On the Pointwise Behavior of Iterates of a Self-adjoint Operator

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The question with which this paper is concerned is the following: Let T be a self-adjoint, linear operator on L_1 of a measure space of finite measure. Assume that $\|T\|_1 \leq 1$. Then if f is in L_1 does $T^{2^n}f$ converge pointwise a.e.?

A case of special interest arises in the following way: Let T_1 and T_2 be operators defined by taking the conditional expectation with respect to σ fields θ_1 and θ_2 . (Then $T_1 T_2 T_1 = T$ is selfadjoint and $T^n = (T_1 T_2)^n T_1$.) One would then expect $(T_1 T_2)^n T_1 f$ and $(T_2 T_1)^n f$ to converge pointwise a.e. (and that the limit would equal the conditional expectation with respect to $\theta_1 \wedge \theta_2$).

Burkholder and Chow [1] showed that the answer to these questions is yes if f is in L_2 . Rota [2] and E. M. Stein [3] showed the result to be true for functions in L_p , $p > 1$. (Rota's method gives something much more general.) In this paper we will show that the answer is no, even in the special case mentioned above, if f is only assumed to be in L_1 .

Theorem. Let I be the unit interval. Then there are 2 σ -fields, θ_1 and θ_2 consisting of Borel sets in I , and a function f , defined on I , $f \geq 0$, $\int f \leq \frac{1}{8}$. If T_1 and T_2 are the operators: conditional expectation with respect to θ_1 and θ_2 respectively, then $\lim_{n \rightarrow \infty} (T_1 T_2)^n f \geq 1$ a.e. on I .

Remark 1. The above theorem implies that $(T_1 T_2)^n f$ does not converge pointwise a.e..

Remark 2. It will follow easily from the construction that $(T_1 T_2 T_1)^{2^n} g$ does not converge pointwise a.e. for some g in L_1 .

Definition. By an (X, r, s) system we will mean the following: X is a measure space of finite measure, r a finite collection of disjoint subsets of X whose union is X , s a finite collection of disjoint subsets of X (whose union is not X).

r will give rise to an operator, R ; conditional expectation with respect to r . s will give rise to an operator, S as follows; S will be the identity on functions whose support is disjoint from $\cup s$ and for functions with support in $\cup s$, S will be conditional expectation with respect to s .

We will say that (X, r, s) is ergodic if for any function g , $(RS)^n g$ tends to a constant.

Lemma 1. Suppose we are given a system (X, r, s) , a function f , (f is assumed to be measurable with respect to $r \cup s$, and hence integrable and bounded), an integer M and an $\epsilon > 0$. Then we can find a $\delta > 0$ such that if $(\bar{X}, \bar{r}, \bar{s})$ is a system with the following properties: $\bar{X} = X \cup E$ (E is a measure space disjoint from X) $\bar{r}$ is obtained by adding sets in E to r . $\bar{s}$ is obtained by adding sets in E and sets C_i $i = 1 \cdots K$ (not in E or X) to s . (Each C_i must be disjoint from any set in s if $(\bar{X}, \bar{r}, \bar{s})$ is to be a system) $\sum_{i=1}^K m(C_i \cap X) < \delta$. ($m(F)$ = measure of F). Then

$$(\bar{R}\bar{S})^i f > (RS)^i f - \epsilon \text{ on } X \text{ for } i < M.$$

Proof. $\bar{S}f \geq Sf + g$ where $g \leq 0$ has support in X and $\int |g|$ can be made arbitrarily small by making δ small, since f is bounded.

$$\bar{R}\bar{S}f \geq \bar{R}Sf + \bar{R}g = RSf + Rg$$

$$(\bar{R}\bar{S})^2 f \geq \bar{R}\bar{S}RSf + \bar{R}\bar{S}Rg \geq (RS)^2 f + Rg_1 + \bar{R}\bar{S}Rg$$

where g_1 can be assumed to have the same properties as g since $\sup RSf \leq \sup f$. Continuing in this way we get that $\int_X |(\bar{R}\bar{S})^i f - (RS)^i f|$ can be made as small as we want by making δ small enough. *L1* now follows because $(\bar{R}\bar{S})^i f$ and $(RS)^i f$ are constant on each set in r .

