Uniform Approximation by Tchebycheffian Spline Functions

LARRY L. SCHUMAKER*

Communicated by G. BIRKHOFF

§1. Introduction. In this paper we consider the problem of approximating a given continuous function $f(x) \in C[a, b]$ in the uniform norm by spline functions with fixed knots. In particular, for fixed $a = x_0 < x_1 < \dots < x_{k+1} = b$ we consider the class of Tchebycheffian Spline Functions (TSF's)

$$(1.1) \quad \mathcal{S}_{n,k}(x_1, \dots, x_k) = \{s(t) \in C^{n-1}[a, b] \mid s(t) \in \pi_n \text{ in each of the intervals } (x_i, x_{i+1}), i = 0, 1, \dots, k\}.$$

Here π_n denotes the class of u -polynomials $\sum_{i=0}^n a_i u_i(t)$ of degree n generated by an Extended Complete Tchebycheff (ECT) system $\{u_i(t)\}_0^n$ of the form

$$(1.2) \quad \begin{aligned} u_0(t) &= w_0(t) \\ u_1(t) &= w_0(t) \int_a^t w_1(\xi_1) d\xi_1 \\ &\dots \\ u_n(t) &= w_0(t) \int_a^t w_1(\xi_1) \int_a^{\xi_1} w_2(\xi_2) \dots \int_a^{\xi_{n-1}} w_n(\xi_n) d\xi_n \dots d\xi_1, \end{aligned}$$

where $w_i(t) \in C^{n-i}[a, b]$ and $w_i(t) > 0$ for $t \in [a, b]$. Properties of TSF's have been studied in [1], [2], and [3].

The class (1.1) includes ordinary spline functions since π_n reduces to the ordinary polynomials for the choice $\{w_i(t) \equiv 1\}_0^n$.

Every $s(t) \in \mathcal{S}_{n,k}(x_1, \dots, x_k)$ has a unique representation of the form (see [3])

$$(1.3) \quad s(t) = \sum_{i=0}^n a_i u_i(t) + \sum_{i=1}^k c_i \varphi_n(t; x_i),$$

where

$$(1.4) \quad \varphi_n(t; x) = \begin{cases} w_0(t) \int_x^t w_1(\xi_1) \int_x^{\xi_1} w_2(\xi_2) \dots \int_x^{\xi_{n-1}} w_n(\xi_n) d\xi_n \dots d\xi_1, & t \geq x \\ 0, & t < x. \end{cases}$$

* Sponsored by the Mathematics Research Center, United States Army, Madison, Wisconsin, under Contract No. DA-31-124-ARO-D-462.

In view of the representation (1.3), it is clear that the spline approximation problem (with fixed knots $\{x_i\}_1^k$) reduces to the standard linear approximation problem of determining the best approximation of a given continuous function in the uniform norm by a linear combination of the $n + k + 1$ functions

$$(1.5) \quad \{u_0(t), u_1(t), \dots, u_n(t), \varphi_n(t; x_1), \dots, \varphi_n(t; x_k)\}.$$

By the general linear theory (cf. eg., [4]) we have

Theorem 1.1. *There exists a best approximation $s^*(t) \in S_{n,k}(x_1, \dots, x_k)$.*

The system of functions (1.5) are linearly independent, but do not form a Tchebycheff (Haar) system. In fact, Th. 3.2 of [5] (cf. Th. 2 of [3]) asserts that

Lemma 1.2.

$$(1.6) \quad \Phi \begin{pmatrix} 0 & \cdots & 0 & x_1 & \cdots & x_k \\ t_1 & \cdots & & & & t_{n+k+1} \end{pmatrix} \\ = \begin{vmatrix} u_0(t_1) & u_1(t_1) & \cdots & u_n(t_1) & \varphi_n(t_1; x_1) & \cdots & \varphi_n(t_1; x_k) \\ u_0(t_2) & u_1(t_2) & \cdots & & & & \\ \vdots & & & & & & \\ u_0(t_{n+k+1}) & u_1(t_{n+k+1}) & \cdots & u_n(t_{n+k+1}) & \varphi_n(t_{n+k+1}; x_1) & \cdots & \varphi_n(t_{n+k+1}; x_k) \end{vmatrix}$$

is nonnegative for all $a \leq t_1 < t_2 < \cdots < t_{n+k+1}$, and strictly positive if and only if

$$(1.7) \quad t_i < x_i < t_{n+i+1} \quad i = 1, 2, \dots, k, (a < t_1).$$

Since the functions (1.5) do not form a Tchebycheff system, the converse of the Haar uniqueness theorem (see e.g., [4], p. 17) implies that there are functions $f \in C[a, b]$ with more than one best approximation. It is very easy to construct examples where f is of class $C^\infty[a, b]$ but has infinitely many best approximations.

