*An Existence Theorem for a Class of Non-linear Integral Equations with Applications to a Non-linear Elliptic Boundary Value Problem**

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In [5], Nehari proved that if Ω is a bounded real interval, K a symmetric positive definite kernel which is continuous on $\Omega \times \Omega$, and F a non-negative continuous function on $\bar{R}_+ \times \Omega$ satisfying, for some $\epsilon > 0$,

$$(1) \quad 0 < \eta_1^{-\epsilon} F(\eta_1, x) \leq \eta_2^{-\epsilon} F(\eta_2, x), \quad 0 < \eta_1 < \eta_2 < \infty,$$

for all $x \in \Omega$, then the integral equation

$$(2) \quad y(x) = \int_{\Omega} K(x, t) y(t) F(y^2(t), t) dt,$$

has a non-trivial solution which is continuous on Ω . The proof uses variational methods. In this note we shall prove, using arguments similar to those of [5], that if Ω is a bounded region in R^n (n -dimensional Euclidean space), if K is a symmetric function on $\Omega \times \Omega$ such that for some pair of conjugate indices p, q , $1 < q \leq 2 \leq p < \infty$,

$$(3) \quad [Au](x) = \int_{\Omega} K(x, t) u(t) dt,$$

defines a completely continuous operator A from $L^q(\Omega)$ to $L^p(\Omega)$, which is positive definite in the sense that

$$\int_{\Omega} \int_{\Omega} K(x, t) u(x) u(t) dx dt > 0, \quad u \in L^q(\Omega) \setminus \{0\},$$

and if in addition to (1), F satisfies the appropriate polynomial growth inequality so that the mapping

$$y(x) \rightarrow y(x) F(y^2(x), x)$$

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is continuous from $L^p(\Omega)$ to $L^q(\Omega)$, then (2) has a non-trivial solution $y \in L^p(\Omega)$.

In the concluding section we apply the result described above to the boundary value problem

$$\Delta u + uF(u^2, x) = 0 \text{ in } \Omega \quad u|_{\partial\Omega} = 0,$$

where Δ is the Laplace operator and Ω is a bounded region in R^n for which the Dirichlet problem is solvable.

Related to the problem (2) and the boundary value problem above are the eigenvalue problems,

$$(4) \quad y(x) = \lambda \int_{\Omega} K(x, t)y(t)F(y^2(t), t) dt,$$

and

$$(5) \quad \Delta u + \lambda uF(u^2, x) = 0 \text{ in } \Omega \quad u|_{\partial\Omega} = 0.$$

The first of these problems is treated in [3] and in [7]; problems of the form (5) are treated in [4] and in [6]. Also Berger, [1], has investigated a problem similar to (5) but with Δ replaced by a more general elliptic operator. All of these results contain the linear cases, the results obtained here are strictly non-linear in character.

2. Statement of results. Let Ω be a bounded region in Euclidean n -space and let K be a real valued symmetric function on $\Omega \times \Omega$. We assume that $K(x, \cdot)$ is measurable for almost all $x \in \Omega$, and that: (H) the operator A defined by (3) is completely continuous from $L^q(\Omega)$ to $L^p(\Omega)$ for some pair of conjugate indices p and q with

$$(6) \quad 1 < q \leq 2 \leq p.$$

Finally we assume that A is positive definite on $L^q(\Omega)$, i.e.

$$(7) \quad \int_{\Omega} \int_{\Omega} K(x, t)u(x)u(t) dt dx > 0,$$

for $u \in L^q(\Omega)$, $u(x)$ not almost everywhere equal to zero.

Let F be a real-valued function on $\bar{R}_+ \times \Omega$, ($\bar{R}_+ = \{\eta : \eta \text{ real}, \eta > 0\}$) which satisfies the following conditions *i*) the Carathéodory hypothesis (i.e. $F(\cdot, x)$ is continuous on $\bar{R}_+$ for almost all $x \in \Omega$ and $F(\eta, \cdot)$ is measurable for all $\eta \in \bar{R}_+$), *ii*) there is a positive constant ϵ such that (1) holds for almost all (fixed) $x \in \Omega$, *iii*) there are positive constants c, γ and there is a non-negative function $\sigma \in L^1(\Omega)$ such that for almost all (fixed) $x \in \Omega$,

$$(8) \quad F(\eta, x) \leq c\eta^\gamma + \sigma(x), \quad 0 \leq \eta < \infty.$$

Theorem 1. Let Ω , K and F be as above, in particular assume that (H) holds, and that (7) holds for all $u \in L^q(\Omega)$, $u \neq 0$. If

$$(9) \quad 2\gamma \leq p - 2, \sigma \in L^r(\Omega), r = p/(p - 2),$$

then the integral equation (2) has at least one non-trivial solution in $L^p(\Omega)$.

