of functionals. Among these are Rayleigh functionals for a class of nonlinear eigenvalue problems. Those quantities $x = \lambda$ causing a parameter-dependent operator L_x to be noninvertible are referred to as *nonlinear eigenvalues*. If L_λ is known simply to have no continuous inverse, then λ is said to be in the *nonlinear spectrum* of L_x . The classical, linear eigenvalue problem has $L_x = xI - A$ and the spectrum is attributed to A .

A minimax theory for nonlinear eigenvalues has been exhibited by Duffin [6, 7], Rogers [8, 9], and Turner [10]. Duffin and Turner have explored the case of L_x quadratic in x for applications which are finite and infinite dimensional, respectively. In [8] the finite dimensional case was presented for a wide class of L_x and here the theory for spaces of arbitrary dimension will be presented. Both were considered in [9], where Theorems 1 and 3 on minimax properties of the spectrum and an effect of constraints were developed. In addition, Theorem 2 demonstrates "inf sup = sup inf" dualities which replace a simpler duality on spaces of finite dimension.

Theorems 4, 5, 6 develop some special consequences of compactness of $I - L_x$.

In particular, excepting points of discontinuity of L_x in x , the nonlinear spectrum is shown to be a point spectrum whose members have finite multiplicity. Those eigenvalues in the range of the Rayleigh functional are shown to be minimax values.

2. A law of inertia and a fundamental lemma. Let W be a real or complex inner product space and let J be an interval of the reals. Now we consider a linear operator L_x which

- (a) is self-adjoint from W to W
- (b) is a continuous operator
- (c) has a derivative $L'_x = dL_x/dx$ which is both continuous in x and continuous on W

for all x in J . Even more must be demanded in producing a "Rayleigh functional". We assume that a continuous functional p with values in J may be defined on all of W (except usually 0) so that

$$(d) \quad (L_{p(u)}u, u) = 0$$

and

$$(e) \quad (L'_{p(u)}u, u) > 0$$

for all u . Noting

$$(L_x\alpha u, \alpha u) = |\alpha|^2(L_xu, u) \text{ and } (L'_{p(u)}\alpha u, \alpha u) = |\alpha|^2(L'_{p(u)}u, u) > 0,$$

we assume further that $p(\alpha u) = p(u)$ for all scalars $\alpha \neq 0$. As a result we may think of p as a functional on the unit sphere of W . This is often helpful.

From the identity (d) we can deduce

$$0 = (L'_{p(u)}u, u) \frac{\partial}{\partial \alpha} p(u + \alpha v) \Big|_{\alpha=0} + \overline{(L_{p(u)}u, v)}$$

and

$$0 = (L'_{p(u)}u, u) \frac{\partial}{\partial \bar{\alpha}} p(u + \alpha v) \Big|_{\alpha=0} + (L_{p(u)}u, v)$$

so that p is stationary at u if and only if $L_{p(u)}u = 0$ or $p(u)$, u is an eigenpair of L_x .

Our use of the quadratic form (L_xu, u) is eased by a form of Sylvester's Law of Inertia [11, 12]. In as much as this author has found this theorem only in conjunction with spaces of finite dimension, a proof will be given.

First, we define

$$C^+(x) = \{u \mid (L_xu, u) > 0\} \cup \{0\}$$

$$C^0(x) = \{u \mid (L_xu, u) = 0\}$$

$$C^-(x) = \{u \mid (L_xu, u) < 0\} \cup \{0\}$$

and

$c(x)$ = the largest of the dimensions of the subspaces of W contained in $C^+(x)$
 $d(x)$ = the largest of the dimensions of the subspaces of W contained in $C^-(x)$.

Morse theory would have us call $d(x)$ the "index" of x while this would be a violation of the algebraic use of the word [12]. The term "signature" is usual in algebraic contexts [13]. We will refer to $d(x)$ as the *signature* and to $c(x)$ as the *co-signature* of L_x . A duality between C^+ and C^- will be observed in the material to follow. Verbal excess will be avoided by treating only one case in detail.

