A Weighted Volume-Diameter Inequality for N -Cubes

WILLIAM R. DERRICK

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1. Introduction. Let A be the topological image of a square and let A_1, A_2, A'_1, A'_2 be the images of the sides of the square traversed in a counter-clockwise direction. Denote the distance between the "opposite" sides A_1, A'_1 and A_2, A'_2 of A by a_1 and a_2 respectively. Then it is well known that

$$\text{Area}(A) \geq a_1 a_2 .$$

The inequality appears first in the work of Teichmüller [T] although it was probably known earlier. Teichmüller's proof shows more, namely that for any nonnegative real-valued Borel-measurable function w

$$(*) \quad \int_A w^2 \, dx \, dy \geq \left(\inf \int_{c_1} w \, ds \right) \left(\inf \int_{c_2} w \, ds \right) ,$$

where the infimums are taken over all arcs c_i joining A_i to A'_i in A , $i = 1, 2$. For $w \equiv 1$ we have

$$(**) \quad \text{Area}(A) \geq d_1 d_2 ,$$

where d_i is the "interior" distance between A_i and A'_i , $i = 1, 2$.

Teichmüller's solution depends heavily on the Riemann Mapping Theorem, but there is really no intrinsic reason why conformal mapping techniques need be used. In 1952, Besicovitch [Be] obtained a proof of (**) without making use of conformal mappings and Taylor [Tr] extended (**) to E^3 . In [D, 4.8] the author generalized (**) to E^n and proved that

$$\text{Volume}(\text{Interior } A) \geq \prod_{i=1}^n d_i .$$

These proofs all make use of Area-theoretic techniques.

Apparently no work has been done on inequality (*) beyond the two-dimensional solution presented by Teichmüller. This paper presents a proof of the generalization of (*) to topological images of n -cubes in n -dimensional space; the result is found in Theorem (4.4). In Theorem (4.5) we extend this last

result and prove

$$\int_A \left(\prod_{i=1}^n w_i \right) dL_n x \geq \prod_{i=1}^n \left(\inf \int_{c_i} w_i ds \right),$$

thus answering a conjecture of Duffin [Du] in the affirmative. This last inequality is then used to prove Theorem (5.3):

$$\prod_{i=1}^n M(\Gamma_i) \geq 1,$$

where $M(\Gamma_i)$ is the module of the family Γ_i of arcs joining A_i to A'_i in A , $i = 1, \dots, n$.

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2. Preliminaries.

2.1 Definitions. Let E^n be the n -dimensional Euclidean space consisting of all n -tuples $x = (x_1, \dots, x_n)$ of real numbers, with the inner product

$$x \circ y = \sum_{i=1}^n x_i y_i,$$

for x, y in E^n and the norm

$$|x| = (x \circ x)^{1/2}.$$

Denote the *boundary* and *interior* of a set A in E^n by $\text{Bdry } A$ and $\text{Int } A$ respectively, let the *distance* between the points x, y in E^n be given by

$$\text{dist } (x, y) = |x - y|,$$

and let the *diameter* of the set A be

$$\text{diam } A = \sup \text{dist } (x, y)$$

for all x, y in $A \neq \emptyset$, and $\text{diam } \emptyset = 0$.

r_{xy} is the straight line segment from x to y in E^n and $S_\epsilon(x) = \{y \text{ in } E^n : \text{dist } (x, y) < \epsilon\}$.

Let $I^n = [0, 1]^n$ be the *closed unit n -cube* in E^n and $I^{n-k}(i, j)$ denote the $(n - k)$ -cube obtained by setting the i_m -th coordinate equal to j_{i_m} , $m = 1, \dots, k$, where

$$i = (i_1, i_2, \dots, i_k), \quad 1 \leq i_1 < i_2 < \dots < i_k \leq n,$$

$$j = (j_{i_1}, \dots, j_{i_k}), \quad j_{i_m} = 0, 1 \quad \text{for } m = 1, \dots, k.$$

Set $I_m^n = \{x \text{ in } I^n : \text{dist } (x, \text{Bdry } I^n) \geq 1/m\}$, $m \geq 3$, and define $I_m^{n-k}(i, j)$ to be the $(n - k)$ -cube obtained by setting the i_p -th coordinate equal to $j_{i_p} = 1/m, (m - 1)/m$, for $p = 1, \dots, k$. $Q_\epsilon^n(x) = \{y \text{ in } E^n : |y_i - x_i| \leq \epsilon/2, i = 1, \dots, n\}$.

Denote the determinant of a square matrix A by $\det A$, and if A is an $m \times n$ -matrix ($m \leq n$) let

$$\Delta(A) = \left\{ \sum_x (\det A_x)^2 \right\}^{1/2},$$

where x is an m -tuple of integers $(x_1, \dots, x_m)$ with $1 \leq x_1 < x_2 < \dots < x_m \leq n$ and A_x is the minor of A made up from the m columns whose indices are elements of x .

If $f: E^n \rightarrow E^m$ is *differentiable* at x , then there exists a linear transformation $Df(x): E^n \rightarrow E^m$ satisfying

$$\lim_{|h| \rightarrow 0} \frac{|f(x+h) - f(x) - Df(x)(h)|}{|h|} = 0.$$

f is *approximately differentiable* at x if for some linear transformation $D_{ap}f(x): E^n \rightarrow E^m$ and for every $r > 0$ the point x is a point of density of the set

$$E(r) = \{x: |f(x+h) - f(x) - D_{ap}f(x)(h)| \leq r|h|\}$$

The *Jacobian* of f , Jf , is defined on the set of all points of approximate differentiability of f and is given by

$$Jf(x) = \Delta(D_{ap}f(x)).$$

For $m = 1$, the gradient of f at x satisfies

$$\text{grad } f(x) = D_{ap}f(x)$$

at all points of approximate differentiability of f and

$$|\text{grad } f(x)| = Jf(x).$$

Hadamard's Inequality (c.f. [BB, page 64]) states that

$$Jf(x) \leq \prod_{i=1}^n |\text{grad } f^i(x)|,$$

where f^i is the i -th coordinate function of f .

