*Upper and Lower Bounds of the Radii of Gyration of Convex Bodies II**

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Communicated by TRACY Y. THOMAS

1. Introduction. This note is a continuation of the development presented in [1] and to show how the problems considered there can be further generalized. Consider a bounded convex body K of dimension $n > 1$ in an Euclidean space. With respect to a rectangular Cartesian coordinate system, $X = (x_1, x_2, \dots, x_n)$, the volume $V(K)$ of K is given by the integral,

$$V(K) \equiv \int_K dX, \quad dX \equiv dx_1 dx_2 \cdots dx_n;$$

and the j -th coordinate of the centroid C_K of K is defined by the formulas,

$$\bar{x}_j(K) \equiv m(K, x_j)/V(K), \quad m(K, x_j) \equiv \int_K x_j dX,$$

for $j = 1, 2, \dots, n$. Let $\mathbf{u}$ be a unit vector emanating from C_K such that for some number $\epsilon > 0$ the other end point of $\epsilon\mathbf{u}$ lies on the surface of a solid ball contained in K . Let $l(C_K, \mathbf{u})$ be the line passing through the centroid C_K and having the direction $\mathbf{u}$; and let $\rho(X, l(C_K, \mathbf{u}))$ be the distance from the point X to the line l . As a generalization of the usual notions, we define for all real numbers $\lambda > 1$,

$$(1.1) \quad I_\lambda(K, C_K, \mathbf{u}) \equiv \int_K [\rho(X, l(C_K, \mathbf{u}))]^\lambda dX,$$

$$g_\lambda(K, C_K, \mathbf{u}) \equiv [I_\lambda(K, C_K, \mathbf{u})/V(K)]^{1/\lambda},$$

and call $I_\lambda(K, C_K, \mathbf{u})$ the moment of inertia of K of order λ about the line $l(C_K, \mathbf{u})$ and $g_\lambda(K, C_K, \mathbf{u})$ the radius of gyration of K of order λ about the line $l(C_K, \mathbf{u})$. Further, for a given line l , we define

$$(1.2) \quad \delta(K, C_K, \mathbf{u}) = \sup \{ \rho(X, l(C_K, \mathbf{u})) ; X \text{ in } K \}.$$

Also, we define the relative radius of gyration of order λ of K about the line

* Work partially supported by the National Science Foundation under Grant No. GP-7374.

$l(C_K, \mathbf{u})$ by the following formula,

$$(1.3) \quad \gamma_\lambda(K, C_K, \mathbf{u}) \equiv g_\lambda(K, C_K, \mathbf{u})/\delta(K, C_K, \mathbf{u}).$$

Clearly, for a given bounded convex body K and for a fixed number $\lambda > 1$, $\gamma_\lambda(K, C_K, \mathbf{u})$ is a continuous function defined on a unit sphere of dimension $n - 1$. Hence, there are two directions, $\mathbf{u}_M(K, \lambda)$ and $\mathbf{u}_m(K, \lambda)$, such that

$$(1.4) \quad \gamma_\lambda(K, C_K, \mathbf{u}_M(K, \lambda)) = \text{maximum of } \gamma_\lambda(K, C_K, \mathbf{u}) \text{ for all } \mathbf{u},$$

$$(1.5) \quad \gamma_\lambda(K, C_K, \mathbf{u}_m(K, \lambda)) = \text{minimum of } \gamma_\lambda(K, C_K, \mathbf{u}) \text{ for all } \mathbf{u}.$$

Theorem. *If K is a bounded convex body of dimension $n \geq 2$, then for every number $\lambda > 1$*

$$(1.6) \quad [\gamma_\lambda(K, C_K, \mathbf{u}_M(K, \lambda))]^\lambda \leq (n - 1)/(n + \lambda - 1),$$

$$(1.7) \quad \gamma_\lambda(K, C_K, \mathbf{u}_m(K, \lambda)) \geq C(n, \lambda),$$

where the constant $C(n, \lambda)$ depends only on n and λ . Further, the equality sign in (1.6) holds only when K is a right circular cylinder [1]; while the equality sign in (1.7) holds only when $\lambda = 2$ and K is a right triangle and in all other cases $C(n, \lambda)$ is the greatest lower bound for all bounded convex bodies of dimension n and its value is given by the formula,

$$(1.8) \quad C(n, \lambda) = \frac{n+1}{n} \left[n \int_0^1 \left| x - \frac{1}{n+1} \right|^\lambda (1-x)^{n-1} dx \right]^{1/\lambda}.$$

Analogous to the technique adopted in [1, 3], the proof is to compare the relative radii of gyration of a given convex body with that of the solution figure which has been conjectured in advance by means of constructing intermediate bodies.