We will say that vectors u, v are x -orthogonal if $(L_x u, v) = 0$.

Law of Inertia. *If $c(x)$ (respectively, $d(x)$) is finite, then W has a decomposition of the form $X \oplus Y \oplus Z$ where*

- (i) $X \subset C^+(x)$, $Y \subset C^0(x)$, $Z \subset C^-(x)$
- (ii) X, Y, Z are x -orthogonal subspaces
- (iii) $\dim X = c(x)$ (resp., $\dim Z = d(x)$).

A lemma is convenient here.

Lemma 1. *If $e_1, \dots, e_n$ is any family of linearly independent vectors which generate a subspace $[e_1, \dots, e_n]$ in $C^+(x)$ (resp., in $C^-(x)$), then there is a unique subspace U of W which is x -orthogonal to the space $[e_1, \dots, e_n]$ and satisfies $W = [e_1, \dots, e_n] \oplus U$.*

Proof. Observe that L_x is positive definite on $[e_1, \dots, e_n]$ so that a Gram-Schmidt orthonormalization can convert the e_i into e'_i obeying $(L_x e'_i, e'_i) = \delta_{ii}$. Now we define

$$U = \{u \mid (L_x u, e'_i) = 0, 1 \leq i \leq n\}.$$

Clearly U is a subspace of W . It is independent of $e_1, \dots, e_n$ for if $w = u + \sum_{j=1}^n \beta_j e'_j = 0$ for some u in U , then $\beta_j = (L_x w, e'_j) = 0$ for all j and thus $u = 0$ as well. In fact, if w is any member of W , then $w - \sum_{j=1}^n (L_x w, e'_j) e'_j$ is x -orthogonal to $[e_1, \dots, e_n]$ and so is in U .

Should V be another subspace of W , x -orthogonal to $[e_1, \dots, e_n]$ and satisfying $W = [e_1, \dots, e_n] \oplus V$, then $V \subset U$. If u is in U but not in V , then u has a unique decomposition as $v + f$ where v is in V and f is in $[e_1, \dots, e_n]$. Now since u is not in V , $(L_x u, f) \neq 0$ and since u is in U , $(L_x u, f) = 0$. This contradiction implies the uniqueness of U .

Having proved the lemma, we turn to the proof of the Law of Inertia.

Let X be a subspace of $C^+(x)$ of dimension $c(x)$. Then by the lemma $W = X \oplus U$ where U is a unique subspace, x -orthogonal to X . Let $Y = U \cap C^0(x)$ and let u, v be in Y . If $(L_x u, v) \neq 0$, then $u + (L_x u, v)v$ is in $C^+(x)$ and is x -orthogonal to X . Consequently, $X \oplus [u + (L_x u, v)v]$ is a subspace of $C^+(x)$ of dimension $c(x) + 1$, a contradiction of the maximality of $c(x)$. Therefore $(L_x u, v) = 0$ for all u, v in Y . This leads us directly to the closure of Y under addition and scalar multiplication or that Y is a subspace of W . We may write

$U = Y \oplus Z$ where necessarily $Z \subset C^-(x)$. Finally, suppose u in Y and v in Z are not x -orthogonal. Then $u + \beta(L_x u, v)v$ is in $C^+(x)$ when $0 < \beta < -2/(L_x v, v)$, contradicting the maximality of $c(x)$.

Corollary A. *Subspace Y is the null space of L_x .*

Corollary B. *If $c(x)$ (resp., $d(x)$) is finite and V is a subspace of $C^+(x)$ (resp., $C^-(x)$), then there is a subspace U of dimension $c(x)$ (resp., $d(x)$) such that $V \subset U \subset C^+(x)$ (resp., $C^-(x)$).*

Proof. Since $\dim V \leq c(x) < \infty$, there is a unique x -orthogonal complement T such that $W = V \oplus T$. Applying the Law of Inertia to T , we may write $T = X \oplus Y \oplus Z$ or $W = (V \oplus X) \oplus Y \oplus Z$. Necessarily $U = V \oplus X \subset C^+(x)$ and $\dim U = c(x)$.

