Submersions of Codimension 1

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Communicated by S. S. CHERN

1. Introduction. A differentiable map $f : M \rightarrow X$ is called a *submersion* provided the differential $df : M_p \rightarrow X_{f(p)}$ is surjective for all $p \in M$. This implies of course that the dimension of M is greater than or equal to the dimension of X . Setting $r = \dim M - \dim X$ (r is the *codimension* of f) we note that $f^{-1}(x)$ constitutes a regularly imbedded r -dimensional submanifold of M for all $x \in X$. These submanifolds will be called the *fibres* of f . We shall assume that M and X are paracompact Hausdorff C^∞ -manifolds, and that X is connected. When M is compact, or more generally when the fibres of f are all compact, f is known to be a fibration (cf. [2]), and for this case one has a fairly extensive theory. On the other hand, very little is known as yet regarding submersions which are not fibrations. One notable result in this direction has recently been obtained by Anthony Phillips, who has studied the spaces of all submersions $f : M \rightarrow X$ for given pairs (M, X) (cf. [4]). In contrast to these investigations of Phillips we shall be interested to study the structure of individual submersions which map a given space M onto a target X . Moreover, the present paper confines itself to the case of codimension 1.

The fibres $f^{-1}(x)$ consist now of one or more connected components (we call these the *leaves* of f) each of which is diffeomorphic to the circle or to the real line, depending on whether it is compact or noncompact. The submersion will be called *simple* if all its fibres are connected. It will be called *orientable* if its fibres can be oriented in a continuous manner. As it turns out, the homology structure of a simple orientable f can be described in terms of a generalized Thom-Gysin sequence as follows (cf. [7]). We have learned from a private communication that the sequence given by Theorem A was found independently by Glen Bredon.)

Theorem A. *Let $f : M \rightarrow X$ be a simple orientable submersion of codimension 1, and let U denote the set of all $x \in X$ such that $f^{-1}(x)$ is compact. There exists an exact sequence*

$$\cdots \rightarrow H_q(M) \xrightarrow{f^*} H_q(X) \xrightarrow{\alpha} H_{q-2}(U) \xrightarrow{\beta} H_{q-1}(M) \rightarrow \cdots$$

relating the integral homology groups.

¹ Supported by National Science Foundation Grant GP-6648

Some of the consequences entailed by Theorem A are particularly relevant to the questions with which we shall be concerned. We will briefly consider three of these corollaries. E^n shall denote Euclidean n -space; S^n an n -sphere.

A1. *Let $f : E^3 \rightarrow X$ be a simple submersion of codimension 1. Then U is connected, and $X = E^2$ or S^2 , depending on whether U is empty or not.*

To see this, we note that f is orientable (because E^3 is simply connected) and one has an exact sequence

$$0 \rightarrow H_2(X) \xrightarrow{f_*} H_0(U) \rightarrow 0 \rightarrow H_1(X) \rightarrow 0$$

by Theorem A. Thus $H_1(X) = 0$ and $H_2(X) \approx H_0(U)$, which immediately implies A1.

A2. *Let $f : E^{n+1} \rightarrow S^n$ be a simple submersion. Then $n = 2$.*

Suppose U to be empty. Then $H_q(E^{n+1}) \approx H_q(S^n)$ for all q by Theorem A, which is impossible. Therefore U is not empty, and now exactness of

$$H_2(S^n) \xrightarrow{\alpha} H_0(U) \rightarrow 0$$

implies that $n = 2$.

It should be noted that simple submersions $f : E^3 \rightarrow S^2$ do exist. For example, taking S^3 to be the 1-point compactification of E^3 and $\psi : E^3 \rightarrow S^3$ the corresponding inclusion, let $f = h \circ \psi$, where $h : S^3 \rightarrow S^2$ denotes the Hopf fibration.

A3. *Let $f : M \rightarrow X$ be a simple orientable submersion of codimension 1. If for some $x \in X$, $f^{-1}(x)$ is compact and contractible in M , then $H_2(X)$ has positive rank.*

Consider $H_2(X) \xrightarrow{\alpha} H_0(U) \xrightarrow{\beta} H_1(M)$. The fibre $f^{-1}(x)$ determines a free generator in $\ker \beta$, and this implies A3 by exactness.