2. Proof of the inequality (1.6). First, we state the following three lemmas which will be used in our proof.

Lemma 1. *Let $f(x)$ be a bounded concave function for $0 \leq x \leq x_1$ and let $F(x) \equiv y - f(x)$ for $0 \leq x \leq x_1$. Let K be the convex set,*

$$K = \{(x, y); 0 \leq x \leq x_1, y \geq 0, F(x, y) \leq 0\}.$$

Then for every integer $n \geq 2$, there is a unique trapezoid T ,

$$T = \{(x, y); 0 \leq x \leq x_1, y \geq 0, y - (y_0 + \alpha x) \leq 0\},$$

with the following properties,

$$(2.1) \quad y_0 \geq f(0), \quad y_0 + \alpha x_1 \geq f(x_1), \quad m_{n-2}(T) = m_{n-2}(K),$$

where α and y_0 are constants and where

$$m_{n-i}(T) \equiv \int_0^{x_1} x^{n-i} (y_0 + \alpha x) dx, \quad m_{n-i}(K) \equiv \int_0^{x_1} x^{n-i} f(x) dx, \quad j = 1, 2.$$

The proof of the above lemma has been presented in [1]. The next lemma is a generalization of Lemma 3* in [1] and its proof was suggested by our colleague R. P. Kaufman.

Lemma 2. *Let K be the plane convex set specified in Lemma 1 and let T be the trapezoid constructed there. Then, for every number $r > n - 1$, we have*

$$(2.2) \quad \int_0^{x_1} x^r (y_0 + \alpha x) dx \geq \int_0^{x_1} x^r f(x) dx,$$

where the equality sign holds only when K is a trapezoid.

Proof. First, we consider the case that $f(x)$ is twice continuously differentiable in the closed interval $[0, x_1]$. Write

$$(2.3) \quad g(x) \equiv y_0 + \alpha x - f(x), \quad 0 \leq x \leq x_1.$$

Then, $g(x)$ is a convex function of the class C^2 on $[0, x_1]$. Hence, by means of integration by parts, we find

$$(2.4) \quad \begin{aligned} m_{n-2}(g) &\equiv \int_0^{x_1} x^{n-2} g(x) dx \\ &= \frac{1}{n-1} x_1^{n-1} g(x_1) - \frac{1}{n(n-1)} x_1^n g'(x_1) + \frac{1}{n(n-1)} \int_0^{x_1} x^n g''(x) dx, \end{aligned}$$

$$(2.5) \quad \begin{aligned} m_{n-1}(g) &\equiv \int_0^{x_1} x^{n-1} g(x) dx \\ &= \frac{1}{n} x_1^n g(x_1) - \frac{1}{n(n+1)} x_1^{n+1} g'(x_1) + \frac{1}{n(n+1)} \int_0^{x_1} x^{n+1} g''(x) dx. \end{aligned}$$

Upon solving the equations (2.4) and (2.5) for the constants $g(x_1)$ and $g'(x_1)$, it gives

$$(2.6) \quad \begin{aligned} g(x_1) &= x_1^{-n} \left\{ n(n+1)m_{n-1}(g) - n(n-1)x_1 m_{n-2}(g) \right. \\ &\quad \left. + \int_0^{x_1} (x_1 x^n - x^{n+1}) g''(x) dx \right\}, \end{aligned}$$

$$(2.7) \quad \begin{aligned} g'(x_1) &= x_1^{-(n+1)} \left\{ n^2(n+1)m_{n-1}(g) - n(n^2-1)x_1 m_{n-2}(g) \right. \\ &\quad \left. + \int_0^{x_1} [(n+1)x_1 x^n - n x^{n+1}] g''(x) dx \right\}. \end{aligned}$$

On the other hand for every number $r > n - 1$ we also have by partial integration

$$(2.8) \quad \begin{aligned} m_r(g) &\equiv \int_0^{x_1} x^r g(x) dx \\ &= \frac{1}{r+1} x_1^{r+1} g(x_1) - \frac{1}{(r+1)(r+2)} \left[x_1^{r+2} g'(x_1) - \int_0^{x_1} x^{r+2} g''(x) dx \right]. \end{aligned}$$