A lemma fundamental to the global minimax theory is next.

Lemma 2. *If x is a real number in the range of the functional p , then*

$$C^+(x) = \{u \mid p(u) < x\} \cup \{0\}$$

$$C^0(x) = \{u \mid p(u) = x\} \cup \{0\}$$

$$C^-(x) = \{u \mid p(u) > x\} \cup \{0\}.$$

Proof. Let u_0 be a non-zero member of $C^0(x)$. Suppose there is a u_2 in $C^+(x) \cup C^0(x)$ with $p(u_2) > x$. Now we let u_1 be u_2 if $(L_x u_0, u_2) = 0$ and $u_1 = (L_x u_0, u_2)u_2$ otherwise. Then

$$(L_x u_0, u_1) \geq 0, \quad p(u_1) = p(u_2) > x$$

and u_1 is in $C^+(x) \cup C^0(x)$.

The path $u_\beta = (1 - \beta)u_0 + \beta u_1$, real β in $0 \leq \beta \leq 1$, from u_0 to u_1 is in $C^+(x) \cup C^0(x)$ because

$$\begin{aligned} (L_x u_\beta, u_\beta) &= (1 - \beta)\beta[(L_x u_0, u_1) + (L_x u_1, u_0)] + \beta^2(L_x u_1, u_1) \\ &\geq (1 - \beta)\beta[(L_x u_0, u_1) + (u_1, L_x u_0)] \\ &= 2(1 - \beta)\beta(L_x u_0, u_1) \geq 0. \end{aligned}$$

Since $p(u_1) > x$, there is a largest β , call it γ , for which $p(u_\gamma) = x$. From the relations

$$\begin{aligned} (L_x u_\beta, u_\beta) &\geq 0, & (L_{p(u_\beta)} u_\beta, u_\beta) &= 0 \\ (L'_{p(u_\beta)} u_\beta, u_\beta) &> 0, & \text{and } p(u_\beta) &> x \end{aligned}$$

for $\gamma < \beta \leq 1$, we conclude that $(L'_x u_\beta, u_\beta) = 0$ for some x_β in $x < x_\beta < p(u_\beta)$. Now we observe

$$0 = \lim_{\beta \rightarrow \gamma^+} (L'_{x_\beta} u_\beta, u_\beta) = (L'_x u_\gamma, u_\gamma)$$

in contradiction of hypothesis (e).

Consequently, $p(u) \leq 0$ for u in $C^+(x) \cup C^0(x)$. In a similar fashion $p(u) \geq 0$ for u in $C^0(x) \cup C^-(x)$.

3. A variational principle. Those values of x for which L_x does not have a continuous inverse on its range compose the *nonlinear spectrum* of L_x .

Lemma 3. *If for some x the co-signature $c(x)$ (resp., the signature $d(x)$) is finite, then for all $j > c(x)$ (resp., $j > d(x)$)*

$$k_j = \inf_{\dim U=j} \max_{u \in U} p(u) \quad (\text{resp., } h_j = \sup_{\dim U=j} \min_{u \in U} p(u))$$

exists, $k_j \geq x$ (resp., $h_j \leq x$), and $j > c(k_j)$ (resp., $j > d(h_j)$).

Proof. Convenience will prompt the use of the notation

$$P(U) = \sup_{u \in U} p(u) \quad \text{and} \quad Q(U) = \inf_{u \in U} p(u).$$

Lemma 2 implies $P(U) \geq x$ for all subspaces U of dimension $j > c(x)$. But the continuity of p and finite dimensionality of U guarantee $P(U)$ finite. Thus k_j exists and $k_j \geq x$. Finally, $j > c(k_j)$ for otherwise a subspace U of dimension j would lie in $C^+(k_j)$ so that $k_j \leq P(U) < k_j$ by Lemma 2.

Now we are ready to show that the minimax values are in the nonlinear spectrum of L_x .

Theorem 1. *A finite real k equals k_j (resp., h_j) for some j if and only if $c(k)$ (resp., $d(k)$) is finite and*

$$\lim_{n \rightarrow \infty} L_k v_n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} p(v_n) = k$$

for a sequence of unit vectors v_n .