Lemma 1'. Lemma 1 is still true if we replace the condition $\sum_{i=1}^K m(C_i \cap X) < \delta$ by:

$$\sum_{i=1}^K m(C_i \cap E) < \delta.$$

Proof. The proof is almost the same as that of *L1* and will be omitted. (In our application each $C_i \cap X$ will be in the field generated by $r \cup s$).

Lemma 2. Suppose $(\bar{X}, \bar{r}, \bar{s})$ and $(\bar{\bar{X}}, \bar{\bar{r}}, \bar{\bar{s}})$ are both ergodic and that we form (X, r, s) as follows:

$$X = \bar{X} \cup \bar{\bar{X}}, \bar{r} = \bar{r} \cup \bar{\bar{r}}, \quad s = \bar{s} \cup \bar{\bar{s}} \cup C$$

where C is a set such that $m(C \cap \bar{X}) \neq 0$ and $m(C \cap \bar{\bar{X}}) \neq 0$. (C must be disjoint from $\bar{s} \cup \bar{\bar{s}}$.) Then (X, r, s) is ergodic.

Proof. Since r and s are finite $\lim_{n \rightarrow \infty} (RS)^n g$ exists. Call the limit l . $Sl = l$ because otherwise $\|Sl\|_2 < \|l\|_2$. Similarly $Rl = l$. If l were not constant we would have 2 disjoint sets E and F , $E \cup F = X$ and every set in r or s would be in either E or F . Since $(\bar{X}, \bar{r}, \bar{s})$ is ergodic $E \cap \bar{X} = \bar{X}$ or $E \cap \bar{X} = \phi$. Hence $E = \bar{X}$ or $\bar{\bar{X}}$ and $F = \bar{\bar{X}}$ or $\bar{X}$, but this contradicts the condition on C .

We will say that (A) holds for α , ($\alpha < 1$) if given an ergodic system (X, r, s) and $\epsilon > 0$ we can find a new ergodic system (X', r', s') with the following properties:

(a) $X' = (\bigcup_{i=1}^M X_i) \cup A$ where A is a measure space, (X_i, r_i, s_i) is an exact copy of (X, r, s) . ($m(X_i) = m(X)$ and unions are assumed to be disjoint).

(b) $m(A) < \epsilon m(X')$.

- (c) $r' = (\bigcup_{i=1}^M r_i) \cup t$ where t is a collection of disjoint sets whose union is A .
 (d) $s' = (\bigcup_{i=1}^M s_i) \cup t_1 \cup t_2$ where t_1 is a collection of disjoint sets contained in A and t_2 is a collection of disjoint sets (each of which is disjoint from anything in $(\bigcup_{i=1}^M s_i) \cup t_1$). The union of the sets in t_2 intersects $\bigcup_{i=1}^M X_i$ is a set of measure less than ϵ .
 (e) There is an $f > 0$, such that $\int f = \alpha m(X')$ (the support of f will be in A and will be disjoint from any set in s' , and f will be measurable with respect to $r' \cup s'$) and $\sup_n (R'S')^n f > 1 - \epsilon$ on X' .

Lemma 3. (A) holds for $\alpha = 1$.

Proof. Let $X' = X \cup A_1 \cup A_2$, $m(A_1 \cup A_2) < \epsilon/2$. Add the set $A_1 \cup A_2$ to r to get r' . Add a set C to s to get s' where $m(C) < \epsilon$ and $m(C \cap X) \neq 0$, $m(C \cap A_2) = 0$ and $m(C \cap A_1) = A_1$. Define f to be constant on A_2 , 0 elsewhere and $\int f = 1$. L3 now follow from the fact that (X', r', s') is ergodic.

The following is our key lemma.

Lemma 4. If (A) holds for α then (A) holds for $\alpha' = \alpha - \alpha^2/4$.