Except for the first assertion contained in A1 (i.e., that U is connected) we shall recover propositions A1, A2 and A3 by an entirely different approach, i.e., as corollaries of the first main result of this paper (Theorem B). The second main result (Theorem C) concerns nonsimple submersions $f : E^3 \rightarrow X$.

2. Simple submersions.

Theorem B. *Let $f : M \rightarrow X$ be a simple submersion of codimension 1. There exists a 1-sphere bundle $p : B \rightarrow X$ and an imbedding $\psi : M \rightarrow B$ such that $f = p \circ \psi$.*

To begin with, there exists an open covering $\{W_\alpha\}$ of X with local cross-sections $\sigma_\alpha : W_\alpha \rightarrow M$. As might be expected, the desired 1-sphere bundle will be constructed from coordinate transformations $g_{\alpha\beta} : W_\alpha \cap W_\beta \rightarrow G$, where G denotes the group of homeomorphisms $g : S^1 \rightarrow S^1$ with compact-open topology (cf. [9]; §§2, 3). This will give us a bundle $p : B \rightarrow X$ with fibre S^1 and group G . With the aid of a Riemannian metric on B one can then effect a reduction of the structure group to the orthogonal group O_2 .

To define the coordinate transformations we shall avail ourselves of a complete Riemannian metric μ on M . Let U denote the set of all $x \in X$ such that $f^{-1}(x)$ is compact, and let V denote the complement of U . Thus $f^{-1}(x)$ is homeomorphic to S^1 or E^1 , depending on whether x belongs to U or V . For $x \in U$ let $2\pi r(x)$ denote the circumference of $f^{-1}(x)$ measured with respect to μ , and let a function $\lambda : X \rightarrow (0, \frac{1}{2}]$ be defined by setting

$$\lambda(x) = \begin{cases} \frac{1}{\pi} \arctan \pi r(x) & \text{for } x \in U, \\ \frac{1}{2} & \text{for } x \in V. \end{cases}$$

Since μ is complete, the fibres $f^{-1}(x)$ have infinite length for $x \in V$, and this makes λ continuous. For each set W_α in the given covering of X we can orient the fibres $\{f^{-1}(x) : x \in W_\alpha\}$ in a continuous manner, and this permits us to define real-valued functions $S_\alpha : f^{-1}(V \cap W_\alpha) \rightarrow R$ by taking $S_\alpha(y)$ to be the signed distance (measured with respect to μ) from $\sigma_\alpha \circ f(y)$ to y along the given fibre. Regarding S^1 as the closed interval $[-\pi, \pi]$ with end points identified, we define maps $h_\alpha : f^{-1}(V \cap W_\alpha) \rightarrow S^1$ by setting

$$(2.1) \quad h_\alpha(y) = \frac{1}{\lambda \circ f(y)} \arctan S_\alpha(y).$$

To define corresponding functions $h_\alpha : f^{-1}(U \cap W_\alpha) \rightarrow S^1$ we first observe that when $x \in U \cap W_\alpha$, every point $y \in f^{-1}(x)$ can be joined to $\sigma_\alpha(x)$ by two distinct arcs in the given fibre. Let $q_\alpha(x)$ denote the point on $f^{-1}(x)$ for which these two arcs have equal length, and consider a point $y \in f^{-1}(x)$ such that $y \neq q_\alpha(x)$. The signed distance from $\sigma_\alpha(x)$ to y along the given fibre is obviously defined, and we again denote this quantity by $S_\alpha(y)$. A map $h_\alpha : f^{-1}(U \cap W_\alpha) \rightarrow S^1$ can now be defined by setting

$$(2.2) \quad h_\alpha(y) = \begin{cases} \frac{1}{\lambda \circ f(y)} \arctan S_\alpha(y) & \text{for } y \neq q_\alpha \circ f(y), \\ [\pi] & \text{for } y = q_\alpha \circ f(y). \end{cases}$$

Moreover, the functions $h_\alpha : f^{-1}(W_\alpha) \rightarrow S^1$ defined by (2.1) and (2.2) are clearly continuous and injective.