Hence, by substituting the expressions in (2.6) and (2.7) into (2.8) for $g(x_1)$ and $g'(x_1)$, it gives

$$(2.9) \quad m_r(g) = k_1 m_{n-1}(g) + k_2 m_{n-2}(g) + \int_0^{x_1} x^n h(x, r) g''(x) dx,$$

where

$$(2.10) \quad k_1 = n(n+1)(r+2-n)(r+1)^{-1}(r+2)^{-2} x_1^{r+1-n},$$

$$(2.11) \quad k_2 = n(n-1)(r+1-n)(r+1)^{-1}(r+2)^{-2} x_1^{r+2-n},$$

$$(2.12) \quad h(x, r) = [(r+1-n)x_1 - (r+2-n)x + x_1(x/x_1)^{r+2-n}] x_1^{r+1-n/(r+1)(r+2)}.$$

Since the trapezoid T was so constructed that the conditions in (2.1) are all satisfied, it follows that

$$(2.13) \quad m_{n-2}(g) = 0, \quad m_{n-1}(g) = 0$$

Hence, the equation (2.9) is reduced to the simple form,

$$(2.14) \quad m_r(g) = \int_0^{x_1} x^n h(x, r) g''(x) dx.$$

Now, direct computation based on the expression in (2.12) shows that for all values $r > n-1$

$$h(x_1, r) = 0 \quad \text{and} \quad \partial h / \partial x < 0 \quad \text{for} \quad 0 \leq x < x_1$$

and that $\partial h / \partial x = 0$ at $x = x_1$. Consequently,

$$(2.15) \quad h(x, r) > 0 \quad \text{for} \quad 0 \leq x < x_1 \quad \text{and} \quad r > n-1.$$

Furthermore, the convexity of the function $g(x)$ implies that

$$(2.16) \quad g''(x) \geq 0, \quad 0 \leq x \leq x_1.$$

Thus, we conclude from (2.14)–(2.16) that $m_r(g) \geq 0$ and that the equality sign holds only where $g''(x)$ vanishes identically. But then $f(x)$ must be a linear function of x and it follows from (2.3) and (2.13) that $g(x)$ must vanish identically. This proves that the lemma is true if $f(x)$ is of class C^2 on $[0, x_1]$.

For the general case, there is a sequence, $\{f_\nu(x)\}$, of functions with the following properties [4]: (i) for each integer ν , $f_\nu(x)$ is defined on some interval containing $[0, x_1]$, (ii) $f_\nu(x)$ is a concave analytic function of x and (iii)

$$(2.17) \quad \int_0^{x_1} |f(x) - f_\nu(x)| dx \rightarrow 0 \quad \text{as} \quad \nu \rightarrow \infty.$$

Consequently, for the sequence of analytic convex functions,

$$(2.18) \quad g_\nu(x) \equiv y_0 + \alpha x - f_\nu(x), \quad 0 \leq x \leq x_1, \quad \nu = 1, 2, \dots,$$

the following two limiting relations hold,

$$(2.19) \quad \lim_{\nu \rightarrow \infty} m_{n-1}(g_\nu) = 0, \quad \lim_{\nu \rightarrow \infty} m_{n-2}(g_\nu) = 0.$$

On the other hand the same analysis as for deriving (2.9) gives that for every integer ν and for every number $r > n - 1$,

$$(2.20) \quad m_r(g_\nu) = k_1 m_{n-1}(g_\nu) + k_2 m_{n-2}(g_\nu) + \int_0^{x_1} x^n h(x, r) g_\nu''(x) dx,$$

where k_1 , k_2 and $h(x, r)$ are given by the expressions in (2.10)–(2.12) respectively. Since for all integers ν , $g_\nu''(x)$ is non-negative for $0 \leq x \leq x_1$, the inequality in (2.15) insures that the value of the integral in (2.20) is nonnegative. By letting $\nu \rightarrow \infty$ in (2.20) and taking the limit relations in (2.19) into account, we find for all numbers $r > n - 1$

$$(2.21) \quad m_r(g) = \lim_{\nu \rightarrow \infty} m_r(g_\nu) \geq 0.$$

We proceed to show that if the equality sign in (2.21) holds for some number $r > n - 1$ then $g(x)$ must vanish identically. Indeed, for every two numbers a and b with $0 < a < b < x_1$, we have the following Taylor's formula with Lagrange's remainder,

$$g_\nu(b) = g_\nu(a) + (b - a)g_\nu'(a) + \frac{1}{2} \int_a^b (b - x)^2 g_\nu''(x) dx,$$

for $\nu = 1, 2, \dots$. Hence, for all integers ν ,

$$(2.22) \quad \frac{g_\nu(b) - g_\nu(a)}{b - a} = g_\nu'(a) + \frac{1}{2(b - a)} \int_a^b (b - x)^2 g_\nu''(x) dx.$$