Set

$$\alpha(n) = \Gamma(\frac{1}{2})^n / \Gamma(1 + (n/2)), \quad n \geq 0,$$

which, for integral n , is the n -dimensional volume of $S_1(0)$.

Denote by $H_n^k(A)$ the k -dimensional Hausdorff measure of A which equals the limit, as $r \rightarrow 0+$, of

$$\inf_F \sum_{S \in F} 2^{-k} \alpha(k) (\text{diam } S)^k$$

over all countable coverings F of A such that $\text{diam } S < r$ for S in F . Then $H_n^0(A)$, $H_n^1(A)$, $H_n^n(A)$ represent the cardinality, length, and Lebesgue n -dimensional measure of A (see [D, 2.8, 2.12, 2.7]) respectively.

A function $f : E^n \rightarrow E^m$ is *Lipschitzian* on a set A in E^n if there is a constant M such that

$$|f(x) - f(y)| \leq M |x - y|$$

for all x, y in A . If each coordinate function of f is Lipschitzian, then so is f .

2.2 Theorem. Let F be a continuous function mapping I^n into itself such that

$$F(I^{n-k}(i, j)) \subset I^{n-k}(i, j),$$

for every i, j , and k . Then F is onto.

Proof. See [D, 3.2].

2.3 Theorem. Let $f : E^n \rightarrow E^k, n \geq k$, be Lipschitzian and A be an H_n^n -measurable subset of E^n . Then $v : E^k \rightarrow E^1$, given by

$$v(y) = H_n^{n-k}(A \cap f^{-1}\{y\}),$$

is an H_k^k -measurable function and

$$\int_A Jf(x) dH_n^n x = \int_{E^k} H_n^{n-k}(A \cap f^{-1}\{y\}) dH_k^k y.$$

Proof. See [F, 3.1 and Part 2 of 3.1]. The theorem also holds for f Lipschitzian on an open set S in E^n with A an H_n^n -measurable subset of S .

3. Solution for an upper semi-continuous weight function. Let g be a homeomorphism of I^n onto the subset A of E^n and let $g(I^{n-1}(i, 0)) = A_i, g(I^{n-1}(i, 1)) = A'_i$. Throughout this section we shall assume we are given a nonnegative upper semi-continuous weight function $w : E^n \rightarrow E^1$.

3.1 Definition. The *weighted distance* between the points x and y in A is given by

$$\lambda(x, y) = \inf \int_c w(x) dH_n^1 x,$$

where the infimum is taken over all arcs c joining x to y which lie entirely in A . Let

$$\lambda_i = \lambda(A_i, A'_i) = \inf \lambda(x, x'),$$

for all x in A_i, x' in A'_i , be the weighted distance between A_i and A'_i , and call

$$V(A) = \int_A w^n(x) dH_n^n x$$

the *weighted volume* of A .

If $w = 1$ in A , then the weighted distance becomes the interior distance, $d(x, y)$, between x and y in A . d_i is then the interior distance between A_i and A'_i and $V(A) = H_n^n(A)$.

In [D, 4.4] we proved the following theorem.

3.2 Theorem. $0 < d_i < \infty$, $i = 1, \dots, n$.

3.3 Lemma. $0 \leq \lambda_i < \infty$, $i = 1, \dots, n$.

Proof. By (3.2) there exist points x in A_i , x' in A'_i , and a rectifiable arc c joining x to x' such that $c - (x \cup x')$ lies entirely in $\text{Int } A$. Since w is upper semi-continuous and A is compact, it attains its maximum on A , and since it is finite, it follows that $w(x) \leq M < \infty$, for x in A . Then

$$0 \leq \lambda_i \leq \lambda(x, x') \leq \int_c w(x) dH_n^1 x \leq MH_n^1(c) < \infty.$$

3.4 Definition. Let $A(m) = g(I_m^n)$, $A_i(m) = g(I_m^{n-1}(i, 0))$, $A'_i(m) = g(I_m^{n-1}(i, 1))$, and $\lambda_i(m) = \lambda(A_i(m), A'_i(m))$, for $i = 1, \dots, n$. Lemma (3.3) implies that $0 \leq \lambda_i(m) < \infty$, $i = 1, \dots, n$. Define the function

$$f^m : \text{Int } A \rightarrow E^n \text{ by } f^m(x) = (f_1^m(x), \dots, f_n^m(x)),$$

where

$$f_i^m(x) = \min(\lambda_i(m), \lambda(A_i(m), x)), \text{ for } x \text{ in } \text{Int } A.$$

3.5 Lemma. f^m is locally Lipschitzian on $\text{Int } A$.

Proof. Let x be any interior point of A and $2\epsilon < \text{dist}(x, \text{Bdry } A)$. Then for every y in $S_\epsilon(x)$ we have

$$f_i^m(x) \leq \lambda(A_i(m), x) \leq \lambda(A_i(m), y) + \lambda(x, y),$$

and

$$f_i^m(x) \leq \lambda_i(m) \leq \lambda_i(m) + \lambda(x, y).$$

Thus

$f_i^m(x) \leq f_i^m(y) + \lambda(x, y)$, and since $S_\epsilon(y) \subset \text{Int } A$, we also have $f_i^m(y) \leq f_i^m(x) + \lambda(x, y)$, implying

$$|f_i^m(x) - f_i^m(y)| \leq \lambda(x, y) \leq \int_{r_{xy}} w dH_n^1 < M |x - y|.$$

Therefore each coordinate function is locally Lipschitzian, so f^m is locally Lipschitzian with constant $= Mn^{1/2}$ on $\text{Int } A$. In particular, f^m is continuous on $\text{Int } A$.

3.6 Theorem. Let A and λ_i be as defined in (3.1). Then

$$V(A) \geq V(\text{Int } A) \geq \prod_{i=1}^n \lambda_i.$$

Proof. We divide the proof into 7 parts.