In §2 we characterize best approximations in terms of an alternation property. Section 3 is devoted to developing conditions to insure that the best approximation of f is unique and the corresponding error function possesses the characteristic type of alternating behavior.

We close with a short section in which we state the de la Vallée Poussin algorithm and consider the continuity of the dependence of the best approximation (s) on the function $f \in C[a, b]$.

§2. Characterization. Let $f \in C[a, b]$ be a fixed function in $C[a, b]$ and let $a = x_0 < x_1 < \cdots < x_{k+1} = b$. A continuous function $\varphi(t)$ is said to alternate r times on an interval $[\alpha, \beta]$ provided there exist $r + 1$ points

$$\alpha \leq t_1 < \cdots < t_{r+1} \leq \beta$$

such that

$$\varphi(t_i) = (-1)^i \max_{\alpha \leq t \leq \beta} |\varphi(t)|$$

for $i = 1, 2, \dots, r+1$, where $\xi = \pm 1$. Henceforth we require $n \geq 1$.

Theorem 2.1. *If $s \in S_{n,k}(x_1, \dots, x_k)$ is such that $f - s$ alternates $n + j + 1$ times on any subinterval $[x_m, x_{m+j+1}] \subset [x_0, x_{k+1}]$, then s is a best approximation of f on $[a, b]$ in the class $S_{n,k}(x_1, \dots, x_k)$.*

Proof. Assume to the contrary that there exists an $\tilde{s} \in S_{n,k}(x_1, \dots, x_k)$ with $\|f - \tilde{s}\| < \|f - s\|$. Since by assumption $(f - s)(t)$ attains $\|f - s\|$ at $n + j + 2$ points in $[x_m, x_{m+j+1}]$ with alternating sign, it is clear that $\Delta = s - \tilde{s}$ has at least $n + j + 1$ distinct zeros on $[x_m, x_{m+j+1}]$, and since $\|f - s\| < \|f - \tilde{s}\|$ these can be chosen so that no two lie on an interval in which $\Delta \equiv 0$. Now $\Delta \in S_{n,j}(x_{m+1}, \dots, x_{m+j})$ and by Lemma 2.1 of [2] it can have at most $n + j$ distinct zeros on $[x_m, x_{m+j+1}]$ (where no two lie on an interval with $\Delta \equiv 0$). This is a contradiction, and the proof is complete.

The converse of Theorem 2.1 is proved below in Theorem 2.3. Together they completely characterize best approximations in terms of alternation. Since $\{u_i\}_0^n$ is a T -system we clearly have

Lemma 2.2. *If $s \in S_{n,k}(x_1, \dots, x_k)$ is a best approximation of f , then $f - s$ alternates at least $n + 1$ times.*

In the following we denote by $A_{n,k}$ the deviation of the best approximation of f , i.e.,

$$(2.1) \quad A_{n,k} = \min_{S_{n,k}(x_1, \dots, x_k)} \|f - s\|.$$

Theorem 2.3. *If $s \in S_{n,k}(x_1, \dots, x_k)$ is such that $f - s$ does not alternate $n + m - l$ times on some interval $[x_l, x_m]$, $0 \leq l < m \leq k + 1$, then it is not a best approximation of f on $[a, b]$.*