Proof. Let $k = k_j$. There must be a sequence U_n , $n = 1, 2, \dots$, of j dimensional subspaces of W such that $\lim P(U_n) = k$. Suppose that k is not an eigenvalue. Then $W = X \oplus Z$ where $X \subset C^+(k)$, $Z \subset C^-(k)$ and X and Z are k -orthogonal. Let u_{nm} , $m = 1, \dots, j$, be a basis for U_n . For each n and m there is a decomposition $u_{nm} = v_m + w_m$ where v_m is in X and w_m is in Z . Because $\dim X = c(k) < j$, the v_m , $m = 1, \dots, j$, are dependent and there are scalars β_m , not all zero, such that

$$\sum_{m=1}^j \beta_m u_{nm} = \sum_{m=1}^j \beta_m w_m \in Z.$$

Thus for each n , a unit vector u'_n may be found in $U_n \cap Z$. Since $k < p(u'_n) \leq P(U_n)$, $\lim p(u'_n) = k$ and $\lim (L_k u'_n, u'_n) = 0$.

We write $L_k u'_n = w'_n + w''_n$, w'_n in X and w''_n in Z . If $\|L_k u'_n\|^2 = (L_k u'_n, w'_n)$ does not tend to zero as n becomes infinite, then for some $\rho > 0$ a subsequence satisfies

$$(L_k w''_n, w''_n) \geq \rho > 0, \quad \text{where} \quad u''_n = \exp[-i \arg (L_k u'_n, w''_n)] u'_n.$$

If $(L_k w_n'', w_n'')$ is bounded, we may observe that the term $2\beta(L_k u_n'', w_n'')$ dominates

$$(L_k(u_n'' + \beta w_n''), u_n'' + \beta w_n'') = (L_k u_n'', u_n'') + 2\beta(L_k u_n'', w_n'') + \beta^2(L_k w_n'', w_n'').$$

Thus for β sufficiently small and all n sufficiently large, this is a positive quantity. Hence $u_n'' + \beta w_n''$ is in $C^+(k) \cap Z = \{0\}$. Consequently, $w_n'' = -u_n''/\beta$ and

$$|(L_k u_n'', w_n'')| = (L_k u_n'', u_n'')/\beta = (L_k u_n'', u_n'')/\beta.$$

The latter tends to zero in contradiction of our assumption that $(L_k u_n'', w_n'') \geq \rho > 0$.

If we show that $(L_k w_n'', w_n'')$ is in fact bounded, then the above contradiction will assure us that $\lim L_k u_n'' = 0$.

Suppose a subsequence of the $(L_k w_n'', w_n'')$ becomes infinite. Without changing notation we will consider this subsequence. The boundedness of the linear operator L_k then implies that $\|w_n''\| \rightarrow \infty$. But $\|L_k u_n''\|$ is bounded and $\|w_n''\| \geq \|L_k u_n''\| - \|w_n''\|$ so that $\|w_n''\| \rightarrow \infty$ as well. We define $a_n^{-1} = \|L_k u_n''\| + \|w_n''\| \geq \|w_n''\|$. Then $a_n w_n'' + a_n w_n'' = a_n L_k u_n'' \rightarrow 0$ while $\|a_n w_n''\| \rightarrow 1$ as $n \rightarrow \infty$. The continuity of p then assures us that $p(a_n w_n'') - p(a_n w_n'') \rightarrow 0$. But $p(a_n w_n'') < k < p(a_n w_n'')$ so that they have common limit k . However, this implies $Q(X) = k$ in spite of $\dim X = c(k)$ and $X \subset C^+(k)$ which implies $Q(X) < k$. Therefore, $(L_k w_n'', w_n'')$ must be bounded and sufficiency is proved.