Theorem 2. Let Ω , K and F satisfy the hypotheses of Theorem 1. If P is a non-negative real-valued function

$$(10) \quad P \in L^r(\Omega), \quad r = p/(p-2),$$

and if the least eigenvalue λ of the symmetrizable linear integral equation

$$(11) \quad u(x) = \lambda \int_{\Omega} K(x, t)P(t)u(t) dt,$$

is larger than 1, then the integral equation

$$(12) \quad y(x) = \int_{\Omega} K(x, t)y(t)(P(t) + F(y^2(t), t)) dt,$$

has a non-trivial solution in $L^p(\Omega)$.

We note here that the hypothesis (H) is implied by the condition

$$(13) \quad \operatorname{ess\,sup}_{x \in \Omega} \int_{\Omega} |K(x, t)|^a dt < \infty, \quad a > \frac{p}{2},$$

see [2], Theorem 9.5.6, p. 658. The positive definiteness of A in $L^q(\Omega)$ is equivalent, in the presence of hypothesis (H), to positive definiteness of A restricted to $L^2(\Omega)$ together with the density of the range of A in $L^p(\Omega)$. To see this we use the fact, [7], p. 189, that when (H) holds and A is positive definite in $L^2(\Omega)$ then for $u, v \in L^q(\Omega)$,

$$(13') \quad \int_{\Omega} \int_{\Omega} K(x, t)u(x)v(t) dx dt = \sum_{k=1}^{\infty} \lambda_k^{-1} \int_{\Omega} u(t)\varphi_k(t) dt \int_{\Omega} v(t)\varphi_k(t) dt,$$

where the λ_k are the eigenvalues of $A|_{L^2(\Omega)}$ and, for each k , φ_k is the normalized [in $L^2(\Omega)$] eigenfunction of A corresponding to λ_k ; the φ_k actually belongs to $L^p(\Omega)$. If A has dense range in $L^p(\Omega)$ it follows, using (13'), that for non-zero $u \in L^q(\Omega)$, $\int_{\Omega} u(t)\varphi_k(t) dt$ cannot vanish for all k . Putting $v = u$ in (13') we conclude, since u was arbitrary, that A is positive definite on $L^q(\Omega)$ when $A|_{L^2}$ is positive definite and A has dense range in $L^p(\Omega)$. The other direction of the equivalence is obvious. One can construct a symmetric kernel satisfying (13) for some $a > 2$, which is positive definite on L^2 , but is not positive definite on L^q for any $q < 2$.

The next theorem gives conditions under which an L^p -solution of (2) or of (12) will be essentially bounded.

Theorem 3. Suppose that the symmetric kernel K satisfies (13) for some $p > 2$, and let F be a non-negative Carathéodory function on $\bar{R}_+ \times \Omega$ satisfying (8) with $\gamma < (p-2)/2$, and $\sigma \in L^{\infty}$. Then every L^p solution of (2) is essentially bounded. If we assume the same hypothesis, and in addition assume that $P \in L^{\infty}$ then every L^p solution of (12) is essentially bounded.

3. Formulation of the variational problem. We shall show that the existence of a non-trivial solution of (2) is implied by the existence of a solution to the variational problem formulated below.

We define a function G with the same domain as that of F

$$(14) \quad G(\eta, x) = \int_0^\eta F(s, x) ds.$$

The variational problem is formulated in terms of functionals $J(u, v)$, $N(y)$, $H(y)$ which are defined, for $u, v, y \in L^p$, as follows,

$$(15) \quad J(u, v) = \int_\Omega \int_\Omega K(x, t) u(x) F(u^2(x), x) v(t) F(v^2(t), t) dx dt,$$

$$(16) \quad H(y) = \int_\Omega [y^2(x) F(y^2(x), x) - G(y^2(x), x)] dx$$

$$(17) \quad N(y) = \int_\Omega y^2(x) F(y^2(x), x) dx - J(y, y).$$

Notice that H vanishes identically for the linear problem.