Part 1.

$$f^m(\text{Int } A) = \prod_{i=1}^n [0, \lambda_i(m)].$$

Proof. Let the mappings

$$h^m : \prod_{i=1}^n [0, \lambda_i(m)] \rightarrow I^n, \quad s^m : I^n \rightarrow I_m^n,$$

be given by

$$\begin{aligned} h^m(x_1, \dots, x_n) &= (x_1/\lambda_1(m), \dots, x_n/\lambda_n(m)), \\ s^m(x_1, \dots, x_n) &= (y_1, \dots, y_n), \end{aligned}$$

where

$$y_i = \frac{1}{m} + x_i - \frac{2x_i}{m},$$

for all $i = 1, \dots, n$. s^m merely shrinks I^n onto I_m^n . Then $h^m f^m g s^m : I^n \rightarrow I^n$ is continuous and satisfies

$$h^m f^m g s^m(I^{n-k}(i, j)) \subset I^{n-k}(i, j),$$

for every i, j , and k . By Theorem (2.2) we have that $h^m f^m g s^m$ is onto, and since h^m , g , and s^m are all homeomorphisms this implies

$$f^m : A(m) \rightarrow \prod_{i=1}^n [0, \lambda_i(m)]$$

is onto.

But since each coordinate function is truncated, we have

$$f^m(\text{Int } A) = \prod_{i=1}^n [0, \lambda_i(m)].$$

Part 2. If w is a continuous function, then

$$V(\text{Int } A) \geq \prod_{i=1}^n \lambda_i(m).$$

Proof. Let $\epsilon(x) = \frac{1}{2} \text{dist}(x, \text{Bdry } A)$ for every x in $\text{Int } A$. Since

$$\text{Int } A = \bigcup_{x \in \text{Int } A} S_{\epsilon(x)}(x),$$

and E^n satisfies the second axiom of countability, by Lindelöf's Theorem, some countable subcollection $\{S_i\}$ covers $\text{Int } A$. Furthermore, f^m is Lipschitzian in each S_i , so Hadamard's inequality

$$Jf^m(x) \leq \prod_{i=1}^n |\text{grad } f_i^m(x)|$$

holds *a.e.* in S_i .

But for $i = 1, 2, \dots, n$,

$$\begin{aligned} |\text{grad } f_i^m(x)| &= \limsup_{y \rightarrow x} \frac{|f_i^m(y) - f_i^m(x)|}{|y - x|} \\ &\leq \limsup_{y \rightarrow x} \frac{\lambda(x, y)}{|y - x|} \\ &\leq \limsup_{y \rightarrow x} \frac{\int_{x, y} w \, dH_n^1}{|y - x|}. \end{aligned}$$

Since w is continuous and by the mean-value theorem we have

$$\begin{aligned} \limsup_{y \rightarrow x} \frac{\int_{x, y} w \, dH_n^1}{|y - x|} &= \limsup_{y \rightarrow x} w(x + t(y - x)), \quad 0 < t = t(y) < 1, \\ &= w(x). \end{aligned}$$

Hence

$$\begin{aligned} V(S_1) &= \int_{S_1} w^n \, dH_n^n \geq \int_{S_1} Jf^m(x) \, dH_n^n x \\ &= \int_{E^n} H_n^0(S_1 \cap (f^m)^{-1}\{y\}) \, dH_n^n y, \end{aligned}$$

and, for $j > 1$,

$$\begin{aligned} V\left(S_i - \bigcup_{i=1}^{j-1} S_i\right) &\geq \int_{S_i - \bigcup_{i=1}^{j-1} S_i} Jf^m(x) \, dH_n^n x \\ &= \int_{E^n} H_n^0\left(\left(S_i - \bigcup_{i=1}^{j-1} S_i\right) \cap (f^m)^{-1}\{y\}\right) \, dH_n^n y. \end{aligned}$$

Thus

$$\begin{aligned} V(\text{Int } A) &= V\left(\bigcup_{j=1}^{\infty} \left(S_j - \bigcup_{i=1}^{j-1} S_i\right)\right) \\ &= \sum_{j=1}^{\infty} V\left(S_j - \bigcup_{i=1}^{j-1} S_i\right) \\ &\geq \sum_{j=1}^{\infty} \int_{E^n} H_n^0\left(\left(S_j - \bigcup_{i=1}^{j-1} S_i\right) \cap (f^m)^{-1}\{y\}\right) \, dH_n^n y. \end{aligned}$$

Since the integrands are nonnegative measurable functions (see (2.3)) we may interchange the sum and integral signs and

$$V(\text{Int } A) \geq \int_{E^n} H_n^0(\text{Int } A \cap (f^m)^{-1}\{y\}) \, dH_n^n y$$

$$\begin{aligned}
&\geq \int_{E_n} \chi_{f^m(\text{Int } A)}(y) dH_n^n y \\
&= \int_{f^m(\text{Int } A)} dH_n^n y = H_n^n \left(\prod_{i=1}^n [0, \lambda_i(m)] \right) \\
&= \prod_{i=1}^n \lambda_i(m).
\end{aligned}$$

Part 3. If w is a continuous function, then

$$V(\text{Int } A) \geq \prod_{i=1}^n (\liminf_m \lambda_i(m)).$$

Proof. By Part 2 we have

$$\begin{aligned}
V(\text{Int } A) &\geq \sup_m \prod_{i=1}^n \lambda_i(m) \geq \limsup_m \prod_{i=1}^n \lambda_i(m) \\
&\geq \liminf_m \prod_{i=1}^n \lambda_i(m) = \lim_k \left(\inf_{m \geq k} \prod_{i=1}^n \lambda_i(m) \right),
\end{aligned}$$

and since every $\lambda_i(m) \geq 0$,

$$V(\text{Int } A) \geq \lim_k \left(\prod_{i=1}^n (\inf_{m \geq k} \lambda_i(m)) \right) = \prod_{i=1}^n (\liminf_m \lambda_i(m)).$$