Proof. In view of Lemma 2.2, $f - s$ alternates at least $n + 1$ times on $[a, b]$. Let $(f - s)(t_i) = (-1)^i \zeta A_{n,k}$, $i = 1, 2, \dots, n + l + 1$, ($1 \leq l \leq k$, $\zeta = \pm 1$), where $a \leq t_1 < t_2 < \dots < t_{n+l+1} \leq b$. For $i = 1, 2, \dots, n + l + 1$ define $[\bar{t}_i, \bar{l}_i]$ to be the maximal interval containing t_i such that $(f - s)(\bar{t}_i) = (f - s)(\bar{l}_i) = (f - s)(t_i) = (-1)^i \zeta A_{n,k}$ while $(f - s)(t) \neq (-1)^{i+1} \zeta A_{n,k}$ for all $t \in [\bar{t}_i, \bar{l}_i]$. Our first observation is that for any r and p satisfying $1 \leq p < n + r + p \leq n + l + 1$ there are at least r of the knots $\{x_i\}_1^k$ in the interval $(\bar{l}_p, \bar{t}_{n+r+p})$; otherwise there exists l and m , $m - l - 1 < r$, with $x_l < \bar{l}_p < \bar{t}_{n+p+r} < x_m$ and $f - s$ alternates at least $n + m - l$ times on $[x_l, x_m]$ contradicting the hypothesis of the lemma. We now claim there exist $\{y_i\}_1^l \subset \{x_i\}_1^k$ such that

$$(2.2) \quad \bar{l}_i < y_i < \bar{t}_{n+i+1}, \quad i = 1, 2, \dots, l.$$

Indeed, since one of the knots $\{x_i\}_1^k$ is in $(\bar{l}_1, \bar{t}_{n+2})$ we may choose y_1 as the smallest of these. We proceed by induction. Suppose $\{y_i\}_1^{q-1}$ satisfying (2.2) up to $q - 1$ have been chosen, where we always take the smallest possible y_i 's. Now set

$$(2.3) \quad y_q = \min \{x_i | \bar{l}_q < x_i < \bar{t}_{n+q+1} \text{ and } y_{q-1} < x_i\}.$$

We claim the set (2.3) is non-empty. If $y_{q-1} < \underline{l}_q$ this is clear since there is some x_i in $(\underline{l}_q, \underline{t}_{n+q+1})$ by the above observation. Suppose now $\underline{l}_q \leq y_{q-1}$ and (2.3) is empty. Since $\{y_i\}_1^{q-1}$ were chosen as small as possible, we conclude that $y_v, \dots, y_{q-1}$ are the only knots of $\{x_i\}_1^k$ in $(\underline{l}_v, \underline{t}_{n+q+1})$ for some $1 \leq v \leq q-1$, which contradicts the requirement that there are $q-v+1$ knots of $\{x_i\}_1^k$ in $(\underline{l}_v, \underline{t}_{n+q+1})$. Thus (2.3) is non-empty and this completes the inductive proof that it is possible to choose $\{y_i\}_1^l$ from $\{x_i\}_1^k$ satisfying (2.2).

We now choose $\{z_i\}_1^{n+l}$ satisfying

$$(2.4) \quad \max(y_{i-n}, \underline{l}_i) < z_i < \min(y_i, \underline{t}_{i+1}),$$

where the y 's are ignored when they have subscripts less than 1 or greater than l . This is possible by (2.2), and we have

$$(2.5) \quad z_i < y_i < z_{n+i}, \quad i = 1, 2, \dots, l. \quad (a < z_1).$$

Next consider (cf. (1.6))

$$\Delta(t) = \Phi \begin{pmatrix} 0, 0, \dots, 0, y_1, \dots, y_l \\ z_1, \dots, z_{n+l}, t \end{pmatrix}.$$

Then $\Delta \in \mathcal{S}_{n,l}(y_1, \dots, y_l)$ and clearly exhibits the zeros $\{z_i\}_1^{n+l}$. We claim that $\Delta(t) \neq 0$ for any $t \neq z_i$, since if $\{\xi_i\}_1^{n+l+1}$ are the points $\{z_i\}_1^{n+l} \cup t$ arranged in increasing order, then by (2.5)

$$\xi_i < y_i < \xi_{n+i+1}, \quad i = 1, 2, \dots, l,$$

and by Lemma 1.2, $\Delta(t) \neq 0$. Using Lemma 2.1 of [2] we conclude that $\Delta(t)$ has simple zeros with sign change at each of the points $\{z_i\}_1^{n+l}$.

Choosing η sufficiently small and with the correct sign, we obtain

$$|f(t) - [s(t) + \eta\Delta(t)]| < A_{n,k},$$

i.e., $s(t)$ is not a best approximation. This contradiction completes the proof of Theorem 2.3.

As a corollary of the above it is possible to show that there exists some $s \in \mathcal{S}_{n,k}(x_1, \dots, x_k)$ which alternates at least $n+k+1$ times on $[a, b]$. Since the proof is somewhat detailed we omit it. (Simple examples show that not every best approximation alternates $n+k+1$ times on $[a, b]$).