Lemma 1. *The functionals H and N are continuous on L^p .*

Proof. In view of (8), (9), the mapping f defined by

$$[fy](x) = y(x) F(y^2(x), x)$$

is a continuous mapping from L^p to L^q ; see [7], Theorem 19.1, p. 154. It then follows from hypothesis (H) that Af is a continuous mapping of L^p into itself; the continuity of J follows from the continuity of the inner product on $L^p \times L^q$. Appealing again to (8), (9) and [7], Theorem 19.1, we easily see that the mappings $y(x) \rightarrow y^2(x) F(y^2(x), x)$, and $y(x) \rightarrow G(y^2(x), x)$ are continuous from L^p to L^1 ; the continuity of H and N on L^p follows.

A function $y \in L^p$ will be called *admissible* if it is not almost everywhere equal to zero and can be represented in the form

$$(18) \quad y = Au, \quad u \in L^q.$$

If $y \in L^p$ and y is not almost everywhere equal to zero (but not necessarily admissible), then it follows from (1) and the positive definiteness of A that v , defined by

$$v(x) = \int_\Omega K(x, t) y(t) F(y^2(t), t) dt,$$

is admissible.

(*) *If y is an admissible function then there is a positive constant α such that*

$$(19) \quad N(\alpha y) = 0.$$

(**) If y is admissible and satisfies

$$(20) \quad N(y) = 0,$$

and if

$$(21) \quad v(x) = \alpha \int_{\Omega} K(x, t)y(t)F(y^2(t), t) dt,$$

where $\alpha > 0$ is chosen so that

$$(22) \quad N(v) = 0,$$

then (v is admissible and)

$$(23) \quad \alpha^2 \int_{\Omega} y^2(x)F(y^2(x), x) dx \leq \int_{\Omega} v^2(x)F(y^2(x), x) dx,$$

and

$$(24) \quad H(v) \leq H(y).$$

Equality holds in (24) if and only if y is a solution of (2). Finally, if $y \in L^p$, then,

$$(25) \quad H(y) \geq \epsilon(1 + \epsilon)^{-1} \int_{\Omega} y^2(x)F(y^2(x), x) dx.$$

Results analogous to the assertions of (*) and (**) are proved in [5]; the proofs are essentially the same for the case considered here.

The variational problem which we consider is that of minimizing the functional $H(y)$ within the class of admissible functions and subject to the side condition $N(y) = 0$. It is clear from (**) that a solution y of this variational problem must be a solution of the integral equation (2).

4. Solution of the variational problem. In this section we prove the existence of a solution to the variational problem posed above. We assume throughout that the hypotheses of Theorem 1 are satisfied.

Lemma 2. *There is a positive constant m such that for any admissible function, y , satisfying (20),*

$$(26) \quad \int_{\Omega} y^2(x)F(y^2(x), x) dx \geq m.$$

Moreover there are positive constants k_1, k_2 such that

$$(27) \quad \left(\int_{\Omega} \left| \int_{\Omega} K(x, t)y(t)F(y^2(t), t) dt \right|^p dx \right)^{1/p} \leq \left(k_1 + k_2 \int_{\Omega} y^2(t)F(y^2(t), t) dt \right)^{1/q}.$$

Proof. Because of hypothesis (H), (17) and (20) we have

$$(28) \quad \int_{\Omega} y^2(x)F(y^2(x), x) dx \leq M \left(\int_{\Omega} |y(x)F(y^2(x), x)|^q dx \right)^{2/q},$$

where M is the norm of the operator A . By Hölder's inequality

$$(29) \quad \int_{\Omega} |y(x)F(y^2(x), x)|^q dx \leq \left(\int_{\Omega} |F(y^2(x), x)|^r dx \right)^{q/2r} \left(\int_{\Omega} y^2(x)F(y^2(x), x) dx \right)^{q/2}$$

where, as before, $r = q/(2 - q)$. Combining (28) and (29) we get, since the term on the left in (28) is positive,

$$(30) \quad 1 \leq M \left(\int_{\Omega} |F(y^2(x), x)|^r dx \right)^{1/r}.$$

The first assertion of the lemma is now proved as follows. Suppose the assertion is false and that $\{y_n(x)\}$ is a sequence of admissible functions satisfying (20) and

$$\int_{\Omega} y_n^2(x)F(y_n^2(x), x) dx \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