Similarly, for all integers ν and for every number $c \neq b$, $0 < a < c < x_1$,

$$(2.23) \quad \frac{g_\nu(c) - g_\nu(a)}{c - a} = g_\nu'(a) + \frac{1}{2(c - a)} \int_a^c (c - x)^2 g_\nu''(x) dx.$$

Since the hypothesis, $m_r(g) = 0$, implies that $g_\nu''(x)$ converges to zero almost everywhere on $[0, x_1]$, it follows from (2.22) and (2.23) that

$$\lim_{\nu \rightarrow \infty} \frac{g_\nu'(b) - g_\nu'(a)}{b - a} = \lim_{\nu \rightarrow \infty} \frac{g_\nu'(c) - g_\nu'(a)}{c - a}.$$

Since these limit relations hold for any three points a , b and c in the interval $(0, x_0)$, it follows that $g_\nu(x)$ converges to a straight line. But then we conclude from (2.3) that $f(x)$ is linear in x . This fact together with the two conditions in (2.13) implies that $g(x)$ must vanish identically. The lemma is now completely established.

In applying this lemma for the proof of (1.6), we shall set $r = \lambda + n - 1$. The following lemma is an integral form of Jensen's inequality which describes a global property of the centroid of a cone like convex body [1].

Lemma 3. *Let $f(x_1, x_2, \dots, x_{n-1})$ be a bounded concave function defined in a bounded convex domain D of dimension $n - 1$ and let*

$$\bar{x}_i \equiv \int_D x_i f(X) dX / \int_D f(X) dX, \quad i = 1, 2, \dots, n - 1.$$

Then

$$f(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{n-1}) \geq \int_D f(X) dX \Big/ \int_D dX,$$

where the equality sign holds if, and only if, f is a constant.

To prove the inequality (1.6), one first verifies that the equality sign holds for all right circular cylinders. Secondly, for a given bounded convex body H , one constructs a convex body K by means of Steiner symmetrization with respect to the hyper-plane perpendicular to $\mathbf{u}_M(H, \lambda)$ such that (i) K is a cone like convex body as defined in [1] and (ii) the maximum relative radius of gyration of order λ is greater than or equal to that of H . Let K_θ be the intersection of K with the half-plane bounded by the line passing through C_K and perpendicular to the "base" of K . Finally, one applies Lemma 1 to transform each section K_θ of K into a trapezoid T_θ . In this way, the cone-like convex body K is transformed into a body B which is a cylinder with the same base as that of K but truncated from above by a "helical surface". The details of all these constructions can be found in [1].

Consider the relative radius of gyration, $\gamma_\lambda(B, z)$, of B about the line passing through C_B and perpendicular to its base. By applying Lemmas 1 and 2, one shows that $\gamma_\lambda(B, z) \geq \gamma_\lambda(H, C_H, \mathbf{u}_M(H, \lambda))$. Furthermore, by applying Lemmas 2 and 3 one proves that $\gamma_\lambda(B, z)$ is less than or equal to $(n-1)/(n+\lambda-1)$ and that the equality is achieved only when H is a right circular cylinder. The details of all these arguments can be carried out in the same way as shown in [1] with some minor modifications.

3. Proof of the inequality (1.7). Let F be the family of all functions A each defined continuous and non-negative on some closed interval and having $[A(x)]^{1/(n-1)}$ concave on that interval. For each $A(x)$, $a \leq x \leq b$, in F and for every real number $\lambda \geq 0$ we define

$$m_\lambda(A) \equiv \int_a^b x^\lambda A(x) dx, \quad \bar{x}(A) \equiv m_1(A)/m_0(A),$$

$$\delta(A) \equiv \max \{ \bar{x}(A) - a, b - \bar{x}(A) \}.$$

Define for $\lambda > 1$,

$$I_\lambda(A) \equiv \int_a^b \{ [x - \bar{x}(A)]^2 \}^{\lambda/2} A(x) dx,$$

$$g_\lambda(A) \equiv [I_\lambda(A)/m_0(A)]^{1/\lambda},$$

$$(3.1) \quad \mu = \text{infimum of } \gamma_\lambda(K, C_k, \mathbf{u}_m(K, \lambda)) \text{ for all convex bodies of dimension } n.$$

$$(3.1') \quad \mu' = \text{infimum of } g_\lambda(A)/\delta(A) \text{ for all } A \text{ in } F.$$