Proof. We are given (X, r, s) and ϵ . We want to find (X', r', s') satisfying (a)–(e) but with α replaced by $\alpha' = \alpha - \alpha^2/4$. By assumption we can find $(\bar{X}, \bar{r}, \bar{s})$, and $\bar{f}$, satisfying (a)–(e) (for α) but with ϵ replaced by $\epsilon/10$. Let $(\bar{X}, \bar{r}, \bar{s})$, $\bar{f}$ be an exact (disjoint) copy of $(\bar{X}, \bar{r}, \bar{s})$, $\bar{f}$.

(1) Pick a set C , $m(C \cap \bar{X}) = m(C \cap \bar{\bar{X}}) = \epsilon' > 0$, $\epsilon' < \epsilon/10$, and form $(\bar{X} \cup \bar{\bar{X}}, \bar{r} \cup \bar{\bar{r}}, \bar{s} \cup \bar{\bar{s}} \cup C)$. Let $\bar{R}$ and $\bar{S}$ be the associated operators. Then, if ϵ' is small enough $\sup_n (\bar{R}\bar{S})^n \bar{f} > 1 - 3\epsilon/10$ on $\bar{X}$. The above statement is also true if we replace $\bar{f}$ by $\bar{\bar{f}}$ and $\bar{X}$ by $\bar{\bar{X}}$. (This follows easily from L1 and the fact that there is an M such that $\sup_{i < M} (\bar{R}\bar{S})^i \bar{f} > 1 - 2\epsilon/10$ on $\bar{X}$).

(2) $\lim_{n \rightarrow \infty} (\bar{R}\bar{S})^n \bar{f}$ is a constant, l , by L2. $l = \alpha/2$. Pick K so that $(\bar{R}\bar{S})^n \bar{f} > \alpha/2 - \epsilon/10$ for all $n > K$.

(3) Let the sets on which $\bar{f}$ is non-zero be $D_1 \cdots D_N$. For each D_i pick a measure space $A_i = \bigcup_{j=1}^{2K} A_i^j$ (The A_i^j are disjoint from each other and from $\bar{X} \cup \bar{\bar{X}}$). Let $m(A_i^{i+1}) = \delta m(A_i^i)$ and $m(A_i^1) = \delta m(D_i)$ for all $1 \leq i \leq N$, $1 \leq j \leq 2K - 1$ (δ will be determined later).

Let $X' = \bar{X} \cup \bar{\bar{X}} \cup \bigcup_{i=1}^N A_i$.

Let $r' = \bar{r} \cup \bar{\bar{r}} \cup t_3$ where t_3 is the collection of all sets of the form $A_i^j \cup A_i^{j+1}$ for j odd.

Let $s' = C \cup \bar{s} \cup \bar{\bar{s}} \cup t_4$ where t_4 is the collection of all sets of the form $D_i \cup A_i^1$ or $A_i^j \cup A_i^{j+1}$ for j even.

Define $\bar{f}$ to be 0 outside of $\bigcup_{i=1}^N A_i^{2K}$, constant on each A_i^{2K} and $\int_{A_i^{2K}} \bar{f} = \int_{D_i} \bar{f}$.

Let $f' = \bar{f} + (1 - \alpha/2)\bar{\bar{f}}$.

(4) We must now show that f' and (X', r', s') , satisfy (a)–(e) with α replaced by $\alpha' = 1 - \alpha^2/2$, if we chose δ small enough. (a) and (c) are clearly satisfied. (b) is satisfied if δ is chosen so small that $m(\bigcup_{i=1}^N A_i) < \epsilon/10$ $m(\bar{X} \cup \bar{\bar{X}})$. (d) is satisfied because the only set in t_2 and not in $\bar{s}$ or $\bar{\bar{s}}$ is C . (because of the

choice of the support of $\bar{f}$ the D_i are disjoint from the copies of X . Hence t_4 is disjoint from the copies of X).

(5) We will now check (e).

(5.1) It is easy to see that $S'(R'S')^K \bar{f} = (1 - \gamma) \bar{f}$ on $\bar{X} \cup \bar{X}$ and that γ can be made as small as we want by making δ small enough.