Part 4. If w is a positive continuous function, then

$$\liminf_m \lambda_i(m) \geq \lambda_i.$$

Proof. Suppose that for some i

$$\liminf_m \lambda_i(m) = \lambda_i^* < \lambda_i.$$

Let $\{c_m\}$ be a sequence of arcs in A such that c_m joins a point of $A_i(m)$ to a point of $A'_i(m)$ and

$$\lambda_i(m) \leq \int_{c_m} w(x) dH_n^1 x = r_m < \lambda_i(m) + (1/m).$$

Since

$$\liminf_m r_m = \lambda_i^*,$$

some subsequence $\{r_k\}$ satisfies

$$\lim_k r_k = \lambda_i^*.$$

Without loss of generality, we may assume $0 < r_k < 2\lambda_i$. As A is compact and w is positive and continuous there exists ϵ satisfying

$$0 < \epsilon \leq w(x), \quad x \text{ in } A.$$

Thus each c_k is rectifiable, because by (3.3)

$$0 < q_k = H_n^1(c_k) \leq \frac{1}{\epsilon} \int_{c_k} w \, dH_n^1 = \frac{r_k}{\epsilon} < \frac{2\lambda_i}{\epsilon} < \infty.$$

Let

$$s(t) = \frac{H_n^1(c_k[0, t])}{q_k}, \quad 0 \leq t \leq 1,$$

then $0 \leq s \leq 1$ and we may reparametrize c_k by letting

$$c_k^*(s(t)) = c_k(t).$$

But

$$|(c_k^*(s) - c_k^*(s'))| \leq |s - s'| q_k < \frac{2\lambda_i}{\epsilon} |s - s'|,$$

so the sequence of functions $\{c_k^*\}$ is equicontinuous. Since A is compact there are points y, y' in A such that

$$|y - y'| = \text{diam } A = L < \infty.$$

Hence $|c_k^*(s) - y| \leq L$, for every k and s and $\{c_k^*\}$ is uniformly bounded. By Ascoli's theorem $\{c_k^*\}$ has a uniformly convergent subsequence $\{c_i^*\}$, thus

$$c^*(s) = \lim_i c_i^*(s)$$

is continuous, and by compactness of A it follows that c^* is a curve joining A_i to A'_i in A .

By the lower semi-continuity of length (c.f. [C, page 12])

$$H_n^1(c^*) \leq \liminf_i q_i < \frac{2\lambda_i}{\epsilon} < \infty$$

so c^* is rectifiable. Given any $\eta > 0$ there exists, by definition of

$$\int_{c^*} w(x) \, dH_n^1 x (< \infty),$$

a finite set of points $0 = s_0 < s_1 < s_2 < \dots < s_m = 1$ such that

$$\sum_{k=1}^m m_k |c^*(s_k) - c^*(s_{k-1})| > \int_{c^*} w(x) \, dH_n^1 x - (\eta/3),$$

where $m_k = \min w(x)$ for all x in $c^*[s_{k-1}, s_k]$. Let $m_k(j) = \min w(x)$ for all x in $c_i^*[s_{k-1}, s_k]$. Because $c_i^*(s) \rightarrow c^*(s)$ uniformly on $0 \leq s < 1$ and w is uniformly continuous on A there exists an integer J such that

$$|m_k - m_k(j)| < a\eta, \quad a = \frac{\epsilon}{6\lambda_i},$$

$$|c^*(s_k) - c_i^*(s_k)| < b\eta, \quad b = \frac{1}{6} \min \left[1, \left(\sum_{k=1}^m m_k \right)^{-1} \right],$$

for all $j \geq J$, $k = 0, 1, \dots, m$. Thus

$$|c^*(s_k) - c^*(s_{k-1})| \leq |c_j^*(s_k) - c_j^*(s_{k-1})| + 2b\eta, \quad k = 1, \dots, m.$$

Finally

$$\begin{aligned} r_j &= \int_{c_j^*} w(x) dH_n^1 x \geq \sum_{k=1}^m m_k(j) |c_j^*(s_k) - c_j^*(s_{k-1})| \\ &> \sum_{k=1}^m m_k |c_j^*(s_k) - c_j^*(s_{k-1})| - a\eta H_n^1(c_j^*) \\ &> \sum_{k=1}^m m_k |c^*(s_k) - c^*(s_{k-1})| - 2b\eta \sum_{k=1}^m m_k - \frac{2\lambda_i}{\epsilon} a\eta \\ &> \int_{c^*} w(x) dH_n^1 x - \eta, \quad \text{for all } j \geq J. \end{aligned}$$

Hence, by Kaluszay's theorem (c.f. [P, p. 17]), c^* contains an arc joining A_i to A'_i , so

$$\lambda_i > \lambda_i^* = \lim_i r_j \geq \int_{c^*} w(x) dH_n^1 x \geq \lambda_i,$$

a contradiction. Therefore

$$\liminf_m \lambda_i(m) \geq \lambda_i, \quad i = 1, \dots, n.$$

Part 5. If w is a positive continuous function, then there exists an arc c in A joining A_i to A'_i such that

$$\int_c w dH_n^1 = \lambda_i.$$

Proof. Let $\{c_k\}$ be a sequence of arcs in A joining a point of A_i to a point of A'_i such that

$$r_k = \int_{c_k} w dH_n^1 < 2\lambda_i, \quad k = 1, 2, \dots,$$

is a nonincreasing sequence of real numbers satisfying

$$\lim_k r_k = \lambda_i.$$

Precisely as in the proof of Part 4, each c_k is rectifiable and satisfies $H_n^1(c_k) < 2\lambda_i/\epsilon$, where $\epsilon = \min w(x)$, for all x in A . Parametrizing each c_k by arc-length it follows that $\{c_k^*\}$ has a subsequence $\{c_i^*\}$ which converges uniformly to a curve c^* joining A_i to A'_i in A . As in the last part of the argument in Part 4

$$\lambda_i = \lim_i r_i \geq \int_{c^*} w dH_n^1 \geq \lambda_i.$$

Since $w > 0$, c^* can have no multiple points (c.f. [B, p. 263]), hence $c^* = c$.