Given $x \in W_\alpha \cap W_\beta$, there exists a unique transformation $g_{\alpha\beta}(x) \in G$ such that

$$(2.3) \quad h_\alpha(y) = g_{\alpha\beta}(x) h_\beta(y)$$

for all $y \in f^{-1}(x)$. For if $x \in U$ the functions $h_\alpha, h_\beta : f^{-1}(x) \rightarrow S^1$ are homeomorphisms, so that our assertion certainly holds in this case. On the other hand, for $x \in V$ the functions h_α, h_β map $f^{-1}(x)$ homeomorphically onto the complement of $[\pi]$ in S^1 . The corresponding transformation $g_{\alpha\beta}(x) : S^1 \rightarrow S^1$ must consequently be defined by setting

$$g_{\alpha\beta}(x)\theta = \begin{cases} h_\alpha \circ h_\beta^{-1}(\theta) & \text{for } \theta \neq [\pi], \\ [\pi] & \text{for } \theta = [\pi]; \end{cases}$$

and this gives a homeomorphism of S^1 onto itself. The verification that the functions $g_{\alpha\beta} : W_\alpha \cap W_\beta \rightarrow G$ thus defined are continuous is straightforward and will be omitted. Moreover, one observes that the conditions

- (i) $g_{\alpha\alpha}(x) = \text{identity}$ for $x \in W_\alpha$;
- (ii) $g_{\alpha\beta}(x)g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$ for $x \in W_\alpha \cap W_\beta \cap W_\gamma$;

for a system of coordinate transformations are obviously satisfied. By the usual construction one now obtains a bundle $p : B \rightarrow X$, together with natural projections $\rho_\alpha : W_\alpha \times S^1 \rightarrow B$ such that

$$\rho_\alpha(x, g_{\alpha\beta}(x)\theta) = \rho_\beta(x, \theta) \text{ for } (x, \theta) \in W_\alpha \cap W_\beta \times S^1.$$

It follows by (2.3) that one can define a map $\psi : M \rightarrow B$ by setting

$$\psi(y) = \rho_\alpha(f(y), h_\alpha(y)) \text{ for } y \in f^{-1}(W_\alpha).$$

This gives an imbedding of M into B such that $f = p \circ \psi$.

A few remarks are called for at this point. (1) If $f : M \rightarrow X$ is simple (orientable) and $U = X$, then f is the projection of a (orientable) 1-sphere bundle. (2) If $f : M \rightarrow X$ is simple and U empty, then by the preceding considerations f is the projection of a bundle with fibre R and group consisting of affine transformations $t \rightarrow \epsilon t + a$, with $\epsilon = \pm 1$ and $a \in R$. (3) If further $f : M \rightarrow X$ is orientable, then the group of this bundle can be reduced to the additive group R , and consequently f is the projection of a trivial bundle. We proceed to consider some corollaries of Theorem B.

B1. *Let $f : M \rightarrow X$ be a simple submersion of codimension 1. If $f^{-1}(x)$ is compact and contractible in M for some $x \in X$, then $\pi_2(X, x)$ has positive rank.*

To show this we note that f factors through a 1-sphere bundle $p : B \rightarrow X$. It follows that the fibre $p^{-1}(x)$ is contractible in B , and consequently the bundle is not equivalent to a product. The characteristic homomorphism $\chi : \pi_2(X, x) \rightarrow \pi_1(U_2)$ (cf. [9], §17) of the bundle must therefore be nontrivial, and this implies B1.

B2. *Let $f : E^3 \rightarrow X$ be a simple submersion of codimension 1. Then $X = E^2$ or S^2 , depending on whether U is empty or not. (We have thus recovered a part of proposition A1.)*

For if U is empty, then $X = E^2$ by remark (3). On the other hand, if U is not empty, then $\pi_2(X, x)$ has positive rank, and this implies that $X = S^2$.

B3. *Let $f : M \rightarrow X$ be a simple orientable submersion of codimension 1. If $f^{-1}(x)$ is compact and contractible in M for some $x \in X$, then the integral homology group $H_2(X)$ has positive rank. (This is precisely A2.)*

To see this, we consider the 1-sphere bundle ξ given by Theorem B. Since f is orientable, ξ is likewise orientable and may consequently be regarded as a

principal bundle of the rotation group R_2 . Our hypothesis implies that the fibre over x bounds in B , which implies that ξ is not a trivial bundle. It follows (cf. [9], §8.3) that ξ does not admit a cross-section. On the other hand, since $\pi_n(S^1)$ is trivial for $n > 1$, ξ will admit a cross-section unless its Euler class χ admits a nonzero period. (This follows by standard obstruction theory. See [9], part III.) Since χ is an integral cohomology class of dimension 2, this implies that $H_2(X)$ has positive rank.

