*On Hölder Continuity at the Boundary in Conformal Maps**

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1. While there are various criteria for differentiability of the Riemann mapping function at the boundary of its domain of definition, it is useful to have a condition which requires less than these and yields the Hölder continuity of this function. Such a condition is presented here and moreover the Hölder exponent obtained is the best possible under the hypotheses.

Let C be a closed rectifiable Jordan curve with bounded arc length-chord length ratio, *i.e.* for any two points $z_1, z_2 \in C$, the length Δs of the shorter arc of C between them satisfies $\Delta s \leq c |z_1 - z_2|$, where $c > 1$ is a constant. If φ maps the interior of C conformally onto the unit disk $|w| < 1$, we show that $\varphi \in \text{Lip}(\mu)$ on C where $\mu = \frac{1}{2}(1 - (1/\pi) \arcsin 1/c)^{-1}$ and that this exponent is sharp.

Domains with a boundary curve of bounded arc length-chord length ratio play a useful role in various applications of conformal mapping. If f denotes the *inverse* function of φ , this property, for example, ensures that f satisfies the Smirnov condition, *i.e.* that $\log |f'(w)|$ is represented by its Poisson integral in $|w| < 1$ [5, p. 183]; this condition is important in the theory of Szegő polynomials connected with arbitrary rectifiable Jordan curves. Furthermore, $f(w)$ also satisfies a Hölder condition on $|w| = 1$, where the exponent depends on c [8]. Other properties which this mapping possesses have been given in [4] (see Remark 2 below) and [9].

2. Theorem. *Suppose C is a closed rectifiable Jordan curve in the z -plane such that*

(a) *for any two points $z_1, z_2 \in C$ and the length Δs of the (shorter) arc of C between them*

$$(1) \quad \Delta s \leq c |z_2 - z_1|, \quad c > 1, \quad c = \text{constant}$$

(b) *the interior Ω of C contains the disk $|z| \leq 2\sigma, \sigma > 0$.*

Let $w = \varphi(z)$ map Ω conformally onto the unit disk such that $\varphi(0) = 0$. Let $\rho_0 =$

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$\min[\sigma/2c, \exp(-4\pi^2)]$. Then, if $z_1, z_2 \in \bar{\Omega}$ are both separated from $z = 0$ by a circular crosscut of Ω with radius $\rho < \frac{1}{2}\rho_0$ and center on C ,

$$(2) \quad |\varphi(z_1) - \varphi(z_2)| \leq M\rho^\mu.$$

Here

$$\mu = \frac{1}{2} \left(1 - \frac{1}{\pi} \arcsin \frac{1}{c} \right)^{-1}, \quad M = 6\pi\rho_0^{-\mu}.$$

Furthermore, for any two points $z_1, z_2 \in C$

$$(3) \quad |\varphi(z_1) - \varphi(z_2)| \leq H |z_1 - z_2|^\mu$$

where $H = 2cM$; (3) remains valid for $z_1 \in C, z_2 \in \Omega$.

Remarks 1. Our proof below shows that this theorem is a *local* result, i.e. if C is assumed to be rectifiable only in a neighborhood of a point z_0 and (1) is satisfied for z_1 and z_2 only in this neighborhood, the conclusion (2) is still valid with a crosscut on $|z - z_0| = \rho$ for all sufficiently small ρ . This consideration indicates that the exponent μ is the best possible. If C has a "reentrant" corner at z_0 , formed by two straight line segments of equal length, with the exterior angle α , $0 < \alpha < \pi$, we have $c = (\sin \alpha/2)^{-1}$ in a neighborhood of z_0 . Then $\varphi(z) - \varphi(z_0) \sim a(z - z_0)^{\pi/(2\pi - \alpha)}$ as $z \rightarrow z_0$ in $C \cup \text{Int}(C)$, where $a \neq 0$ is a constant, so that $\mu = \frac{1}{2}(1 - \alpha/2\pi)^{-1}$ cannot be improved. The exponent is also sharp in the two limiting cases: if $c = 1$ in a neighborhood of z_0 , e.g. when C contains a straight line segment with z_0 as interior point, then $\mu = 1$; if condition (1) is omitted, i.e. $c = \infty$, then $\mu = \frac{1}{2}$ and our result becomes a special case of a known theorem¹, see [2, p. 78] and [7, Appendix B], also [3].

2. Under the hypothesis (1) M. Lavrientiev [4] has shown that there exists a $\delta > 0$ such that for any measurable set $E \subset C$ with $m(E) < \epsilon$, $m(\varphi(E)) \leq 2\pi(\epsilon/2\sigma)^\delta$. However, since no information on the size of δ (beyond $\delta > 0$) is given, this does not contain our theorem.