Part 6. If w is a nonnegative continuous function, then

$$V(\text{Int } A) \geq \prod_{i=1}^n \lambda_i .$$

Proof. Define $w_i(x) = \max(w(x), 1/j)$, for x in A . Then $\{w_i(x)\}$ is a non-increasing sequence of positive continuous functions whose uniform limit over A is $w(x)$. By Part 5 there exists an arc c_{ij} joining A_i to A'_i in A such that

$$\lambda_{ij} = \int_{c_{ij}} w_i dH_n^1, \quad i = 1, \dots, n, \quad j = 1, 2, \dots,$$

where λ_{ij} is the weighted distance corresponding to w_i . But

$$\int_{c_{ij}} w_i dH_n^1 \geq \int_{c_{ij}} w dH_n^1,$$

implying that $\lambda_{ij} \geq \lambda_i$, $i = 1, \dots, n$, $j = 1, 2, \dots$. By Parts 3 and 4 each w_j satisfies the inequality

$$V_j(\text{Int } A) \geq \prod_{i=1}^n \lambda_{ij} \geq \prod_{i=1}^n \lambda_i,$$

where V_j is the weighted volume corresponding to w_j . Hence

$$V(\text{Int } A) = \lim_j V_j(\text{Int } A) \geq \prod_{i=1}^n \lambda_i .$$

Part 7. If w is a nonnegative upper semicontinuous function, then

$$V(A) \geq V(\text{Int } A) \geq \prod_{i=1}^n \lambda_i .$$

Proof. Since w is nonnegative and upper semicontinuous there exists a non-increasing sequence $\{w_i\}$ of continuous functions such that

$$w(x) = \lim_i w_i(x)$$

and $w_i(x) \leq M$ for all x in A , where M is the maximum of $w(x)$ on A . Since $w_i \geq w$,

$$\int_c w_i dH_n^1 \geq \int_c w dH_n^1$$

on every arc c joining A_i to A'_i in A , hence $\lambda_{ij} \geq \lambda_i$. By Part 6 and Lebesgue's bounded convergence theorem (c.f. [S, p. 29]) we have

$$V(\text{Int } A) = \lim_j V_j(\text{Int } A) \geq \lim_j \prod_{i=1}^n \lambda_{ij} \geq \prod_{i=1}^n \lambda_i .$$

4. The general case. In this section we remove the condition that the weight function w be upper semicontinuous. We retain the notations introduced in Section 3.

4.1 Theorem. *Let w be a nonnegative Borel-measurable weight function such that $V(\text{Int } A) < \infty$. Then*

$$\lambda_i < \infty, \text{ for } i = 1, \dots, n.$$

Proof. Let w^* agree with w on $\text{Int } A$ and vanish elsewhere. For $\epsilon > 0$ and x in $\text{Int } A$ we have, by Hölder's inequality and Fubini's theorem,

$$\begin{aligned} \int \left(\int w^* dH_n^1 \right)^n dH_n^{n-1} &\leq \int \left(\epsilon^{n-1} \int w^{*n} dH_n^1 \right) dH_n^{n-1} \\ &= \epsilon^{n-1} \int_{Q_{\epsilon^n}(x)} w^{*n} dH_n^n \\ &\leq \epsilon^{n-1} V(\text{Int } A) < \infty. \end{aligned}$$

hence

$$\int w^* dH_n^1 < \infty$$

on almost every line segment in $Q_\epsilon^n(x)$ parallel to some coordinate axis.

By [D, 4.4] there exist points x, x' in $\text{Int } A_i, \text{Int } A'_i$ respectively, and a rectifiable arc c joining x to x' such that $c - \{x, x'\}$ lies entirely in $\text{Int } A$.

Let

$$\delta = \min_{i \neq i'} \{ \text{dist}(c, A_i), \text{dist}(c, A'_i) \},$$

and choose $\epsilon < \delta/2$. Parametrize c by arc-length

$$c : c(t), \quad 0 \leq t \leq L = H_n^1(c),$$

with $c(0) = x$ and $c(L) = x'$. Set $t_0 = 0$ and let t_{n+1} be the smallest number, if any, such that

$$(i) \quad t_n < t_{n+1} \leq L, \quad n \geq 0, \text{ and}$$

$$(ii) \quad c(t_{n+1}) \notin \{c - \text{Int } Q_\epsilon^n(c(t_n))\}.$$

Clearly $t_{n+1} \geq t_n + \epsilon/2$, hence only a finite set of such numbers exist: $t_0, t_1, \dots, t_N$. Set $L = t_{N+1}$. Observe that $Q_\epsilon^n(c(t_n))$ contains the points $c(t_{n-1}), c(t_{n+1}), 1 \leq n \leq N$, hence

$$H_n^n(Q_\epsilon^n(c(t_n)) \cap Q_\epsilon^n(c(t_{n+1}))) \geq (\epsilon/2^n) > 0.$$

By the continuity of g and the construction above, there exists a polygonal curve K joining a point y of $\text{Int } A_i$ to a point y' of $\text{Int } A'_i$ such that $K - \{y, y'\} \subset \text{Int } A$, K is composed of a finite number of line segments each parallel to some coordinate axis, and

$$\int_K w^* dH_n^1 = \int_K w dH_n^1 < \infty,$$

Thus $\lambda_i < \infty$.