We can then conclude, using (1), that a subsequence of $\{y_n(x)\}$, which can be assumed to be the full sequence, converges almost everywhere to zero. Since Ω has finite measure we can assume that $\gamma = (p - 2)/2$, and then $\sigma^{1/2\gamma} \in L^p$. Let E_n denote the subset of Ω where $|y_n(x)| \geq \sigma^{1/2\gamma}$, by (8) and (9) we then have, on E_n ,

$$\begin{aligned} |F(y_n^2(x), x)|^r &= |F(y_n^2(x), x)| |F(y_n^2(x), x)|^{r-1} \\ &\leq c_1^{r-1} y_n^2(x) F(y_n^2(x), x), \end{aligned}$$

where $c_1 = (c + 1)$. Thus

$$\int_{E_n} |F(y_n^2(x), x)|^r dx \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

If $E'_n = \Omega/E_n$, then $|F(y_n^2(x), x)|^r \leq |F((\sigma(x))^{1/\gamma}, x)|^r$ for $x \in E'_n$. Thus since $y_n(x) \rightarrow 0$ a.e. in Ω , it follows from the dominated convergence theorem that

$$\int_{E'_n} |F(y_n^2(x), x)|^r dx \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Thus our supposition has led to a contradiction of (30) and (26) is proved.

For an arbitrary $y \in L^p$ we have, for almost all $x \in \Omega$,

$$|F(y^2(x), x)|^r \leq c_1 y^2(x) F(y^2(x), x) + |F((\sigma(x))^{1/\gamma}, x)|^r.$$

Upon integrating this inequality over Ω we obtain

$$\int_{\Omega} |F(y^2(x), x)|^r dx \leq c_1 \int_{\Omega} y^2(x) F(y^2(x), x) dx + \int_{\Omega} |F((\sigma(x))^{1/\gamma}, x)|^r dx.$$

Using this inequality in (29) and using the continuity of A from L^q to L^p we obtain (27).

We now show that the problem

$$(31) \quad H(y) = \min., \quad N(y) = 0,$$

has a solution within the class of admissible functions. First we observe that (25) and (26) imply

$$(32) \quad \mu = \inf \{H(y) : y \text{ is admissible and } N(y) = 0\} > 0.$$

Let y , v and α be as in (**). The function $F(\eta, x)$ is increasing in η for almost all x , therefore

$$0 \leq \int_{\Omega} (v^2(x) - y^2(x))(F(v^2(x), x) - F(y^2(x), x)) \, dx,$$

and this implies, in view of (24) and (25),

$$\begin{aligned} \int_{\Omega} v^2(x)F(y^2(x), x) \, dx &\leq \int_{\Omega} (v^2(x)F(v^2(x), x) + y^2(x)F(y^2(x), x)) \, dx \\ &\leq 2\epsilon^{-1}(1 + \epsilon)H(y). \end{aligned}$$

Using (23) this gives,

$$\alpha^2 \int_{\Omega} y^2(x)F(y^2(x), x) \, dx \leq 2\epsilon^{-1}(1 + \epsilon)H(y).$$

From (26) follows

$$(33) \quad \alpha^2 \leq CH(y), \quad C = 2(m\epsilon)^{-1}(1 + \epsilon),$$

and finally from (21), (27), (33) and (25) we have

$$(34) \quad \|v\|_p \leq C_0(1 + H(y))^{(2+q)/2q}.$$

For $B \subset L^p/\{0\}$ we denote by $\phi(B)$ the set of all (admissible) functions of the form (21), where $\alpha > 0$ is chosen so that (22) holds,

$$\phi(B) = \{v : v = \alpha Afy, y \in B; \alpha \in R_+, N(v) = 0\}.$$

We choose $\mu_1 > \mu$ and take

$$B = \{y : y \text{ admissible, } N(y) = 0, H(y) \leq \mu_1\}.$$

Then by (34),

$$(35) \quad \phi(B) \subset \{y \in L^p(\Omega) : \|y\|_p \leq c_0(1 + \mu_1^{(2+q)/2q})\},$$

and since f maps bounded sets in L^p into bounded sets in L^q it follows from (35) that

$$(36) \quad \phi^2(B) = \phi(\phi(B)) \subset A\{u \in L^q : \|u\|_q \leq \text{const.}\}.$$

The ball $\{u \in L^q : \|u\|_q \leq \text{const.}\}$ is weakly compact and A is completely continuous so the set on the right in (36) is compact (in L^p). Thus B_1 , the closure of $\phi^2(B)$ in L^p , is compact in L^p and any non-zero function in B_1 is admissible. By Lemma 1,

$$(37) \quad \inf \{H(y) : y \in B_1\} = \inf \{H(y) : y \in \phi^2(B)\},$$

and since $\phi^2(B)$ consists entirely of admissible functions

$$(38) \quad \mu \leq \inf \{H(y) : y \in \phi^2(B)\} \leq \inf \{H(y) : y \in B\} = \mu;$$

the second inequality in (38) follows from (**) and the definition of ϕ . From the first inequality of (38) and from (37) it follows, using (32), that $0 \notin B_1$, therefore all functions in B_1 are admissible. Since B_1 is compact it follows from (37), (38), and Lemma 1 that there is a $y_0 \in B_1$ with $H(y_0) = \mu$. By Lemma 1, N vanishes identically on B_1 , thus the admissible function y_0 is a solution of the variational problem (31). As we have already observed, an admissible solution of (31) satisfies (2). Thus we have proved Theorem 1.