B4. *Let $f : E^3 \rightarrow S^2$ be a simple submersion. There exists an imbedding $\psi : E^3 \rightarrow S^3$ such that $f = h \circ \psi$, where $h : S^3 \rightarrow S^2$ is the projection of the Hopf bundle.*

It will suffice to show that the bundle ξ given by Theorem B is equivalent to the Hopf bundle, i.e., that the two bundles have the same characteristic homomorphism. To this end we note that $f^{-1}(x)$ must be compact for some $x \in S^2$, so that $Y = p^{-1}(x)$ must be contractible in B , where $p : B \rightarrow S^2$ denotes the projection of ξ . Consider the homotopy sequence

$$\cdots \rightarrow \pi_2(S^2) \xrightarrow{\Delta} \pi_1(Y) \xrightarrow{i_*} \pi_1(B) \rightarrow \cdots$$

It follows that $\pi_1(Y) = \ker i_*$, which implies by exactness that $\Delta : \pi_2(S^2) \cong \pi_1(Y)$. But this gives the characteristic homomorphism of the Hopf bundle (cf. [9], §26.2).

3. Submersions of Euclidean space. Consider a regular 1-dimensional foliation Θ on E^3 and let l denote a compact leaf. By the stability theorem for regular foliations ([3], chapter I) l admits a tubular neighborhood saturated by compact leaves. A leaf l' of Θ (distinct from l) is said to be *near* l if it belongs to such a neighborhood. Assuming E^3 and Θ to be oriented one obtains a linking number $Lk(l, l')$ for all l' near l (cf. [8], p. 361). It is clear that $Lk(l, l')$ does not depend on the choice of l' , and we shall denote this index by $\lambda(l)$.

Theorem C. *Let $f : E^3 \rightarrow X$ be a submersion of codimension 1. Then $\lambda(l)$ is odd for every compact leaf l of f .*

Let (r, θ, z) denote cylindrical coordinates in R^3 and $D \subset R^3$ the unit disc given by $r \leq 1, z = 0$. Let ϕ denote a diffeomorphism mapping the boundary $\dot{D}$ of D onto a compact leaf l of f . By a result of Smale ([5], Theorem 4.2) ϕ can be extended to an immersion $\phi : D \rightarrow E^3$. This in turn can be extended to a local diffeomorphism $\phi : E \rightarrow E^3$, where E denotes a neighborhood of D . Setting $\tilde{f} = f \circ \phi$ one has a submersion $\tilde{f} : E \rightarrow X$ which admits $\dot{D}$ as a compact leaf. We assume that E and E^3 are both oriented and that $\phi : E \rightarrow E^3$ is orientation preserving. Moreover, we take f and $\tilde{f}$ to be oriented and $\phi : \dot{D} \rightarrow l$ to be orientation preserving. It follows by naturality of the linking number ([8], chapter 6) that $\lambda(l) = \lambda(\dot{D})$.

At this point we introduce a second index $\chi(\dot{D})$ associated with the leaf $\dot{D}$. Let e denote a field of unit vectors on E belonging to the kernel of $d\tilde{f}$, and let v denote a nonzero tangent vector to X at the point $x = f(l)$. Clearly v can be

lifted (with respect to $\tilde{f}$) to a field $\bar{v}$ of vectors tangent to E defined over $\dot{D}$ with $\bar{v}$ orthogonal to e . Let $\{e_1, e_2, e_3\}$ denote a frame field on E with $e_3 = e$. We define $\chi(\dot{D})$ to be the winding number $wn(e_1, \bar{v})$ of e_1 with respect to $\bar{v}$ as $\dot{D}$ is traversed once in a positive sense. The proof of Theorem C will be effected in two steps. We will show (1) that $\chi(\dot{D})$ is even; (2) that $\chi(\dot{D})$ and $\lambda(\dot{D})$ have opposite parities. (This fact was noted by Y. H. Clifton and the author in [1].)

Using a triangulation of D such that $\dot{D}$ is covered by an even number of 1-simplexes one obtains a singular integral 2-cycle z in X by the map $\tilde{f} : D \rightarrow X$. We also observe that X must be orientable. Let X be oriented and let χ denote its Euler class. It is not difficult to see that the period of χ on z is $\pm\chi(\dot{D})$, the sign being determined by orientation. (Full details may be found in [6].) If X is not compact, its Euler class is zero. Otherwise $\chi = \epsilon\mu$, where ϵ denotes the Euler characteristic and μ the fundamental class of X . Since $\epsilon = 2(1 - q)$, where q denotes the genus of X , one sees that the periods of χ are even in all cases.

