On the Spectrum of a Class of Toeplitz Operators

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Communicated by GIAN-CARLO ROTA

Let L^p denote the Lebesgue space for the normalized measure $1/2\pi d\theta$ defined on the unit circle $\mathbf{T} = \{e^{i\theta} | 0 \leq \theta < 2\pi\}$ and H^p the corresponding Hardy space of functions in L^p which have zero negative coefficients. Let P be the projection of L^p onto H^p for $1 < p < \infty$; that P is bounded follows from a well-known theorem of M. Riesz. For ϕ in L^∞ we define the Toeplitz operator T_ϕ on H^p ($1 < p < \infty$) by $T_\phi f = P(\phi f)$ for f in H^p . It is clear that T_ϕ is bounded and this class of operators has been much studied especially in the case $p = 2$.

Let $R(\phi)$ denote the essential range of ϕ , that is, the compact set of λ for which $1/(\phi - \lambda)$ fails to be essentially bounded. In [6] Hartman and Wintner showed that $\sigma(T_\phi)$ contains $R(\phi)$, while Brown and Halmos showed in [2] that $\sigma(T_\phi)$ is contained in the convex hull of $R(\phi)$. More recently, Widom showed that $\sigma(T_\phi)$ is connected for ϕ in L^∞ (cf. [10]).

If ϕ is continuous, then $\sigma(T_\phi)$ consists of $R(\phi)$ along with those λ for which the index of λ with respect to the curve determined by ϕ is not zero. Devinatz gave the first proof with no further restrictions on ϕ in [5] (for earlier results see [7]). In this note we give a different proof of this result based on a recent paper of Coburn [3], some facts on the first homotopy group of the punctured plane, and certain properties of the analytical index of Fredholm operators.

Let $\mathfrak{L}$ denote the algebra of bounded operators on H^2 , $\mathfrak{K}$ the uniformly closed two-sided ideal of compact operators, and π the projection of $\mathfrak{L}$ onto $\mathfrak{L}/\mathfrak{K}$. An operator A in $\mathfrak{L}$ is said to be Fredholm if A has closed range and finite dimensional null and defect spaces. It is known [1] that this is equivalent to $\pi(A)$ being an invertible element of $\mathfrak{L}/\mathfrak{K}$. If A is Fredholm, then the analytical index $i_a(A)$ is defined $i_a(A) = \dim \mathfrak{N} - \dim \mathfrak{N}_*$, where $\mathfrak{N}$ and $\mathfrak{N}_*$ are the null spaces of A and A^* respectively. The following is essentially contained in [3].

Lemma 1. *If ϕ is continuous, then $T_\phi - \lambda$ is Fredholm if and only if λ is not in $R(\phi)$.*

We next show that $i_a(T_\phi - \lambda)$ depends only on the homotopy class of ϕ in $\mathbf{C} - \{\lambda_0\}$.

Let λ_0 be a point in the complex plane $\mathbf{C}$. Two continuous functions ϕ and ψ from $\mathbf{T}$ to $\mathbf{C} - \{\lambda_0\}$ are said to be homotopic if there exists a continuous map

$\Phi_t(e^{i\theta})$ from $[0, 1] \times \mathbf{T}$ to $\mathbf{C} - \{\lambda_0\}$ so that $\Phi_0 = \phi$ and $\Phi_1 = \psi$. The equivalence classes of homotopic maps form the first homotopy group $\Pi(\mathbf{C} - \{\lambda_0\})$ of $\mathbf{C} - \{\lambda_0\}$. It is well-known (cf. [7]) that $\Pi(\mathbf{C} - \{\lambda_0\})$ is isomorphic to the group $\mathbf{Z}$ of integers. If ϕ is a continuous function from $\mathbf{T}$ to $\mathbf{C} - \{\lambda_0\}$, then we denote by $i_t(\phi, \lambda_0)$ the integer corresponding to the equivalence class in $\Pi(\mathbf{C} - \{\lambda_0\})$ which contains ϕ .