Remark. Let C denote the cone of almost everywhere non-negative functions in L^p and assume that K is non-negative, or let C denote some other closed convex cone in L^p and assume that K and F are such that $\int_{\Omega} K(x, t)y(t)F(y^2(t), t) dt$ is in C whenever y is. Then one can add to the definition of admissibility the condition that $y \in C$; with this definition of admissibility the argument given above implies that (2) has a non-trivial solution in C . In particular, if we add non-negativity to the definition of admissibility, then the condition that A be positive definite in L^p can be replaced by the condition that K be positive a.e. in $\Omega \times \Omega$, and that $A|L^2$ be non-negative definite.

5. Proof of Theorem 2. The only place in the proof of Theorem 1 where the argument can break down when F is replaced by $F_1 = P + F$ is in the demonstration (for which the reader was referred to [5]) that the normalization (19) is possible for any admissible function y_0 . However if F is replaced by F_1 in (15) and (17) then the normalization (19) is still possible provided the least eigenvalue of (11) exceeds 1. The proof is the same as in [5]. All of the rest of the arguments above remain valid as they stand when F is replaced by F_1 . It should be noted that $H(y)$ remains unchanged when F is replaced by F_1 .

6. Boundedness. Proof of Theorem 3. The following lemmas constitute the proof of Theorem 3.

Lemma 3. Suppose that (13) holds for some $p > 2$, and that F satisfies (8) where

$$(39) \quad 0 \leq \gamma < (p - 2)/2, \sigma \in L^s, s > p/2\gamma.$$

Then A_f maps L^{p_1} into L^{p_2} for

$$(40) \quad p \leq p_1 \leq 2\gamma s, \quad p_1^{-1} > p_2^{-1} > (2\gamma + 1)/p_1 - (a - 1)/a.$$

Proof. For $y \in L^{p_1}$, $p_1 \geq p$, we have by (8) and (39),

$$(41) \quad fy \in L^{q_1},$$

for $q_1 \leq p_1 \min((2\gamma + 1)^{-1}, s/(s + p_1))$. If $p_1 \leq 2\gamma s$ then $s/(s + p_1) \geq$

$(2\gamma + 1)^{-1}$ and thus (41) holds for $q_1 \leq p_1/(2\gamma + 1)$. Now by Theorem 9.5.6, [2], A maps L^{q_1} into L^{p_2} for

$$p_2^{-1} > q_1^{-1} - (a - 1)/a, \quad 1 \leq p_2 \leq \infty,$$

and we conclude that the composite Af maps L^{p_1} into L^{p_2} for p_1, p_2 satisfying (40).

Lemma 4. *Assume the hypothesis of Lemma 3. Then any L^p -solution of (2) belongs to L^{p_2} for every p_2 satisfying*

$$(42) \quad p_2^{-1} > (2\gamma + 1)/2\gamma s - (a - 1)/a, \quad 1 \leq p_2 \leq \infty.$$

Proof. It is clear from Lemma 3 that there is a positive integer k such that $(Af)^k$ maps L^p into L^{p_1} where $p_1 = 2\gamma s$. It then follows, again by Lemma 3, that $(Af)^{k+1}(L^p) \subset L^{p_2}$ for any p_2 satisfying (42).

It follows from Lemma 4 that any L^p solution of (2) will belong to L^∞ if $s > (2\gamma + 1)a/(2\gamma(a - 1))$, we shall prove the following stronger result which, however, is not needed for the proof of Theorem 3.

Lemma 5. *Assume again the hypothesis of Lemma 3. If*

$$(43) \quad s > (4\gamma + 1)a/4\gamma(a - 1),$$

then any L^p solution of (2) belongs to L^∞ .