Theorem 2.4. *Let $f \in C[a, b]$. Then there exists an $s \in \mathcal{S}_{n,k}(x_1, \dots, x_k)$ which alternates at least $n+k+1$ times on $[a, b]$.*

§3. Uniqueness. We begin this section by characterizing uniqueness in terms of an alternation property.

Theorem 3.1. *Suppose $f \in C[a, b]$. Then $s \in \mathcal{S}_{n,k}(x_1, \dots, x_k)$ is a unique best approximation of f if there exist points $a \leq t_1 < \dots < t_{n+k+2} \leq b$ satisfying*

$$(3.1) \quad t_{i+1} < x_i < t_{n+i+1} \quad i = 1, 2, \dots, k$$

with

$$(3.2) \quad (f - s)(t_i) = (-1)^i \zeta A_{n,k} \quad i = 1, 2, \dots, n + k + 2 \quad (\zeta = \pm 1).$$

Proof. If $\bar{s}$ is another best approximation of f then $\Delta = s - \bar{s}$ has $n + k + 1$ zeros $\{z_i\}_1^{n+k+1}$ satisfying

$$(3.3) \quad z_i < x_i < z_{n+i+1}, \quad i = 1, 2, \dots, k,$$

and hence by Lemma 1.2 the system of equations

$$\Delta(z_j) = \sum_{i=0}^n (a_i - \bar{a}_i) u_i(z_j) + \sum_{i=1}^k (c_i - \bar{c}_i) \varphi_n(z_j; x_i) = 0$$

$$j = 1, 2, \dots, n + k + 1$$

has positive determinant. Therefore $s(t) \equiv \bar{s}(t)$.

The converse of Theorem 3.1 does not hold, *i.e.*, there are examples where $\{t_i\}_1^{n+k+2}$ satisfying (3.1) and (3.2) do not exist.

We now derive a sufficient condition involving deviations which insures that the best approximation is unique. Let f be a fixed function in $C[a, b]$, and let $a = x_0 < x_1 < \dots < x_{k+1} = b$. We define

$$(3.4) \quad A_{n,l}[x_j, x_{j+l+1}] = \min_{S_{n,l}(x_{j+1}, \dots, x_{j+l})} \max_{x_j \leq t \leq x_{j+l+1}} |s(t) - f(t)|$$

for $0 \leq j < j + l + 1 \leq k + 1$. Note that $A_{n,k} = A_{n,k}[x_0, x_{k+1}]$. The following lemma is very easy to verify.

Lemma 3.2. For any $l = 0, 1, \dots, k - 1$,

$$A_{n,l}[x_j, x_{j+l+1}] \leq A_{n,l+1}[x_j, x_{j+l+2}], \quad 0 \leq j \leq k - l - 1,$$

$$A_{n,l}[x_j, x_{j+l+1}] \leq A_{n,l+1}[x_{j-1}, x_{j+l+1}], \quad 1 \leq j \leq k - l.$$

Theorem 3.3. Suppose $f \in C[a, b]$ satisfies

$$(3.5) \quad A_{n,k}[x_0, x_{k+1}] > \begin{cases} A_{n,k-1}[x_0, x_k] \\ A_{n,k-1}[x_1, x_{k+1}]. \end{cases}$$

Then there exists a unique best approximation of f on $[a, b]$ and $f - s$ alternates r times with $n + k + 1 \leq r \leq 2n + k + 2$.

Proof. By Theorem 2.4 there is a TSF $s \in S_{n,k}(x_1, \dots, x_k)$ such that $f - s$ alternates at least $n + k + 1$ times. Let $\{t_i\}_1^r$, $r \geq n + 2$ be such that $(f - s)(t_i) = (-1)^i A_{n,k}$, $1 \leq i \leq r$. Now we claim

$$(3.6) \quad t_{i+1} \leq t_{r-n-k+i-1} < x_i < t_{n+i+1} \quad i = 1, 2, \dots, k.$$

Indeed, if $x_i \leq t_{r-n-k+i-1}$ for some $1 \leq i \leq k$, then $s \in S_{n,k-i}(x_{i+1}, \dots, x_k)$ on $[x_i, b]$ and $f - s$ alternates $n + k - i + 1$ times. Thus by Theorem 2.1 s is a best

approximation in the class $\mathcal{S}_{n,k-i}(x_{i+1}, \dots, x_k)$ on $[x_i, b]$ and by Lemma 3.2

$$A_{n,k} = A_{n,k-i}[x_i, b] \leq A_{n,k-1}[x_1, b],$$

contradicting (3.5). The proof of the right hand inequality is similar. The uniqueness follows immediately from Theorem 3.1.