Lemma 2. *If ϕ and ψ are continuous functions from $\mathbf{T}$ to $\mathbf{C} - \{\lambda_0\}$ which are homotopic, then $i_a(T_\phi - \lambda_0) = i_a(T_\psi - \lambda_0)$.*

Proof. If $\Phi_t(e^{i\theta})$ is continuous from $[0, 1] \times \mathbf{T}$ to $\mathbf{C} - \{\lambda_0\}$ so that $\Phi_0 = \phi$ and $\Phi_1 = \psi$, then T_{Φ_t} is continuous in the uniform topology and T_{Φ_t} is Fredholm for $0 \leq t \leq 1$. Then $i_a(T_{\Phi_t} - \lambda_0)$ is continuous, since the analytical index is continuous in the uniform topology [8]. But $i_a(T_{\Phi_t} - \lambda_0)$ is integer-valued which implies it is constant. Thus $i_a(T_\phi - \lambda_0) = i_a(T_{\Phi_0} - \lambda_0) = i_a(T_{\Phi_1} - \lambda_0) = i_a(T_\psi - \lambda_0)$.

We next relate the analytical index and the topological index.

Lemma 3. *If ϕ is a continuous function and λ_0 is not in $R(\phi)$, then $i_a(T_\phi - \lambda_0) = -i_t(\phi, \lambda_0)$.*

Proof. For n an integer let $f_n(e^{i\theta}) = e^{in\theta} + \lambda_0$. Then $T_{f_n} - \lambda_0 = T_{\epsilon_n}^*$ for $n \geq 0$, while $T_{f_n} - \lambda_0 = T_{\epsilon_n}^{*-n}$ for $n < 0$. Further, it is well-known for $n \geq 0$ that $i_a(T_{\epsilon_n}^*) = -n$, while for $n < 0$, $i_a(T_{\epsilon_n}^{*-n}) = -n$. Thus, $i_a(T_{f_n} - \lambda_0) = -n$ for all n . Next we recall that $i_t(t_n, \lambda_0) = n$. Hence, if $n = i_t(\phi, \lambda_0)$, then ϕ is homotopic to f_n so that from the preceding lemma it follows that $i_a(T_\phi - \lambda_0) = i_a(T_{f_n} - \lambda_0) = -n = -i_t(\phi, \lambda_0)$.

To complete the proof we need to show that $i_a(T_\phi) = 0$ implies the invertibility of T_ϕ . The following lemma appears in Coburn [4].

Lemma 4. *If ϕ is in L^∞ , $\mathfrak{N}$ is the null space of T_ϕ , and $\mathfrak{N}_*$ the defect space of T_ϕ (that is, $(T_\phi H^2)^\perp$), then either $\mathfrak{N} = \{0\}$ or $\mathfrak{N}_* = \{0\}$.*

We can now prove the

Theorem. *If ϕ is continuous on $\mathbf{T}$, then $\sigma(T_\phi) = R(\phi) \cup \{\lambda_0 \mid i_t(\phi, \lambda_0) \neq 0\}$. Moreover, for λ_0 not in $R(\phi)$ the operator $T_\phi - \lambda_0$ is Fredholm and $i_a(T_\phi - \lambda_0) = i_t(\phi, \lambda_0)$.*

Proof. If λ_0 is in $R(\phi)$, then λ_0 is in $\sigma(T_\phi)$. If λ_0 is not in $R(\phi)$, then $T_\phi - \lambda_0$ is Fredholm by Lemma 1 and $i_a(T_\phi - \lambda_0) = i_t(T_\phi - \lambda_0)$ by Lemma 3. If $i_a(T_\phi - \lambda_0) \neq 0$, then $T_\phi - \lambda_0$ is not invertible so that λ_0 is in $\sigma(T_\phi)$. If $i_a(T_\phi - \lambda_0) = 0$, then $T_\phi - \lambda_0$ is invertible by Lemma 4 so that λ_0 is not in $\sigma(T_\phi)$.

The analytical expression for the index of a curve gives us the following

Corollary. *If ϕ is continuous and of bounded variation on $\mathbf{T}$, then*

$$\sigma(T_\phi) = R(\phi) \cup \left\{ \lambda_0 \mid \frac{1}{2\pi} \int_0^{2\pi} (\phi(e^{i\theta}) - \lambda_0)^{-1} d\phi \neq 0 \right\}.$$

As our references have been to results about operators on Hilbert space we have held our discussion to that context. We point out, however, that the same proof holds for Toeplitz operators on H^p , $1 < p < \infty$.

Lastly, we add that our result is related to the Atiyah–Singer Index Theorem (cf. [8]) since a Toeplitz operator can be defined in terms of a singular integral. Moreover, results from the theory of singular integral operators would probably be sufficient to give our theorem. We feel that our proof is of interest, in part, because it does not depend on such results.

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Date communicated: FEBRUARY 2, 1968