3. Proof. We draw the circle $|z - z_0| = \rho < \rho_0$ where $z_0 \in C$. Let k_ρ denote a subarc of this circle forming a crosscut of Ω which separates z_0 from $z = 0$. If z'_1 and z'_2 are the endpoints of k_ρ , then the arc z'_1, z_0, z'_2 of C , say b_ρ , and k_ρ form a closed Jordan curve whose interior $D_\rho(\subset \Omega)$ does not contain $z = 0$. The complementary domain, bounded by k_ρ and $C - b_\rho$, thus has $z = 0$ and also the disk $|z| \leq \sigma$ in its interior. Consequently, the length of its boundary curve is at least $2\pi\sigma$ and since the length of k_ρ is $\leq 2\pi\rho$, the length of $C - b_\rho$ is $\geq 2\pi\sigma - 2\pi\rho = 2\pi(\sigma - \rho) > \pi\sigma$. By (1) one of the two arcs of C between z'_1 and z'_2 has a length $\Delta s \leq 2\rho c < \sigma$; hence b_ρ is the shorter of the two arcs of C between z'_1 and z'_2 and (1) holds for b_ρ .

Next we note that the length of the arc b_ρ is $\geq 2\rho$ (since it connects the center z_0 of a circle of radius ρ with two distinct points on its circumference),

¹ This theorem, which asserts (2) with $\mu = \frac{1}{2}$ and suitable M for an arbitrary bounded simply connected domain, can also be obtained by our method (cf. [7], p. 62).

so that by (1), $|z'_1 - z'_2| \geq 2\rho/c$. If we denote the angle subtended by the chord $z'_1 z'_2$ of the circle $|z - z_0| = \rho$ at its center by α , we have $\alpha \geq 2 \arcsin 1/c = 2\omega$ ($\omega < \pi/2$). Hence the length of k_ρ is at most $2\pi\rho - \rho\alpha \leq 2\rho(\pi - \omega)$.

We may assume without loss of generality that $\varphi(z_0) = 1$. Then $\varphi(D_\rho)$ is a Jordan domain in $|w| < 1$ bounded by $\varphi(k_\rho)$ and $\varphi(b_\rho)$, an arc of $|w| = 1$ which contains $w = 1$. The point $w = 0$ is exterior to $\varphi(D_\rho)$. We map now the disk $|w| \leq 1$ conformally onto the half-plane $\text{Im } \zeta \geq 0$ by means of the transformation

$$(4) \quad \zeta = i \frac{1 - w}{1 + w}$$

which carries $w = 0, 1$, and -1 into $\zeta = i, 0$, and ∞ , respectively. By a well known estimate (see *e.g.* [10, p. 217]) $\varphi(D_\rho)$ is contained in the circle

$$(5) \quad |w - 1| \leq \frac{2\pi}{\sqrt{\log \frac{1}{\rho}}} = \delta.$$

A short calculation shows that the image of this circle in the ζ -plane by means of (4) is

$$(6) \quad \left| \zeta - \frac{i\delta^2}{4 - \delta^2} \right| \leq R = \frac{2\delta}{4 - \delta^2}.$$

Now let

$$\psi(z) = i \frac{1 - \varphi(z)}{1 + \varphi(z)};$$

$\psi(D_\rho) = \Delta_\rho$ is a Jordan domain in $\text{Im } (\zeta) > 0$ bounded by $\psi(k_\rho) = \gamma_\rho$ and a segment, $\psi(b_\rho)$ of the real axis which contains $\zeta = 0$. Let λ_ρ denote the length of γ_ρ . Then for $z = z_0 + \rho e^{i\theta}$ ($\rho \leq \rho_0$)

$$(7) \quad \lambda_\rho^2 = \left(\int_{k_\rho} |\psi'(z)| \rho d\theta \right)^2 \leq \int_{k_\rho} |\psi'(z)|^2 \rho d\theta \cdot 2\rho(\pi - \omega).$$

Hence for $0 < r \leq \rho_0$

$$\int_0^r \frac{\lambda_\rho^2}{\rho} d\rho \leq \int_0^r \int_{k_\rho} |\psi'(z)|^2 \rho d\theta d\rho \cdot 2(\pi - \omega) = A(r) \cdot 2(\pi - \omega)$$

where $A(r)$ denotes the area of Δ_r . For almost all ρ , $0 < \rho \leq \rho_0$, $\lambda_\rho < \infty$ and γ_ρ is therefore a rectifiable Jordan arc. We reflect γ_r and Δ_r in the real axis and γ_r and its reflection γ_r^* form a closed Jordan curve of length $2\lambda_r$. The area of its interior is $2A(r)$. By the isoperimetric inequality we have therefore²

$$(8) \quad 2A(r) \leq \frac{(2\lambda_r)^2}{4\pi} \quad \text{or} \quad A(r) \leq \frac{\lambda_r^2}{2\pi}.$$

² The isoperimetric inequality in conjunction with reflection and the "arc length-area" method has been applied earlier by J. Ferrand [1, pp. 167–170], [2, pp. 77–78] in considering the mapping of domains on the Riemann sphere and using the spherical metric. The present writer has employed this method *in the plane* also on previous occasions. See also [7, p. 63].