It is easy to construct examples to illustrate that property (3.5) is not necessary for uniqueness. Clearly condition (3.5) can only be satisfied if $f - s$ alternates $n + k + 1$ times on $[a, b]$ and does not alternate $n + j + 1$ times, $j < k$, on any subinterval $[x_m, x_{j+m+1}] \subset [a, b]$. As a corollary of the proof we have

Corollary 3.4. *Let $f \in C[a, b]$ and suppose $f - s$ alternates $n + k + 1$ times on $[a, b]$ but does not alternate $n + j + 1$ times ($j < k$) on any subinterval $[x_m, x_{m+j+1}] \subset [a, b]$. Then s is the unique best approximation of f on $[a, b]$ in the class $\mathcal{S}_{n,k}(x_1, \dots, x_k)$.*

As an application of Theorem 3.3 we prove the following special theorem for $k = 1$.

Theorem 3.5. *Let $k = 1$ and suppose $f \equiv u_{n+1}$ where $\{u_i\}_0^{n+1}$ is an ECT system ($n \geq 2$). Then the best approximation of f with respect to $\mathcal{S}_{n,1}(x_1)$ is always unique and alternates at least $n + 2$ times.*

Proof. If

$$A_{n,1}[a, b] > \begin{cases} A_{n,0}[a, x_1] \\ A_{n,0}[x_1, b], \end{cases}$$

then Theorem 3.3 applies. Assume now that $A_{n,1}[a, b] = A_{n,0}[a, x_1]$. Then if $s(t)$ is a best approximation of f in $\mathcal{S}_{n,1}(x_1)$, it is also best in the class π_n on $[a, x_1]$, and hence by the Haar uniqueness theorem $f - s$ alternates $n + 1$ times on $[a, x_1]$. But this implies $(f - s)'(t)$ has $n + 1$ distinct zeros on $[a, x_1]$ which contradicts the fact that a u -polynomial in the ECT system $\{u_i\}_0^{n+1}$ can have at most n zeros in its derivative. The case $A_{n,1}[a, b] = A_{n,0}[x_1, b]$ is similar.

Remark 3.6. The situation is entirely different for $k > 1$. Here we can construct an example where the best approximation is not unique although $f \equiv u_{n+1}$ is part of the ECT system $\{u_i\}_0^{n+1}$. Indeed, by Theorem 4.6 of [1] there exists a monospline

$$m(t) = u_{n+1}(t) + \sum_{i=0}^n a_i u_i(t) + \sum_{i=1}^{k-1} c_i \varphi_n(t; x_i)$$

which satisfies

$$\begin{aligned} m(t_i) &= (-1)^i A & i &= 0, 1, \dots, n + 2k - 1 \\ m'(t_i) &= 0 & i &= 1, 2, \dots, n + 2k - 2 \end{aligned}$$

for some constant A and $a = t_0 < t_1 < \dots < t_{n+2k-1} = b$ and some $a < x_1 < \dots < x_{k-1} < b$ with $t_{2i} < x_i < t_{n+2i-1}$, $i = 1, 2, \dots, k - 1$. Consider the problem of approximating $f(t) = u_{n+1}(t)$ on $[a, b]$ by TSF's of class $\mathcal{S}_{n,k}$

$(x_1, \dots, x_k)$, where $\{x_i\}_{i=1}^{k-1}$ are the $k-1$ knots of the monospline $m(t)$, and $t_{n+2k-2} < x_k < b$. Now $s(t) = -\sum_{i=0}^n a_i u_i(t) - \sum_{i=1}^{k-1} c_i \varphi_n(t; x_i)$ is such that $f-s$ alternates $n+2k-1$ times on $[a, b]$, and since $n+2k-1 \geq n+k+1$ for $k \geq 2$, we conclude from Theorem 2.1 that $s(t)$ is a best approximation in $\mathcal{S}_{n,k}(x_1, \dots, x_k)$. Now by Theorem 1.1 of [2], $m'(t)$ has at most $n+2k-2$ zeros, and since $m'(t_i) = 0$, $i = 1, 2, \dots, n+2k-2$, we conclude $m'(t) \neq 0$ for $t_{n+2k-2} < x_k < t \leq b$. Now it is clear that

$$s(t) - \eta \varphi_n(t; x_k)$$

is also a best approximation for an appropriate choice of $\eta > 0$ sufficiently small. Of course, here $A_{n,k}[a, b] = A_{n,k-1}[a, x_k]$.

