*Boundary Correspondence Under Quasiconformal Mappings of Jordan Domains*¹

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As the main result in [3] (Theorem 2) there is given a necessary and sufficient condition, called (A), for a homeomorphism φ of a Jordan curve γ lying in the extended complex plane $\mathbf{C}^* = \mathbf{C} \cup \{\infty\}$ to have a quasiconformal extension to $\mathbf{C}^*$. The present paper has a slightly different point of view, namely, we ask for a necessary and sufficient condition for such a homeomorphism φ to have a homeomorphic extension to a quasiconformal mapping of one of the complementary domains of γ . Actually the existence of an extension of this kind is proved in [3] (Lemma 2) with an assumption made on φ which is a consequence of (A). But, as one can easily see by an example, this assumption is not necessary for the present problem. However, Lemma 1 in [3], which is a kind of stretching result for quadrilaterals, can also here in an essential way be used to establish a reasonable condition on φ which is not only sufficient but necessary as well.

A solution for the very special case where the boundaries of the domains are quasiconformal curves or as some authors call, quasicircles, is already established in Theorem 1 in [3]. This is because in this case, to find an extension of the homeomorphism φ to the whole plane is essentially the same problem as to find an extension to one complementary domain. This special extension problem has also been treated by T. J. Reed [2] who also gives some explicit bounds for the maximal dilatation of the extension.

Let now G and G' be Jordan domains in $\mathbf{C}^*$ with boundaries γ and γ' respectively and let $\varphi: \gamma \rightarrow \gamma'$ be a homeomorphism which preserves the orientation with respect to G and G' . Since we are dealing with quasiconformal mappings which do not necessarily have quasiconformal extensions to $\mathbf{C}^*$, we cannot use cross-ratios as in [3]. This time we express our condition, which is called (B), only in terms of cross-cuts. Recall that a cross-cut of a domain is an open Jordan arc lying in the domain with endpoints on the boundary. If α_1 and α_2 are two Jordan arcs which can be separated by a circular annulus, we denote by $u(\alpha_1, \alpha_2)$ the supremum of the ratios of the radii of such separating annuli. Let Q be a quadrilateral. Then $M(Q)$ is the modulus of Q and we denote by $s_a(Q)$ and

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$s_b(Q)$ the euclidean distances in Q between the a - and b -sides respectively ([1], p. 15, 23). Our condition on the homeomorphism φ is the following:

(B) There exists a constant c , $1 < c < \infty$, such that the following holds: Let $z_1, z_2, z_3, z_4 \in \gamma$ be distinct points following in this order on γ and let λ_1 and λ_2 be cross-cuts of G with endpoints z_1, z_4 and z_2, z_3 respectively such that $u(\lambda_1, \lambda_2) \geq c$. Let l (resp. l_*) be the subarc of γ with endpoints z_1 and z_2 (resp. z_3 and z_4) such that $z_3, z_4 \notin l$ (resp. $z_1, z_2 \notin l_*$). Then there exist cross-cuts λ'_1 and λ'_2 of G' which join $\varphi(l)$ and $\varphi(l_*)$ and for which $u(\lambda'_1, \lambda'_2) > e^{2\pi}$.

The proof of the following lemma is contained in the first part of the proof of Lemma 2 in [3] where the essential step is Lemma 1 in [3].

Lemma. *Let Q be a quadrilateral. For every $d > 1$ there exists a $k > 0$ depending only on d such that if $s_a(Q)/s_b(Q) < k$, then the a -sides of Q are joined by arcs α_1 and α_2 in Q such that $u(\alpha_1, \alpha_2) \geq d$.*

Theorem. *Let G and G' be Jordan domains in $\mathbf{C}^*$ with boundaries γ and γ' respectively and let $\varphi: \gamma \rightarrow \gamma'$ be a homeomorphism which preserves the orientation with respect to G and G' . Then φ has a homeomorphic extension $f: \bar{G} \rightarrow \bar{G}'$ which is quasiconformal in G if and only if φ satisfies (B).*

Proof. Let first $f: \bar{G} \rightarrow \bar{G}'$ be a homeomorphism which is K -quasiconformal in G and which restricts to φ on γ . Let z_1, z_2, z_3, z_4 be four distinct points on γ which follow in this order. We can assume that this sequence of points has a positive orientation with respect to G . Suppose λ_1 and λ_2 are cross-cuts in G which join z_1, z_4 and z_2, z_3 respectively and for which $u(\lambda_1, \lambda_2) \geq p > 1$. Let Q be the quadrilateral $G(z_1, z_2, z_3, z_4)$ and let Q' be its image under f . Then the modulus $M(Q)$ of Q satisfies the inequality $M(Q) \geq \log p/(2\pi)$ and we have $M(Q') \geq \log p/(2\pi K)$. Now Hilfssatz I.4.1 in [1] says that $s_a(Q')/s_b(Q') \leq \nu(M(Q'))$, where ν is a decreasing function of the positive real axis onto itself. Since $\lim_{t \rightarrow \infty} \nu(t) = 0$, there exists by the lemma a constant $c < \infty$ depending only on K such that if $p \geq c$, the a -sides of Q' are joined by cross-cuts λ'_1 and λ'_2 of G' with $u(\lambda'_1, \lambda'_2) > e^{2\pi}$. This proves the necessity.

Assume that φ satisfies (B) with some constant c . Let now Q' be a quadrilateral with domain G' and modulus $M(Q') = 1$, and let Q be the quadrilateral with domain G corresponding to Q' under φ . Suppose $s_a(Q)/s_b(Q) < k$. Then, again by the lemma, the a -sides of Q are joined by cross-cuts λ_1 and λ_2 of G for which $u(\lambda_1, \lambda_2) \geq c$ if k is sufficiently small. Let the endpoints of λ_1 and λ_2 be z_1, z_4 and z_2, z_3 respectively such that the sequence z_1, z_2, z_3, z_4 is positively oriented with respect to G . Denote the quadrilateral $G(z_1, z_2, z_3, z_4)$ by $\tilde{Q}$. By (B) the a -sides of the quadrilateral $\tilde{Q}'$ with domain G' corresponding to $\tilde{Q}$ under φ are joined by cross-cuts λ'_1 and λ'_2 of G' such that $u(\lambda'_1, \lambda'_2) > e^{2\pi}$. This implies $M(\tilde{Q}') > 1$, and we have a contradiction because $M(\tilde{Q}') \leq M(Q')$. Hence there exists a lower bound k_0 for $s_a(Q)/s_b(Q)$ which depends only on c . Hilfssatz I.4.1 in [1] gives thus an upper bound for $M(Q)$ depending only on c . A lower

bound for $M(Q)$ is obtained similarly, and the required extension exists by Satz II.6.1 in [1]. The theorem is proved.

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