To prove the inequality (1.7) we shall apply McShane's idea [2] to reduce the problem in 2-dimensional space through the following lemma. This lemma was

proved in [2] for $n = 3$, $\lambda = 2$. The proof for the present general case can be carried out in the same way as shown in [1] with some minor modifications.

Lemma 4. *The values of μ in (3.1) and μ' in (3.1') are equal.*

In view of this lemma, it suffices, for the proof of (1.7), to show that

$$(3.2) \quad \mu' = c(n, \lambda).$$

To establish (3.2) let $A(x)$ in F be given. Since the value of $g_\lambda(A)/\delta(A)$ remains unchanged under the transformation: $x \rightarrow hx + k$, h, k constants with $h > 0$ we may assume that $A(x)$ is defined on $[0, 1]$. It is also clear that we may assume that $\bar{x}(A) \leq \frac{1}{2}$. Since the n -dimensional body K obtained by rotating the curve,

$$y = [A(x)]^{1/(n-1)}, \quad 0 \leq x \leq 1,$$

about the x -axis is convex, it follows that [1]

$$(3.3) \quad 1/(n+1) \leq \bar{x}(A),$$

where the equality sign holds only when the body K is a conoid, *i.e.*, only when $[A(x)]^{1/(n-1)}$ is a constant multiple of $(1-x)$. Our proof of (3.2) is to construct a function $A^*(x)$ in F such that

$$(3.4) \quad c(n, \lambda) = g_\lambda(A^*)/\delta(A^*) \leq g_\lambda(A)/\delta(A).$$

Clearly, the function $A^*(x)$ defined by

$$(3.5) \quad A^*(x) \equiv [y_0(1-x)]^{n-1}/(1-x_0)^{n-1}, \quad x_0 \leq x \leq 1,$$

belongs to F . Also, it is easy to verify that for $x_0, y_0 > 0$ then $g_\lambda(A^*)/\delta(A^*) = c(n, \lambda)$. It remains to establish the inequality in (3.4). To do this we shall make proper choice of the yet-to-be determined constants x_0 and y_0 . First, we calculate that

$$\begin{aligned} m_0(A^*) &= \int_{x_0}^1 A^*(x) dx = y_0^{n-1}(1-x_0)/n, \\ m_1(A^*) &= \int_{x_0}^1 x A^*(x) dx = y_0^{n-1}(1-x_0)(1+nx_0)/n(n+1). \end{aligned}$$

Next, we choose

$$(3.6) \quad x_0 = [(n+1)\bar{x}(A) - 1]/n,$$

$$(3.7) \quad y_0 = \{n^2 m_0(A)/(n+1)(1-\bar{x}(A))\}^{1/(n-1)}.$$

With these choices of x_0 and y_0 , $A^*(x)$ and $A(x)$ have the same 0-th and 1st moment, *i.e.*,

$$(3.8) \quad m_0(A^*) = m_0(A), \quad \bar{x}(A^*) = \bar{x}(A).$$

Before pursuing further we observe that if $A(x)$ is a constant multiple of $(1-x)^{n-1}$, then $A^*(x)$ and $A(x)$ are identical and in this case the inequality

(3.4) trivially holds. Accordingly, we assume that this is *not* the case and proceed to establish the *strict* inequality in (3.4). To do this we shall make use of the strict concavity of $[A(x)]^{1/(n-1)}$ and the two relations in (3.8). First, we note from (3.3) and (3.6) that

$$(3.9) \quad 0 < x_0 < \bar{x}(A) = \bar{x}(A^*), \quad \delta(A^*) = \delta(A).$$

Secondly, we observe that $A^*(x) \geq A(x)$ if and only if $[A^*(x)]^{1/(n-1)} \geq [A(x)]^{1/(n-1)}$. Thus, to determine the relative positions of the two curves

$$y = A(x), \quad 0 \leq x \leq 1; \quad y = A^*(x), \quad x_0 \leq x \leq 1;$$

it suffices to determine those of the following two curves, $y = [A(x)]^{1/(n-1)}$, $0 \leq x \leq 1$; $y = [A^*(x)]^{1/(n-1)}$, $x_0 \leq x \leq 1$.