§4. Supplements. An argument similar to that used in Theorem 2.1 establishes the following analogue for approximation by spline functions of the de la Vallée Poussin algorithm (cf. [4]).

Theorem 4.1. Suppose there exist $\{t_i\}_{i=1}^{n+k+2}$ and $\{\lambda_i\}_{i=1}^{n+k+2}$ with $\lambda_i > 0$ and $a \leq t_1 < \dots < t_{n+k+2} \leq b$ satisfying

$$(f-s)(t_i) = (-1)^i \zeta \lambda_i \quad i = 1, 2, \dots, n+k+2. \quad (\zeta = \pm 1)$$

Then

$$\min_i \lambda_i \leq A_{n,k} \leq \|f-s\|.$$

We now investigate the manner in which the best approximation depends on f .

Theorem 4.2. Suppose the best approximation $s \in \mathcal{S}_{n,k}(x_1, \dots, x_k)$ of f satisfies (3.1), (3.2) for some choice of $\{t_i\}_{i=1}^{n+k+2}$. Then for every $\tilde{f} \in C[a, b]$

$$\|s - \tilde{s}\| \leq M(f) \|f - \tilde{f}\|,$$

where $\tilde{s} \in \mathcal{S}_{n,k}(x_1, \dots, x_k)$ is any best approximation of $\tilde{f}$ and $M(f)$ is independent of $\tilde{f}$.

Remark. The hypotheses of Theorem 4.2 are satisfied whenever f satisfies the requirements for Theorem 3.3 or Corollary 3.4.

Proof. We may assume $\zeta = +1$ in (3.2). Clearly

$$\begin{aligned} (-1)^i [s - \tilde{s}](t_i) &= (-1)^i [f(t_i) - \tilde{f}(t_i)] - A_{n,k} + (-1)^i [\tilde{f}(t_i) - \tilde{s}(t_i)] \\ &\leq \|f - \tilde{f}\| + \tilde{A}_{n,k} - A_{n,k} \\ &\leq 2 \|f - \tilde{f}\|, \end{aligned}$$

where $\tilde{A}_{n,k}$ is the deviation corresponding to $\tilde{f}$.

Now let $L(\psi) = \sum_{i=1}^{n+k+2} \lambda_i \psi(t_i)$ be the linear functional defined by the requirements $\sum_{i=1}^{n+k+2} |\lambda_i| = 1$ and

$$(4.1) \quad L(b_j) = 0 \quad j = 1, 2, \dots, n+k+1,$$

where $\{b_j(t)\}_1^{n+k+1}$ are the functions of (1.5). The required $\{\lambda_i\}$ exist since by (1.7)

$$(4.2) \quad \Phi \begin{pmatrix} 0, 0, \dots, 0, x_1, \dots, x_k \\ s_1, s_2, \dots, s_{n+k+1} \end{pmatrix} \neq 0,$$

for any choice of $\{s_i\}_1^{n+k+1}$ from $\{t_i\}_1^{n+k+2}$, and hence the same is true of its transpose, which is precisely the determinant arising from the system of equations (4.1). The $\{\lambda_j\}_1^{n+k+2}$ are uniquely determined and alternate in sign.

Now since $\delta = s - \tilde{s} \in \mathcal{S}_{n,k}(x_1, \dots, x_k)$, we have $L(\delta) = 0$ and thus

$$\begin{aligned} -2 \|f - \tilde{f}\| (1 - |\lambda_j|) &\leq - \sum_{\substack{i=1 \\ i \neq j}}^{n+k+2} |\lambda_i| (-1)^i \delta(t_i) \\ &= |\lambda_j| (-1)^j \delta(t_j) \leq 2 \|f - \tilde{f}\| \cdot |\lambda_j| \end{aligned}$$

which implies

$$|\delta(t_i)| \leq 2 \|f - \tilde{f}\| \max_{1 \leq j \leq n+k+2} \left(\frac{1 - |\lambda_j|}{|\lambda_j|} \right).$$