(5.2) If δ is small enough $\sup_n (R'S')^n \bar{f} > 1 - 5\epsilon/10$ on $\bar{X}$. To see this: (1) implies that for some L , $\sup_{n < L} (\bar{R}\bar{S})^n \bar{f} > 1 - 4\epsilon/10$ on $\bar{X}$. (5.2) now follows by using $L1'$.

(5.3) It follows from (5.1) and (5.2) that $\sup_{n > 2K} (R'S')^n \bar{f} > 1 - 6\epsilon/10$ on $\bar{X}$ (for a proper choice of δ).

(5.4) It follows from (5.3), (2) and $L1'$ that if δ is chosen small enough $\sup_{n > 2K} (R'S')^n f' = \sup_{n > 2K} (R'S')^n (\bar{f} + (1 - \frac{1}{8}\alpha)\bar{f}) > 1 - 7\epsilon/10$ on $\bar{X}$.

(5.5) (1) says that $\sup_n (R'S')^n f' > 1 - 3\epsilon/10$ on $\bar{X}$.

(5.6) It is clear that $\sup_n (R'S')^n \bar{f} > 1$ on $\bigcup_{i=1}^N A_i$.

(5.7) $\int f' = \int \bar{f} + (1 - \alpha/2) \int \bar{f}$ and $\int \bar{f} = \alpha m(\bar{X})$. Therefore $\int f' = 2\alpha m(\bar{X}) - \alpha^2/2m(\bar{X}) = (\alpha - \alpha^2/4) m(\bar{X} \cup \bar{X}) < (\alpha - \alpha^2/4) m(X')$.

(5.4)–(5.7) give (e) and it follows easily from $L2$ that (X', r', s') is ergodic.

Lemma 5. (A) holds for arbitrarily small α .

Proof. This is an immediate consequence of $L3$ and $L4$.

Construction of the example. The construction will be given inductively. Suppose we have an ergodic (X_n, r_n, s_n) system, integers $N_i, i = 1 \dots n$ and functions $f_i > 0, i = 1 \dots n, \int f_i = 1/2^{i+3}, \sup_{i < N_i} (R_n S_n)^i f_i > 1$ on X_n .

Pick M (to be determined later) disjoint sets $F_i^n, \bigcup_{i=1}^M F_i^n = X_n, m(F_i^n) = m(F_i^n)$ and each F_i^n is independent of $r_n \cup s_n$. Each $(F_i^n, r_n \cap F_i^n, s_n \cap F_i^n)$ is isomorphic to (X_n, r_n, s_n) .

$L5$ tells us that if M were chosen properly we could form a new system $(X_{n+1}, r_{n+1}, s_{n+1})$ satisfying (a)–(e) with $\alpha = 1/2^{n+4}$. X_{n+1} is formed by adding a set A to X_n .

(e) gives us $f_{n+1}, \int f_{n+1} = 1/2^{n+4}m(X_{n+1})$ and $\sup_{i < N_{n+1}} (R_{n+1}S_{n+1})^i f_{n+1} > 1$ on X_{n+1} for some N_{n+1} (if ϵ were chosen small enough).

$L1$ and (d) tells us that if ϵ were chosen small enough then

$$(1) \quad \sup_{i < N_i} (R_{n+1}S_{n+1})^i f_i > 1 \text{ on } X_{n+1}. (j \leq n+1)$$

We now have a sequence of sets $X_n \subset X_{n+1} \subset \dots$ and operators R_n and S_n defined on X_n . We will now define operators R_∞ and S_∞ on $\bigcup_{n=1}^\infty X_n$. R_∞ will be conditional expectation with respect to $\bigcup_{n=1}^\infty r_n$. S_∞ will be defined as follows: If f has support in the complement of $\bigcup_{n=1}^\infty s_n$ then $S_\infty f = f$. If f has support in some set $E \subset \bigcup_{n=1}^\infty s_n$ and if $E \subset s_n$ and $E \not\subset s_{n-1}$, then $\bigcup_{i=1}^\infty (F_i^n \cap E)$ generates a σ -field on E and $S_\infty f =$ conditional expectation with respect to this σ -field. S_∞ is clearly conditional expectation with respect to some σ -field. (It will be convenient to have R_n and S_n defined on $\bigcup_{n=1}^\infty X_n$. We can do this by letting $R_n g = S_n g = g$ for those g whose support is disjoint from X_n .)