To prove the converse, we suppose $c(k)$ finite and let the unit vectors v_n , $n = 1, 2, \dots$, obey $\lim L_k v_n = 0$ and $\lim p(v_n) = k$. Let X be a subspace of $C^+(k)$ of dimension $c(k)$. Since X is finite dimensional, $\beta = \inf_{0 \neq v \in X} (L_k v, v)/(v, v)$ is positive. For an integer m sufficiently large $\beta > (L_k v_n, v_n)$ for all $n \geq m$. Hence $U_n = X \oplus [v_n]$ has $c(k) + 1$ dimensions when $n \geq m$. Since $\lim L_k v_n = 0$,

$$(L_k(v + \alpha v_n), v + \alpha v_n) = (L_k v, v) + 2 \operatorname{Re} [\alpha (L_k v, v_n)] + |\alpha|^2 (L_k v_n, v_n)$$

becomes $(L_k v, v) \geq 0$ as $n \rightarrow \infty$. Thus $\lim P(U_n) = k$. Since every subspace of dimension $c(k) + 1$ meets $C^0(k)$, $k = \inf_{\dim U = c(k) + 1} \max_{u \in U} p(u)$, proving our theorem.

An immediate corollary is

Corollary. *If k is in the nonlinear spectrum of L_x and for some j*

$-\infty < k_j \leq k \leq k_{j+1} < +\infty$ (resp., $-\infty < h_{j+1} \leq k \leq h_j < +\infty$), then $c(k)$ (resp., $d(k)$) is finite and $k = k_{c(k)+1}$ (resp., $k = h_{d(k)+1}$).

If $W = U \oplus V$ and U is finite dimensional, then $\dim U$ is called the *codimension* of V . This definition is used to extend the spectral characterizations of Theorem 1 by a duality.

Theorem 2. *The quantity*

$$k'_i = \sup_{\operatorname{co-dim} V = i-1} \inf_{u \in V} p(u) \quad (\text{resp., } h'_i = \inf_{\operatorname{co-dim} V = i-1} \sup_{u \in V} p(u))$$

is finite if and only if k_i (resp., h_i) is finite. Moreover, when they exist, they are equal.

Proof. Let $W = U_{i-1} \oplus V_{i-1}$ with $\dim U_{i-1} = j - 1$ and $W = U_i \oplus V_i$ with $\dim U_i = j$. Let u_{ni} , $n = 1, \dots, j$, be a basis for U_i and $u_{ni} = u_{ni-1} + v_{ni-1}$ with u_{ni-1} in U_{i-1} and v_{ni-1} in V_{i-1} . There are scalars β_n , not all zero, such that $\sum_{n=1}^j \beta_n u_{ni-1} = 0$ so that $u = \sum \beta_n u_{ni} = \sum \beta_n v_{ni-1}$. The independence of the u_{ni} guarantees u to be non-zero and in $U_i \cap V_{i-1}$. Consequently, $Q(V_{i-1}) \leq P(U_i)$ for all subspaces V_{i-1} of co-dimension $j - 1$ and all subspaces U_i of dimension j . This immediately yields $k'_i \leq k_i$.

Suppose k'_i is finite. Then $c(k'_i) \leq j - 1$ for otherwise a U_i could be found in $C^+(k'_i)$ and $k'_i \leq P(U_i) < k'_i$ would follow. Let $c(k'_i) = j - i$, for a fixed $i \geq 1$.

If k'_i is an eigenvalue, then picking the v_n of Theorem 1 all equal to an eigenvector, it is immediate that $k'_i = k_i$. Thus we assume k'_i is not an eigenvalue.

Now the case $i = 1$ can be treated relatively simply. Let V_{ni-1} , $n = 1, 2, \dots$, be a sequence of subspaces of co-dimension $j - 1$ such that $Q(V_{ni-1}) \rightarrow k'_i$ as $n \rightarrow \infty$. Next we construct such a sequence of subspaces in $C^-(k'_i)$. The Law of Inertia is applied to each V_{ni-1} to yield $V_{ni-1} = X_{ni-1} \oplus Z_{ni-1}$ with $X_{ni-1} \subset C^+(k'_i)$ and $Z_{ni-1} \subset C^-(k'_i)$. Corollary B assures us that there is a subspace X_n in $C^+(k'_i)$ of dimension $j - 1$ and containing X_{ni-1} . Using this subspace, we express $W = X_n \oplus Z_n$ with $Z_n \subset C^-(k'_i)$. We observe that Z_n has co-dimension $j - 1$ and if $X_{ni-1} \neq \{0\}$,