4.2 Theorem. Let w be a positive lower semi-continuous weight function satisfying $\lambda_i < \infty$, $i = 1, \dots, n$. Then

$$V(\text{Int } A) \geq \prod_{i=1}^n \lambda_i ,$$

and there exist arcs c_i in A joining A_i to A'_i such that

$$\int_{c_i} w dH_n^1 = \lambda_i , \quad i = 1, \dots, n.$$

Proof. Since A is compact there is a number d such that $0 < d \leq w(x)$ for x in A . Then there exists a nondecreasing sequence of positive continuous weight functions $\{w_k\}$ such that $0 < d \leq w_k(x)$ and

$$w(x) = \lim_k w_k(x).$$

By Lebesgue's theorem on integration of monotone sequences of functions (c.f. [S, p. 28]) we have

$$V(\text{Int } A) = \lim_k V_k(\text{Int } A),$$

and

$$\int_c w dH_n^1 = \lim_k \int_c w_k dH_n^1 ,$$

for every curve c in A . By Part 5 of theorem (3.6) there exists a rectifiable arc c_{ik} joining A_i to A'_i in A such that

$$\lambda_{ik} = \int_{c_{ik}} w_k dH_n^1 , \quad i = 1, \dots, n, \quad k = 1, 2, \dots .$$

Further, since

$$0 < dd_i \leq dH_i^1(c_{ik}) \leq \lambda_{ik} \leq \lambda_i < \infty$$

implies that

$$H_n^1(c_{ik}) \leq \lambda_i/d,$$

the proof of Part 4 of Theorem (3.6) applies, and if the arcs c_{ik} are parametrized by arc-length, the sequence $\{c_{ik}^*\}$ has a subsequence $\{c_{ij}^*\}$ which converges uniformly to a rectifiable curve c_i^* joining A_i to A'_i in A . Dropping the $*$, we may assume that $\{c_{ik}\}$ converges uniformly to c_i , $i = 1, \dots, n$, and

$$\int_{c_i} w_i dH_n^1 \leq \liminf_k \int_{c_{ik}} w_i dH_n^1 , \quad i = 1, \dots, n, \quad j = 1, 2, \dots .$$

Since the sequence $\{w_k\}$ is nondecreasing we have

$$\lambda_{ij} \leq \int_{c_{ik}} w_i dH_n^1 \leq \lambda_{ik} \leq \lambda_i < \infty , \quad \text{for } j \leq k.$$

Thus

$$\begin{aligned}\lambda_i &\leq \int_{c_i} w \, dH_n^1 = \lim_i \int_{c_i} w_i \, dH_n^1 \\ &\leq \lim_i \left\{ \liminf_k \int_{c_{ik}} w_i \, dH_n^1 \right\} \\ &\leq \lim_i \{ \lim_k \lambda_{ik} \} = \lim_k \lambda_{ik} \leq \lambda_i ,\end{aligned}$$

$i = 1, \dots, n$, so the weighted distance is attained. Finally, by Theorem (3.6),

$$\begin{aligned}V(\text{Int } A) &= \lim_k V_k(\text{Int } A) \\ &\geq \lim_k \prod_{i=1}^n \lambda_{ik} \\ &= \prod_{i=1}^n \lim_k \lambda_{ik} = \prod_{i=1}^n \lambda_i .\end{aligned}$$

4.3 Corollary. *Let w be a nonnegative lower semicontinuous weight function satisfying $\lambda_i < \infty$, $i = 1, \dots, n$. Then*

$$V(\text{Int } A) \geq \prod_{i=1}^n \lambda_i .$$

Proof. Define $w_k(x) = \max(w(x), 1/k)$ for all x . Then $\{w_k(x)\}$ is a non-increasing sequence of positive lower semicontinuous functions converging to $w(x)$. Clearly

$$\lambda_{ik} \geq \lambda_i , \quad i = 1, \dots, n, \quad k = 1, 2, \dots ,$$

and, letting $S_k = \{x \in \text{Int } A : w(x) = w_k(x)\}$, we have

$$\begin{aligned}V(\text{Int } A) &\leq V_k(\text{Int } A) = \int_{S_k} w^n \, dH_n^n + \frac{1}{k^n} \int_{(\text{Int } A) - S_k} dH_n^n \\ &\leq V(\text{Int } A) + H_n^n(\text{Int } A)/k^n \\ &\rightarrow V(\text{Int } A), \quad \text{as } k \rightarrow \infty .\end{aligned}$$

Thus, if $\lambda_{ik} < \infty$ for all i and sufficiently large k ,

$$V(\text{Int } A) = \lim_k V_k(\text{Int } A) \geq \lim_k \prod_{i=1}^n \lambda_{ik} \geq \prod_{i=1}^n \lambda_i .$$

Otherwise, $\lambda_{ik} = +\infty$ for some i and all k , so (4.1) implies $V_k(\text{Int } A) = \infty$ for all k . Hence

$$V(\text{Int } A) = +\infty \geq \prod_{i=1}^n \lambda_i .$$

4.4 Theorem. *Let w be a nonnegative Borel-measurable weight function. Then*

$$V(\text{Int } A) \geq \prod_{i=1}^n \lambda_i .$$

Proof. By the theorem of Vitali-Caratheodory (c.f. [S, page 75]) there exists a nonincreasing sequence $\{w_k\}$ of nonnegative lower semi-continuous functions such that $w_k(x) \geq w(x)$ and

$$\lim_k \int_{E^n} (w_k^n(x) - w^n(x)) dH_n^n x = 0,$$

using the convention that $w_k^n(x) - w^n(x) = 0$ at any x for which $w_k(x) = w(x) = +\infty$. Clearly

$$\lambda_{ik} \geq \lambda_i, i = 1, \dots, n, k = 1, 2, \dots .$$

There is nothing to prove if $V(\text{Int } A) = +\infty$. If $V(\text{Int } A) < \infty$ then $V_k(\text{Int } A) < \infty$ and by (4.1) $\lambda_{ik} < \infty, i = 1, \dots, n, k = 1, 2, \dots$. Finally by (4.3)

$$V(\text{Int } A) = \lim_k V_k(\text{Int } A) \geq \lim_k \prod_{i=1}^n \lambda_{ik} \geq \prod_{i=1}^n \lambda_i .$$

4.5 Theorem. *Let w_i be a nonnegative Borel-measurable weight function with w_i -weighted volume $V_i(\text{Int } A) < \infty, i = 1, \dots, n$, and let $\lambda_{i,i}$ denote the w_i -weighted distance between A_i and A'_i in A . Then*

$$\int_{\text{Int } A} \left(\prod_{i=1}^n w_i(x) \right) dH_n^n x \geq \prod_{i=1}^n \lambda_{i,i} .$$