Suppose that $y_0^{n-1} = A^*(x_0) \leq A(x_0)$. Then the strict concavity of $[A(x)]^{1/(n-1)}$ implies that $[A^*(x)]^{1/(n-1)} < [A(x)]^{1/(n-1)}$ for $x_0 < x < 1$. Consequently, $A^*(x) < A(x)$ for $x_0 < x < 1$. But this contradicts that $m_0(A^*) = m_0(A)$. Hence $y_0^{n-1} = A^*(x_0) > A(x_0)$. Thus, the line segment joining the points (x_0, y_0) and $(1, 0)$ intersects the curve $y = [A(x)]^{1/(n-1)}$ at a single point P_2 and the vertical segment joining (x_0, y_0) and $(x_0, 0)$ intersects it at a single point P_0 , (see figure 1).

We proceed to establish the following stronger result, namely,

$$(3.10) \quad y_0^{n-1} = A^*(x_0) > \max \{A(x); 0 \leq x \leq 1\}.$$

Suppose that this strict inequality does not hold. Then the horizontal line passing through the point (x_0, y_0) intersects the strictly concave curve, $y = [A(x)]^{1/(n-1)}$, at a point (x_1, y_0) which is the one nearest to (x_0, y_0) . There are only two possibilities for the value of x_1 .

FIGURE 1.

Case (i). $x_0 < x_1 \leq 1$. In this case, the strict concavity of $[A(x)]^{1/(n-1)}$ implies that

$$\begin{aligned} [A(x)]^{1/(n-1)} &> y_0(1-x)/(1-x_1) && \text{for } x_1 < x < 1, \\ [A(x)]^{1/(n-1)} &> y_0(x_1-x)/(x_1-x_0) && \text{for } x_0 < x < x_1. \end{aligned}$$

Hence

$$\begin{aligned} m_0(A) &= \int_0^1 A(x) dx \\ &> \int_{x_0}^{x_1} \left(\frac{y_0}{x_1-x_0} \right)^{n-1} (x_1-x)^{n-1} dx + \int_{x_1}^1 \left(\frac{y_0}{1-x_1} \right)^{n-1} (1-x)^{n-1} dx. \end{aligned}$$

But direct computation shows that the value of the right-hand of the above inequality is equal to $m_0(A^*)$. Thus, $m_0(A) > m_0(A^*)$, which contradicts the 1st relation in (3.8).

Case (ii). $0 \leq x_1 < x_0$. By completely similar analysis we also reach a contradiction with the fact that $m_0(A) = m_0(A^*)$. Since both the case (i) and the case (ii) can never happen, our inequality (3.10) is now established.

So far we have only made use of the fact that $A(x)$ belongs to F and that $m_0(A) = m_0(A^*)$. Now, we derive a consequence from the relation $\bar{x}(A^*) = \bar{x}(A)$. The result which we need is that

$$(3.11) \quad x_0 < \bar{x}(A^*) = \bar{x}(A) < x_2$$

where (x_2, y_2) stands for the intersection point P_2 , (see Figure 1). To establish (3.11) we put $A^*(x) = 0$ for $x \leq x_0$ and note that $A^*(x) > A(x)$ for $x_0 < x < x_2$ and that $A^*(x) \leq A(x)$ otherwise. Consequently, to prove the inequality (3.11) it suffices to show that

$$(3.12) \quad A^*(\bar{x}(A^*)) > A(\bar{x}(A)).$$

To prove this inequality we are going to apply a theorem of Minkowski [4, 5, 6] which we state as

Lemma 5. *Let K be a bounded convex body of dimension n and let C_K be its centroid. Then the distance $\delta(\mathbf{u})$ from C_K to a supporting plane of K which is perpendicular to a vector $\mathbf{u}$ satisfies the following condition,*

$$B(\mathbf{u})/(n+1) \leq \delta(\mathbf{u}) \leq nB(\mathbf{u})/(n+1),$$

where $B(\mathbf{u})$ stands for the breadth of K in the direction $\mathbf{u}$.