$$Q(V_{ni-1}) \leq Q(X_{ni-1}) < k'_i \leq Q(Z_n)$$

(of course, if $X_{ni-1} = \{0\}$, then $V_{ni-1} = Z_n$). Therefore, we may replace the maximizing sequence $\{V_{ni-1}\}$ by $\{Z_n\}$.

From each Z_n a v_n may be extracted so that $|p(v_n) - Q(Z_n)| < 1/n$ and thus $p(v_n) \rightarrow k'_i$ as $n \rightarrow \infty$. The subspaces $U_n = X_n \oplus [v_n]$ are j -dimensional. For arbitrary members $u_n + \alpha v_n$, ($k = k'_i$)

$$(L_k(u_n + \alpha v_n), u_n + \alpha v_n) \rightarrow (L_k u_n, u_n) \geq 0$$

so that $p(u_n + \alpha v_n)$ is eventually less than $k'_i + \epsilon$ ($\epsilon > 0$ arbitrary) as $n \rightarrow \infty$. Thus $P(U_n) \rightarrow k'_i$ as $n \rightarrow \infty$ and because $k'_i \leq k_i \leq P(U_n)$, we have $k'_i = k_i$.

The extension of this argument to the case of any $i \geq 1$ requires that a sequence of subspaces U_n of dimension j be constructed from $X_n \subset C^+(k'_i)$ of dimension $j - i$ and k'_i -orthogonal vectors $v_{1n}, \dots, v_{in}$ with the property that $p(v_{mn}) \rightarrow k'_i$ as $n \rightarrow \infty$. The X_n are constructed just as when $i = 1$. To construct the v_{mn} , we first observe that $k'_{i-i+1} \leq k'_{i-i+2} \leq \dots \leq k'_i$. However, since $c(k'_i) = j - i$, the Law of Inertia allows a subspace Z of co-dimension $j - i$ to fit into $C^-(k'_i)$. Thus $Q(Z) \geq k'_i$ and

$$k'_{i-i+1} = \sup_{\text{co-dim } U = j-1} Q(U) \geq Q(Z) \geq k'_i$$

so that $k'_{i-i+1} = \dots = k'_i$. Let $\{V_{ni-i}\}_{n=1}^\infty$ be a maximizing sequence ($Q(V_{ni-i}) \rightarrow$

k'_j) of subspaces of co-dimension $j - i$. Let the v_{1n} be chosen from V_{ni-i} so that $|p(v_{1n}) - Q(V_{ni-i})| < 1/n$ and thus $p(v_{1n}) \rightarrow k'_j$. Let the subspaces V_{ni-i+1} of V_{ni-i} be the k'_j -orthogonal complements of the v_{1n} . This makes sense because $L_{k'}$ is negative definite on V_{ni-i} . Because

$$Q(V_{ni-i}) \leq Q(V_{ni-i+1}) \leq k'_j \quad \text{and} \quad Q(V_{ni-i}) \rightarrow k'_j,$$

V_{ni-i+1} , $n = 1, 2, \dots$, is a maximizing sequence of subspaces, also. From these are extracted the v_{2n} . Inductively it is clear how to construct subspaces $V_{ni-i+k} \subset V_{ni-i+k-1}$ and for $0 \leq k \leq i - 1$ show that an additional maximizing sequence has been found and that vectors v_{k+1n} obeying $p(v_{k+1n}) \rightarrow k'_j$ may be extracted.

This lengthy derivation has yet to consider the case in which k_i is finite and $k'_j = k_i$ must be shown. Happily this case requires only a paraphrasing of parts of the previous case and may be left to the reader's imagination.