Proof. To begin assume each w_i is positive and continuous. Define $f_m : \text{Int } A \rightarrow E^n$ by letting its i^{th} coordinate

$$f_i^m(x) = \min(\lambda_{i,i}(m), \lambda_i(A_i(m), x)),$$

where $\lambda_{i,i}(m)$ and $\lambda_i(A_i(m), x)$ are the w_i -weighted distance between $A_i(m)$ and $A'_i(m)$, and $A_i(m)$ and x in A respectively. By (3.5) and the proofs of (3.6, Part 1 and 2) we have that f^m is locally Lipschitzian on $\text{Int } A$,

$$f^m(\text{Int } A) = \prod_{i=1}^n [0, \lambda_{i,i}(m)],$$

and

$$|\text{grad } f_i^m(x)| \leq w_i(x).$$

Further, some countable subcollection $\{S_i\}$ of open disks covers $\text{Int } A$ and f^m is Lipschitzian in each S_i . Thus, by Hadamard's inequality and (2.3),

$$\begin{aligned} \int_{S_1} \left(\prod_{i=1}^n w_i \right) dH_n^n &\geq \int_{S_1} \prod_{i=1}^n |\text{grad } f_i^m| dH_n^n \\ &\geq \int_{S_1} Jf^m dH_n^n \end{aligned}$$

$$= \int_{E^n} H_n^0(S_1 \cap (f^m)^{-1}\{y\}) dH_n^n y,$$

and similarly, for $j > 1$,

$$\int_{S_j - \bigcup_{i=1}^{j-1} S_i} \left(\prod_{i=1}^n w_i \right) dH_n^n \geq \int_{E^n} H_n^0 \left[\left(S_j - \bigcup_{i=1}^{j-1} S_i \right) \cap (f^m)^{-1}\{y\} \right] dH_n^n y,$$

implying as in (3.6, Part 2)

$$\begin{aligned} \int_{\text{Int } A} \left(\prod_{i=1}^n w_i \right) dH_n^n &\geq \int_{E^n} H_n^0(\text{Int } A \cap (f^m)^{-1}\{y\}) dH_n^n y \\ &\geq \int_{f^m(\text{Int } A)} dH_n^n = \prod_{i=1}^n \lambda_{i,i}(m). \end{aligned}$$

Then

$$\int_{\text{Int } A} \left(\prod_{i=1}^n w_i \right) dH_n^n \geq \liminf_m \prod_{i=1}^n \lambda_{i,i}(m) \geq \prod_{i=1}^n (\liminf_m \lambda_{i,i}(m)),$$

and, since each w_i is positive and continuous, (3.6, Part 4) applies

$$\liminf_m \lambda_{i,i}(m) \geq \lambda_{i,i},$$

yielding

$$\int_{\text{Int } A} \left(\prod_{i=1}^n w_i \right) dH_n^n \geq \prod_{i=1}^n \lambda_{i,i},$$

the desired result for positive continuous weight functions. Moreover (3.6, Part 5) guarantees the existence of arcs $c_{i,i}$ joining A_i to A'_i in A such that

$$\int_{c_{i,i}} w_i dH_n^1 = \lambda_{i,i}.$$

Assume now that each w_i is a positive lower semi-continuous function satisfying $V_i(\text{Int } A) < \infty$. Then $\lambda_{i,i} < \infty$, $i = 1, \dots, n$, and, since A is compact, there exist number b_i such that

$$0 < b_i \leq w_i(x), \text{ for } x \text{ in } A, i = 1, \dots, n.$$

Then there exist nondecreasing sequences $\{w_{ik}\}_{k=1}^\infty$, $i = 1, \dots, n$, of positive continuous functions satisfying $0 < b_i \leq w_{ik}(x)$ and

$$w_i(x) = \lim_k w_{ik}(x), \quad i = 1, \dots, n.$$

Thus the sequence $\{\prod_{i=1}^n w_{ik}\}_{k=1}^\infty$ is nondecreasing, so by Lebesgue's theorem on integration of monotone sequences of functions we have

$$\int_{\text{Int } A} \left(\prod_{i=1}^n w_i \right) dH_n^n = \lim_k \int_{\text{Int } A} \left(\prod_{i=1}^n w_{ik} \right) dH_n^n,$$

and

$$\int_c w_i dH_n^1 = \lim_k \int_c w_{ik} dH_n^1,$$

for every curve c in A . Let λ_{iik} denote the w_{ik} -weighted distance between A_i and A'_i in A . The proof of (4.2) holds and we have

$$\lim_k \lambda_{iik} = \lambda_{ii}, \quad i = 1, \dots, n,$$

and that there is a curve c_{ii} joining A_i to A'_i in A satisfying

$$\lambda_{ii} = \int_{c_{ii}} w_i dH_n^1.$$

Thus

$$\begin{aligned} \int_{\text{Int } A} \left(\prod_{i=1}^n w_i \right) dH_n^n &= \lim_k \int_{\text{Int } A} \left(\prod_{i=1}^n w_{ik} \right) dH_n^n \\ &\geq \lim_k \left(\prod_{i=1}^n \lambda_{iik} \right) = \prod_{i=1}^n (\lim_k \lambda_{iik}) = \prod_{i=1}^n \lambda_{ii}. \end{aligned}$$