Proof. By (8), (39) and Lemma 4, if y is an L^p solution of (2) then $fy \in L^{q_2}$ for every q_2 satisfying

$$(44) \quad q_2^{-1} > 2s^{-1} + (2\gamma s)^{-1} - (a - 1)/a + 2\gamma \max [0, s^{-1} - (a - 1)/a].$$

Using Hölder's inequality, it follows from (13) that A maps L^{q_2} into L^∞ if

$$(45) \quad q_2^{-1} \leq (a - 1)/a.$$

In order that (44) and (45) be satisfied simultaneously by some q_2 it suffices that

$$0 > 2s^{-1} + (2\gamma s)^{-1} - 2(a - 1)/a + 2\gamma \max [0, s^{-1} - (a - 1)/a],$$

and this holds if and only if (43) holds. Clearly if there is a q_2 satisfying both (44) and (45) then the conclusion of the Lemma follows.

We can conclude from Lemma 4 that, under the hypothesis of Theorem 3, any L^p solution of (2) actually belongs to L^∞ . The condition (1) is not used in this section so the assertion of Theorem 3 concerning solutions of (12) follows by applying Lemma 4 with F replaced by $F_1 = P + F$.

The argument used above to prove boundedness is an integral equations version of the 'bootstrap procedure' for proving regularity of a generalized solution of a boundary value problem.

7. Application to a non-linear elliptic boundary value problem. We consider

the boundary value problem

$$(46) \quad \Delta y + yF(y^2, x) = 0, \text{ in } \Omega, \quad y|_{\partial\Omega} = 0,$$

where Ω is a bounded region in n -space, $n \geq 2$, for which the Dirichlet problem is solvable and Δ is the Laplace operator. If $G(x, t)$ is the Green's function for the Dirichlet problem in Ω then

$$\operatorname{ess\,sup}_{x \in \Omega} \int_{\Omega} |G(x, t)|^a dt < \infty, \quad \text{for } a < n/(n-2),$$

thus if we put $K = G$ the hypothesis H is satisfied for any pair p, q with

$$(47) \quad 2 < p < 2n/(n-2).$$

Let such a p be chosen and let A be the operator from L^q to L^p defined by (3), with $K = G$. Then, as is well known, $A|_{L^2}$ is positive definite and the range of A contains all C^3 functions with compact support in Ω . Thus, in view of the remarks following the statement of Theorem 2, A is positive definite on L^q . If we assume that F satisfies (1) and (8), where

$$(48) \quad \gamma < 2/(n-2), \quad \sigma = \text{const.},$$

then, since p can be chosen anywhere on the range (47) it follows from Theorems 1 and 3 that the integral equation

$$(49) \quad y(x) = \int_{\Omega} G(x, t)y(t)F(y^2(t), t) dt,$$

has a non-trivial solution in L^∞ . From (8), (48) and the properties of the Green's function it follows that an L^∞ solution y of (49) is continuously differentiable in Ω , and continuous in $\bar{\Omega}$ and that $y|_{\partial\Omega} = 0$. If we assume in addition that F is locally Hölder continuous on $\bar{R}_+ \times \Omega$, then y will be twice continuously differentiable in Ω . Thus we have proved the following.

Theorem 4. *Let F be locally Hölder continuous on $\bar{R}_+ \times \Omega$, and satisfy (8) where (48) holds. Assume also that for some $\epsilon > 0$, (1) holds for all $x \in \Omega$. Then the boundary value problem (46) has a non-trivial solution y which is continuous in $\bar{\Omega}$ and of class C^2 in Ω .*

Similarly we can prove the following.

Theorem 5. *Assume the hypothesis of Theorem 4, and assume that P is a bounded non-negative function which is locally Hölder continuous on Ω . If the least eigenvalue of*

$$\Delta u + \lambda P(x)u = 0, \text{ in } \Omega, \quad u|_{\partial\Omega} = 0,$$

exceeds 1, then the boundary value problem

$$\Delta y + y(P(x) + F(y^2, x)) = 0, \quad \text{in } \Omega, \quad y|_{\partial\Omega} = 0,$$

has a non-trivial solution y which is continuous in $\bar{\Omega}$ and of class C^2 in Ω .

Remarks 1. From Remark 1 at the end of Section 4 it follows that the solutions whose existence is obtained in Theorems 4 and 5 can also be asserted to be positive in Ω .

2. In [6] it is shown that if $\Omega = \{x \in R^n : |x| < 1\}$, $n > 2$, and if $F(\eta, x) = \eta^\gamma$ where $\gamma \geq 2/(n - 2)$ then (46) does not have a solution which is positive and of class C^2 in Ω and continuous in $\bar{\Omega}$.

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