4. Some consequences and related matters. (A) Constraints. The restriction of all vectors by orthogonal projection to the subspace $[w]^\perp$ of all vectors in W orthogonal to $w \neq 0$ is a one dimensional, linear constraint. It is easily seen that L_x so confined still satisfies hypotheses (a) through (e) of Section 2. Moreover, the constrained signature d^* and co-signature c^* obey $d^*(k) \leq d(k)$ and $c^*(k) \leq c(k)$. An immediate result is the existence of k_i^* whenever k_i exists. In fact,

$$k_i^* = \inf_{\substack{U \subset [w]^\perp \\ \dim U = j}} P(U) \geq \inf_{\substack{U \subset W \\ \dim U = j}} P(U) = k_i.$$

Moreover, once we observe that the spaces of co-dimension $j - 1$ in $[w]^\perp$ are subspaces of co-dimension j in W , it is clear that

$$k_i^* = k_i^{*'} = \sup_{\substack{V \subset [w]^\perp \\ \text{co-dim } V = j-1}} Q(V) \leq \sup_{\substack{V \subset W \\ \text{co-dim } V = j}} Q(V) = k'_{i+1} = k_{i+1}.$$

A similar argument on h_i^* establishes

Theorem 3. *The portion of the nonlinear spectrum of L_x which has a minimax characterization as given in Theorem 1 possesses the separation property that a member of the nonlinear spectrum of L_x subject to a one dimensional constraint lies in each of the closed intervals $k_i \leq x \leq k_{i+1}$ (resp., $h_{i+1} \leq x \leq h_i$); in particular, $k_i \leq k_i^* \leq k_{i+1}$ (resp., $h_{i+1} \leq h_i^* \leq h_i$).*

The effect of a multi-dimensional, linear constraint may be deduced inductively.

(B) A noteworthy property of the eigenvectors is given by Lemma 2 of [8]: If $u_1, \dots, u_n$ are eigenvectors corresponding to eigenvalues $p(u_1) < \dots < p(u_n)$, then the vectors $u_1, \dots, u_n$ are linearly independent and $p(u_1) < p(u_1 + \dots + u_n) < p(u_n)$. The inductive proof in [8] applies here as well. This notion can be extended to the rest of the spectrum if the eigenvectors $u_1, \dots, u_n$ are replaced by sequences of vectors, "linearly independent" is modified by "eventually", and the final inequality similarly modified.

(C) Compactness. The linear spectrum of compact operators is closed, countable, and consists of eigenvalues except for possible accumulation points at 0 and $\pm \infty$. The nonlinear spectrum of $L_x = I - C_x$, where C_x is compact, possesses analogous properties. In fact, the continuity argument for the linear case (as in [14] for instance) may be applied to show that if L_x is continuous on an interval J of the real axis, then the nonlinear spectrum in J is closed in J . The same may be said if J is a region of the complex plane. This result requires nothing of L_x except continuity and the continuity in x .

By the *multiplicity* of an eigenvalue we mean the maximum number of independent eigenvectors.

Theorem 4. *If L_x has the form $I - C_x$ with C_x compact and continuous in x at $x = k$ and if k is in the nonlinear spectrum of L_x , then k is an eigenvalue of L_x of finite multiplicity.*

Proof. Let $v_n, n = 1, 2, \dots$, be a sequence of unit vectors such that $\lim L_k v_n = 0$. The sequence $C_k v_n, n = 1, 2, \dots$, has a convergent subsequence. If we imagine the v_n re-labelled so that $\lim C_k v_n = v_0$, then

$$\lim_{n \rightarrow \infty} v_n = v_0 \quad \text{and} \quad \|v_0\| = 1$$

because $v_n - C_k v_n = L_k v_n \rightarrow 0$. Consequently, $C_k v_0 = v_0$ or $L_k v_0 = 0$ and k is an eigenvalue.

Now if there corresponds an infinite sequence of independent eigenvectors for k , then they generate a subspace of eigenvectors of infinite dimension. Let $u_n, n = 1, 2, \dots$, represent an orthonormal family in this eigen-space. Then $u_n = C_k u_n$ so that there is a convergent subsequence. This is contradictory for being orthogonal unit vectors, u_m and u_n obey $\|u_m - u_n\| = \sqrt{2}$ for arbitrary, distinct m and n .