(2) $\overline{\lim_{n \rightarrow \infty}} (R_\infty S_\infty)^n \left(\sum_{i=1}^\infty f_i \right) \geq 1$ a.e. (This proves the theorem since $\int \sum_{i=1}^\infty f_i < (1/8)m \left(\bigcup_{n=1}^\infty X_n \right)$ and we could let $T_1 = R_\infty$, $T_2 = S_\infty$ and we could take all the measure spaces in the construction to be subsets of the unit interval).

In order to show (2) we will show that for each i and j (3) $\lim_{n \rightarrow \infty} (R_n S_n)^i f_i = (R_\infty S_\infty)^i f_i$ where the convergence is in the L_1 norm. ((2) will follow easily from (3) and (1)).

We will get (3) by induction on i (keeping j fixed). Using our induction hypothesis, $(R_n S_n)^{i-1} f_i = (R_\infty S_\infty)^{i-1} f_i + g_n$ where $\int |g_n|$ can be made as small as we want by making n large enough. By the Martingale convergence theorem $\lim_{n \rightarrow \infty} S_n((R_\infty S_\infty)^{i-1} f_i) = S_\infty((R_\infty S_\infty)^{i-1} f_i)$ a.e. and since f_i was bounded we have convergence in L_1 . We therefore get $S_n(R_n S_n)^{i-1} f_i = S_\infty(R_\infty S_\infty)^{i-1} f_i + g'_n + S_n g_n$ where $\int |g'_n|$, (and $\int |g'_n + S_n g_n|$) can be made as small as we want by making n large enough. We can now get (3) by repeating the same argument using R instead of S .

We get (2) from (3) and (1) as follows: (1) and (3) imply (4) $\sup_{i < N_i} (R_\infty S_\infty)^i f_i \geq 1$ a.e. (4) and the fact that $\lim_{i \rightarrow \infty} \int f_i = 0$ implies; (5) given K and $\epsilon > 0$ there is an M such that if $j > M$ then $\sup_{K < i < N_i} (R_\infty S_\infty)^i f_i \geq 1 - \epsilon$ except on a set of measure less than ϵ . (5) implies (2).

Proof of Remark 2. (1) $S_\infty f_n = f_n$, $n = 1, 2, \dots$. Because of (e) in property (A) we have that $S_n f_n = f_n$. By making sure that the new elements that we add to s are disjoint from the support of f_n we get (1). From (1) and our theorem we get

(2) $\overline{\lim_{n \rightarrow \infty}} (R_\infty S_\infty)^n R_\infty \sum_{i=1}^\infty f_i \geq 1$ a.e. and hence

(3) $\overline{\lim_{n \rightarrow \infty}} (R_\infty S_\infty R_\infty)^n \sum_{i=1}^\infty f_i \geq 1$ a.e. (3) and the fact that $\int \sum_{i=1}^\infty f_i \leq 1/8$ imply that either $(R_\infty S_\infty R_\infty)^{2n} \sum_{i=1}^\infty f_i$ or $(R_\infty S_\infty R_\infty)^{2n} (R_\infty S_\infty R_\infty) \sum_{i=1}^\infty f_i$ does not converge.

REFERENCES

- [1] D. L. BURKHOLDER & Y. S. CHOW, Iterates of conditional expectation operators, *Proc. Amer. Math. Soc.*, **12** (1961) 490-495.
- [2] G-C. ROTA, An "alternierende Verfahren" for general positive operators, *Bull. Amer. Math. Soc.*, **68** (1962) 95-102.
- [3] E. M. STEIN, On the Maximal Ergodic Theorem, *Proc. Nat. Academy of Sciences*, **47** (1961) 1894-1897.

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