Allow the w_i to be nonnegative lower semi-continuous functions satisfying $V_i(\text{Int } A) < \infty$. Define $w_{ik}(x) = \max(w_i(x), 1/k)$ for all x , yielding n nonincreasing sequences of positive lower semicontinuous functions such that $V_{ik}(\text{Int } A) < \infty$ and

$$\lim_k w_{ik} = w_i, \quad i = 1, \dots, n.$$

Then $\lambda_{iik} \geq \lambda_{ii}$, $k = 1, 2, \dots$, $i = 1, \dots, n$, $\{\prod_{i=1}^n w_{ik}\}$ is a nonincreasing sequence of positive lower semi-continuous functions converging to $\prod_{i=1}^n w_i$, and $\prod_{i=1}^n w_{ik}$ is integrable on $\text{Int } A$ since

$$\left(\int_{\text{Int } A} \left(\prod_{i=1}^n w_{ik} \right) dH_n^n \right)^n \leq \prod_{i=1}^n \int_{\text{Int } A} w_{ik}^n dH_n^n < \infty,$$

by Hölder's inequality (see [BB, p. 20]). By Lebesgue's theorem on term-by-term integration

$$\begin{aligned} \int_{\text{Int } A} \left(\prod_{i=1}^n w_i \right) dH_n^n &= \lim_k \int_{\text{Int } A} \left(\prod_{i=1}^n w_{ik} \right) dH_n^n \\ &\geq \lim_k \prod_{i=1}^n \lambda_{iik} \\ &\geq \prod_{i=1}^n \lambda_{ii}. \end{aligned}$$

Finally, let the w_i be nonnegative Borel-measurable functions with $V_i(\text{Int } A) < \infty$. The theorem of Vitali-Caratheodory yields n nonincreasing sequences

$\{w_{ik}\}$ of nonnegative lower semi-continuous functions satisfying precisely the same properties as the previous sequences $\{w_{ik}\}$. The proof above yields

$$\int_{\text{Int } A} \left(\prod_{i=1}^n w_i \right) dH_n^n \geq \prod_{i=1}^n \lambda_{ii} .$$

5. A module inequality.

5.1 Definition. Let Γ be a family of curves in E^n . Then the *module* of Γ is defined by

$$M(\Gamma) = \inf_w V_w(E^n) / \lambda_w^n ,$$

where the infimum is taken over all nonnegative Borel-measurable weight functions w for which the quotient has meaning and

$$V_w(E^n) = \int_{E^n} w^n dH_n^n , \quad \lambda_w = \inf_{c \in \Gamma} \int_c w dH_n^1 .$$

We say such functions are *admissible* for Γ .

5.2 Remark. Denote by Γ_i , $i = 1, \dots, n$, the family of all arcs in A joining A_i to A'_i , and let λ_{iw} , $V_w(A)$ be the w -weighted distance between A_i and A'_i in A and w -weighted volume of A , respectively. Then

$$M(\Gamma_i) = \inf_w V_w(A) / \lambda_{iw}^n \leq H_n^n(A) / d_i^n ,$$

so we need only consider functions w for which $V_w(A) < \infty$. By the same token, if $V_w(A) = 0$, a proof similar to (4.1) yields $\lambda_{iw} = 0$, for all i , implying that w is not admissible.

If A is a right cylinder and Γ_i is the family of curves joining the base D to the top of A , then for each c in Γ_i perpendicular to D , by Hölder's inequality,

$$\lambda_{iw}^n \leq \left(\int_c w dH_n^1 \right)^n \leq d_i^{n-1} \cdot \int_c w^n dH_n^1 ,$$

which by Fubini's theorem becomes

$$\lambda_{iw}^n H_n^n(A) / d_i = \lambda_{iw}^n H_n^{n-1}(D) \leq d_i^{n-1} \cdot \int_D \left(\int_c w^n dH_n^1 \right) dH_n^{n-1} = d_i^{n-1} V_w(A) ,$$

so for all admissible functions w we have

$$H_n^n(A) / d_i^n \leq V_w(A) / \lambda_{iw}^n ,$$

implying, together with the upper bound of $M(\Gamma_i)$, that

$$M(\Gamma_i) = H_n^n(A) / d_i^n .$$

If $n = 2$ and f maps A conformally onto A^* , let Γ_i^* be the image set of Γ_i , $i = 1, 2$. For any admissible function w on A define w^* on A^* by

$$w^*(f(x)) = w(x) / |f'(x)| , \quad x \text{ in } A .$$

Then for c^* in Γ_i^*

$$\int_{c^*} w^*(y) |dy| = \int_c w(x) |dx|,$$

and

$$\int_{A^*} w^{*2}(y) dH_2^2 y = \int_A w^2(x) dH_2^2 x,$$

implying that w^* is admissible for Γ_i^* and $M(\Gamma_i) \leq M(\Gamma_i^*)$. By considering f^{-1} we obtain the reverse inequality, hence the module of a family of curves is a conformal invariant in E^2 . In particular, since A may be mapped conformally onto a rectangle $[0, a] \times [0, b]$, we have

$$M(\Gamma_1)M(\Gamma_2) = \frac{ab}{a^2} \frac{ab}{b^2} = 1.$$

We prove the following generalization of this identity to E^n .

5.3 Theorem

$$\prod_{i=1}^n M(\Gamma_i) \geq 1.$$

Proof. For each $i = 1, \dots, n$, there exists a sequence $\{w_{ik}\}$ of functions admissible for Γ_i such that the sequence $\{V_{ik}/\lambda_{ik}^n\}$ is nonincreasing and converges to $M(\Gamma_i)$, where

$$V_{ik} = V_{w_{ik}}(A), \lambda_{ik} = \lambda_{w_{ik}},$$

and $0 < V_{ik}, \lambda_{ik} < \infty$. Then, by (4.5) and Hölder's inequality,

$$\begin{aligned} \prod_{i=1}^n M(\Gamma_i) &= \lim_k \prod_{i=1}^n \left(\frac{V_{ik}}{\lambda_{ik}^n} \right) \\ &= \lim_k \left[\frac{\prod_{i=1}^n V_{ik}}{\prod_{i=1}^n \lambda_{ik}^n} \right] \\ &\geq \inf_k \left[\frac{\int_A \left(\prod_{i=1}^n w_{ik} \right) dH_n^n}{\prod_{i=1}^n \lambda_{ik}} \right]^n \geq 1. \end{aligned}$$

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University of Utah

Salt Lake City, Utah

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