Theorem 5. *If L_x satisfies hypotheses (a) through (e) of Section 2 on the range J of p and if L_x has the form $I - C_x$ with C_x compact at $x = k$ in J , then k cannot be a point of accumulation of the nonlinear eigenvalues.*

Proof. Assume the contrary by letting $\lambda_n, u_n, n = 1, 2, \dots$, be a sequence of eigenpairs with $\lambda_n \rightarrow k$ as $n \rightarrow \infty$ and $\|u_n\| = 1$. We define the sequence $v_n = C_k u_n$ and observe that

$$\|u_n - v_n\| \leq \|C_{\lambda_n} - C_k\| \rightarrow 0$$

as $n \rightarrow \infty$. The compactness of C_k guarantees that a subsequence of the u_n converges. We let u_0 be a limit point of the sequence and note that $\|u_0\| = 1$ because $\|u_n\| = 1$.

Because the right side of

$$\|L_k u_0\| \leq \|L_k(u_0 - u_n)\| + \|L_k - L_{\lambda_n}\| \|u_n\|$$

converges to zero through a subsequence of the n , $L_k u_0 = 0$ and k, u_0 is an

eigenpair. Now consider the real-valued function $f_n(x) = (L_x u_n, u_0)$ of the real variable x . It has a continuous derivative on the interval J and satisfies $f_n(\lambda_n) = 0 = f_n(k)$. By Rolle's Theorem its derivative f'_n has a zero at an x_n between λ_n and k . Therefore,

$$0 = \lim_{n \rightarrow \infty} f'_n(x_n) = (L'_k u_0, u_0),$$

the limit being taken through those n for which $u_n \rightarrow u_0$. However, $k = p(u_0)$ contradicting hypothesis (e). Therefore, k cannot be a point of accumulation of the eigenvalues of L_x .

Stronger hypotheses give

Theorem 6. *If L_x satisfies hypotheses (a) through (e) on J , the range of p , and if $L_x = I - C_x$ with C_x compact for x in J , then $d(x)$ is finite for all x in J and the portion of the nonlinear spectrum lying in J is*

- (i) *composed only of eigenvalues of finite multiplicity*
- (ii) *countable*
- (iii) *composed of minimax values h_j , $j = 1, 2, \dots$.*

Proof. The linear eigenvalues of a compact, self-adjoint C_x are necessarily confined so that at most a finite number are greater than one. This means that all but at most a finite number of the linear eigenvalues of $L_x = I - C_x$ are non-negative. Therefore, the quadratic form $(L_x u, u)$ must have a finite signature $d(x)$.

Conclusions (i) and (ii) are special cases of Theorems 4 and 5. Suppose k is in J and the nonlinear spectrum both. Because $d(k) < \infty$, Theorem 1 guarantees that k is an h_j .

5. Remarks. The most serious restriction in the application of the minimax theory given here is the existence of a real, continuous p on all of W (excepting 0). In the case of polynomial nonlinearities, the roots x of $(L_x u, u) = 0$ are in general complex and multiple. Our p is a real branch of the function defined by $(L_x u, u) = 0$. Condition (e) prohibits branch points. This prohibition can be reduced to the demand that the complement of the set of branch points be dense and connected in W [9].

Operator types to which this theory has been shown to apply are

$$\begin{aligned} L_x &= xA - B, & L_x &= x^2 A + xB + C, \\ L_x &= x^3 A + xB + C, & L_x &= xA + \sqrt{1 - x^2} B + C. \end{aligned}$$

In each case, A , B , C must be self-adjoint and their quadratic forms must obey other conditions guaranteeing the existence and regularity of a Rayleigh functional p . The inclusion of $L_x = xA - B$ means (as is really obvious) that this paper generalizes the theory of linear spectra. Also obvious is the fact that nothing new is determined about linear spectra. However, this development is

closer to the topological foundations of critical point theory than usual developments.

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