It remains to show that $\chi(\dot{D})$ and $\lambda(\dot{D})$ have opposite parities. Since clearly $\lambda(\dot{D}) = wn(\bar{v}, \partial/\partial r)$, our problem is reduced to showing that $wn(e_1, \partial/\partial r)$ is odd. We will assume the frame fields $\Phi_0 = \{e_1, e_2, e_3\}$ and $\Phi_1 = \{\partial/\partial r, \partial/\partial \theta, \partial/\partial z\}$ to be positively oriented and let $F(E)$ denote the space of positive frames in E . Choosing a base point $\phi_0 \in F(E)$ permits us to identify $F(E)$ with the proper orthogonal group R_3 , and we observe that the frame fields Φ_i restricted to $\dot{D}$ define corresponding loops ω_i in R_3 , which in turn give elements $[\omega_i] \in \pi(R_3)$. The group $\pi(R_3)$ being cyclic of order 2, one easily sees that $[\omega_0] = [\omega_1]$ if and only if $wn(e_1, \partial/\partial r)$ is even. Since ϕ_0 is defined over D our problem is reduced to showing that ω_1 is not contractible in R_3 . This can be accomplished by considering the well known covering projection $p : S^3 \rightarrow R_3$. (This projection is obtained by identifying S^3 with the space of unit quaternions and letting $p(q)$ denote the inner automorphism by q . See [9], §22.3.) One finds by a simple calculation that ω_1 lifts to a great semicircle in S^3 , showing that it is not contractible.

Theorem C admits the following corollary.

C1. *Let $f : E^3 \rightarrow X$ be a submersion of codimension 1. If f admits a compact connected fibre, then $X = S^2$.*

Let $l = f^{-1}(x)$ denote a compact connected fibre. Using Smale's theorem as in the proof of Theorem C one obtains a submersion $\tilde{f} : E \rightarrow X$ which admits $\dot{D}$ (the boundary of our unit disc) as a fibre. Consider a homotopy $h : I \times I \rightarrow D$ taking $\dot{D}$ to a constant map. More specifically, for each $t \in I$, $h_t : I \rightarrow D$ may denote a circular loop with radius $r = 1 - t$. For $t > 0$, $\omega_t = \tilde{f} \circ h_t$ represents then a loop in the punctured space X_0 obtained from X by removing x . Let v denote a nonzero vector in X_x and $\bar{v}$ denote as before its lift to $\dot{D}$. For $t > 0$ and a suitable orientation of X , $wn(\bar{v}, \partial/\partial r)$ now represents the winding number of ω_t with respect to x . Since $wn(\bar{v}, \partial/\partial r)$ is odd by Theorem C, X_0 must be contractible, and therefore $X = S^2$.

It should be observed that when $f : E^3 \rightarrow X$ is simple, the conclusion can be established by two other independent methods (see A1 and B2 above).

REFERENCES

- [1] Y. H. CLIFTON & J. W. SMITH, The Euler class as an obstruction in the theory of foliations, *Proc. Nat. Acad. Sc.*, **50** (1963) 949–954.
- [2] C. EHRESMANN, Les connexions infinitesimales dans un espace fibré différentiable, *Colloque de Topologie*, Bruxelles, 1950, pp. 29–55.
- [3] R. S. PALAIS, A global formulation of the Lie theory of transformation groups, *Amer. Math. Soc. Memoir*, **22** (1957).
- [4] A. PHILLIPS, Submersions of open manifolds, *Topology*, **6** (1967) 171–206.
- [5] S. SMALE, A classification of immersions of the two-sphere, *Trans. Amer. Math. Soc.*, **90** (1959) 281–290.
- [6] J. W. SMITH, The Euler class of generalized vector bundles, *Acta Math.*, **115** (1966) 51–81.
- [7] J. W. SMITH, An exact sequence for submersions, *Bull. Amer. Math. Soc.*, to appear.
- [8] E. H. SPANIER, *Algebraic Topology*, McGraw-Hill, 1966.
- [9] N. STEENROD, *The Topology of Fibre Bundles*, Princeton University Press, 1951.

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Date communicated: FEBRUARY 10, 1968