Since, for almost all r , $0 \leq r \leq \rho_0$,

$$A'(r) = \int_{kr} |\psi'(z_0 + re^{i\theta})|^2 r d\theta$$

we have from (7)

$$\lambda_r^2 \leq A'(r) 2(\pi - \omega)r$$

and therefore by (8)

$$A(r) \leq A'(r) \left(1 - \frac{\omega}{\pi}\right)r$$

or

$$\frac{1}{\left(1 - \frac{\omega}{\pi}\right)r} \leq \frac{A'(r)}{A(r)}.$$

Integrating between the limits ρ and ρ_1 , $0 < \rho < \rho_1 \leq \rho_0$, we have

$$(9) \quad \frac{A(\rho)}{\rho^{2\mu}} \leq \frac{A(\rho_1)}{\rho_1^{2\mu}}, \quad 2\mu = \left[\frac{1}{1 - \frac{\omega}{\pi}} \right].$$

Since $\varphi(D_\rho)$ is contained in the disk (5), Δ_ρ is contained in the disk (6) and consequently (since $\delta < 1$)

$$\frac{A(\rho_1)}{\rho_1^{2\mu}} \leq \frac{\pi R^2}{\rho_0^{2\mu}} \leq \frac{4\pi}{9\rho_0^{2\mu}} < \frac{\pi}{2\rho_0^{2\mu}}.$$

By (9) we have then for $\rho \leq \rho_0$

$$A(\rho) \leq \frac{\pi}{2\rho_0^{2\mu}} \rho^{2\mu}.$$

Now from (7), for $\rho \leq \frac{1}{2}\rho_0$

$$\int_\rho^{2\rho} \frac{\lambda_r^2}{r} dr \leq 2(\pi - \omega)A(2\rho) < \frac{\pi^2}{\rho_0^{2\mu}} (2\rho)^{2\mu}.$$

Hence there exists a ρ_1 with $\rho \leq \rho_1 \leq 2\rho$,

$$\lambda_{\rho_1}^2 \log 2 \leq \frac{4\pi^2}{\rho_0^{2\mu}} \rho^{2\mu}.$$

Thus, for $\rho \leq \rho_0/2$, the diameter of Δ_ρ is $\leq \lambda_{\rho_1}$ and consequently for $z_1, z_2 \in D_\rho$

$$|\psi(z_1) - \psi(z_2)| \leq M_0 \rho^\mu, \quad M_0 = \sqrt{\frac{4\pi^2}{\rho_0^{2\mu} \log 2}} < \frac{3\pi}{\rho_0^\mu}.$$

To conclude the proof we note that for the function (4)

$$\left| \frac{dw}{d\zeta} \right| = \left| -\frac{2i}{(\zeta + i)^2} \right| \leq 2$$

for $\zeta \in \Delta_\rho$, since then $\operatorname{Im} \zeta \geq 0$ and therefore $|\zeta + i| \geq 1$. Hence

$$|\varphi(z_1) - \varphi(z_2)| \leq 2M_0\rho^\mu.$$

This proves (2). To prove (3) assume first that, for $z_1, z_2 \in C$, $|z_1 - z_2| = d < \rho_0/2c$. Then, by (1), the circle $|z - z_1| = \rho$ with $dc < \rho < \min[\rho_0/2, 2cd]$ contains the shorter arc of C between z_1 and z_2 in its interior and there exists, therefore, a subarc k_ρ of this circle which is a crosscut of Ω and separates z_1 and z_2 from $z = 0$. Hence by (2)

$$|\varphi(z_1) - \varphi(z_2)| \leq M\rho^\mu \leq M(2cd)^\mu = M(2c)^\mu |z_2 - z_1|^\mu.$$

If $d \geq \rho_0/2c$, then

$$|\varphi(z_1) - \varphi(z_2)| \leq 2 \leq \frac{2}{\left(\frac{\rho_0}{2c}\right)^\mu} |z_1 - z_2|^\mu \leq 2\left(\frac{2c}{\rho_0}\right)^\mu \cdot |z_1 - z_2|^\mu.$$

and one easily verifies that $2(2c/\rho_0)^\mu < M(2c)^\mu \leq 2cM = H$.

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