for $i = 1, 2, \dots, n+k+2$. But since the determinant (4.2) is non-zero, the fact that δ is bounded at the points $\{t_i\}_1^{n+k+2}$ implies its coefficients are uniformly bounded and hence $|\delta(t)| \leq B \max |\delta(t_i)|$ for all t , where B is a constant depending only on $\{t_i\}_1^{n+k+2}$, i.e., on f . Thus

$$\|s - \tilde{s}\| \leq M(f) \|f - \tilde{f}\|$$

where

$$M(f) = 2 \cdot B \max_{1 \leq j \leq n+k+2} \left(\frac{1 - |\lambda_j|}{|\lambda_j|} \right)$$

and the proof is complete.

Remark. The linear functional defined in Theorem 4.2 has $\|L\| = 1$, and hence (see [4])

$$|L(f)| \leq A_{n,k}.$$

Numerical lower bounds for $A_{n,k}$ can thus be calculated by constructing the linear functional corresponding to any set of $\{t_i\}_1^{n+k+2}$ satisfying (3.1).

A weaker continuity result can be obtained when f does not satisfy the hypotheses of Theorem 4.2. We have

Theorem 4.3. Suppose f satisfies

$$(4.3) \quad A_{n,k}[x_0, x_{k+1}] = \dots = A_{n,m-l-1}[x_l, x_m] > \begin{cases} A_{n,m-l-2}[x_l, x_{m-1}] \\ A_{n,m-l-2}[x_{l+1}, x_m] \end{cases}$$

Then there exists a constant $M(f)$ depending only on f such that

$$\|s - \tilde{s}\|_{lm} \leq M(f) \|f - \tilde{f}\|_{lm}$$

where s and $\tilde{s}$ are any best approximations of f and $\tilde{f}$, respectively, on $[a, b]$ and

$$\|f - \tilde{f}\|_{lm} = \max_{x_l \leq t \leq x_m} |f(t) - \tilde{f}(t)|.$$

Proof. If $m - l > 2$ the proof proceeds exactly as in Theorem 4.2 except that we restrict attention to $[x_l, x_m]$. Thus $L(\varphi) = \sum_{i=l+1}^{n+m+1} \lambda_i \varphi(t_i)$ for some $\{t_i\}_{i=l+1}^{n+m+1}$ satisfying (3.1), (3.2) and L is required to annihilate $\{u_0(t), \dots, u_n(t), \varphi_n(t; x_{l+1}), \dots, \varphi_n(t; x_{m-1})\}$.

Using Theorem 3.3 it can also be shown that for certain functions with a unique best approximation all continuous functions in a neighborhood also have unique best approximations:

Theorem 4.4. Suppose f satisfies (3.5). Define

$$\epsilon = \min (A_{n,k} - A_{n,k-1}[x_0, x_k], A_{n,k} - A_{n,k-1}[x_1, x_{k+1}]).$$

Then if $\|f - \tilde{f}\| < \epsilon/2$, the function $\tilde{f}$ also has a unique best approximation $\tilde{s} \in \mathcal{S}_{n,k}(x_1, \dots, x_k)$.

It is not true in general, of course, that if f has a unique best approximation then so does every continuous function in a neighborhood of f . Examples are easily constructed.

Most of these results were obtained in the author's dissertation [5] at Stanford University, and I would like to thank Professor S. Karlin for his guidance as my advisor.

REFERENCES

- [1] CARL H. FITZGERALD & L. SCHUMAKER, A differential equation approach to interpolation at extremal points, *MRC Technical Summary Report*, **731** (1967).
- [2] S. KARLIN & L. SCHUMAKER, The fundamental theorem of Algebra for Tchebycheffian monosplines, *J. d'Analyse*, **20** (1967) 233-270.
- [3] S. KARLIN & Z. ZIEGLER, Tchebycheffian spline functions, *J. SIAM Num. Anal.*, **3** (1966), 514-543.
- [4] G. MEINARDUS, *Approximation von Funktionen und Ihre Numerische Behandlung*, Springer-Verlag, Heidelberg, 1964.
- [5] LARRY L. SCHUMAKER, On some problems involving Tchebycheff systems and spline functions, *dissertation, Stanford University* (1966).

Mathematics Research Center
University of Wisconsin
Date communicated: FEBRUARY 5, 1968