Consider the n -dimensional convex body K obtained by rotating the curve $y = [A(x)]^{1/(n-1)}$, $0 \leq x \leq 1$ about the x -axis, i.e. K consists of those points $(x_1, \dots, x_n)$ such that

$$x_1 \in [0, 1], \quad \sum_{i=2}^n (x_i)^2 \leq [A(x_1)]^{2/(n-1)}.$$

Since the centroid of K is $(\bar{x}(A), 0, \dots, 0)$, it follows from the above lemma that for $\mathbf{u}$ being a unit vector perpendicular to x_1 -axis we have

$$[A(\bar{x}(A))]^{1/(n-1)} \leq \delta(\mathbf{u}) \leq \frac{n}{n+1} \max \{ [A(x)]^{1/(n-1)}, 0 \leq x \leq 1 \}.$$

Hence

$$(3.13) \quad A(\bar{x}(A)) \leq \left(\frac{n}{n+1} \right)^{n-1} \max \{ A(x), 0 \leq x \leq 1 \}.$$

By combining (3.13) and (3.10), we obtain that

$$(3.14) \quad A(\bar{x}(A)) < \left(\frac{n}{n+1} \right)^{n-1} y_0^{n-1}.$$

On the other hand, direct computation basing on the expression in (3.5) gives

$$(3.15) \quad A^*(\bar{x}(A^*)) = A^* \left(\frac{1 + nx_0}{n+1} \right) = \left(\frac{n}{n+1} \right)^{n-1} y_0^{n-1}.$$

Thus, (3.12) follows directly from (3.14) and (3.15) and hence (3.11) is established.

Basing on all the results established above, we wish to show that for all numbers $\lambda > 1$.

$$(3.16) \quad \int_0^1 |x - \bar{x}(A)|^\lambda A(x) dx > \int_0^1 |x - \bar{x}(A^*)|^\lambda A^*(x) dx,$$

where we have set $A^*(x) = 0$ for $x \leq x_0$. To this end, we first show that for all numbers $\lambda > 1$

$$(3.17) \quad \int_{\bar{x}(A)}^1 |x - \bar{x}(A)| A(x) dx > \int_{\bar{x}(A^*)}^1 |x - \bar{x}(A^*)| A^*(x) dx.$$

To show this we note that if

$$\int_{\bar{x}(A)}^1 A(x) dx \geq \int_{\bar{x}(A^*)}^1 A^*(x) dx,$$

then it is clear (see Figure 1) that (3.17) must hold. Consequently if

$$(3.18) \quad \int_{\bar{x}(A)}^1 |x - \bar{x}(A)| A(x) dx < \int_{\bar{x}(A^*)}^1 |x - \bar{x}(A^*)| A^*(x) dx,$$

then it is necessary that

$$\int_{\bar{x}(A)}^1 A(x) dx < \int_{\bar{x}(A^*)}^1 A^*(x) dx.$$

Now, this inequality together with the first relation in (3.8) implies that

$$(3.19) \quad \int_0^{\bar{x}(A)} A(x) dx > \int_0^{\bar{x}(A^*)} A^*(x) dx.$$

Furthermore, as a necessary consequence of (3.19) and the 2nd relation in (3.8) we have

$$(3.20) \quad \int_0^{\bar{x}(A)} |x - \bar{x}(A)| A(x) dx > \int_0^{\bar{x}(A^*)} |x - \bar{x}(A^*)| A(x) dx.$$

Since the two inequalities (3.18) and (3.20) contradict the definitions for $\bar{x}(A)$ and $\bar{x}(A^*)$, the validity of (3.17) is proved.

It is now easy to establish the inequality (3.16). To this end, we consider the integral

$$\begin{aligned} I_1 &\equiv \int_{\bar{x}(A)}^1 |x - \bar{x}(A)|^\lambda [A(x) - A^*(x)] dx \\ &= \left\{ \int_{\bar{x}(A)}^{x_2} + \int_{x_2}^1 \right\} |x - \bar{x}(A)|^\lambda [A(x) - A^*(x)] dx. \end{aligned}$$

Since the integrand does not change sign in each of the intervals, $[\bar{x}(A), x_2]$ and $[x_2, 1]$, it follows from the mean-value theorem for integrals that

$$I_1 = \left\{ k_1^{\lambda-1} \int_{\bar{x}(A)}^{x_2} + k_2^{\lambda-1} \int_{x_2}^1 \right\} |x - \bar{x}(A)| [A(x) - A^*(x)] dx,$$

where the constants k_1 and k_2 satisfy the strict inequalities

$$(3.21) \quad 0 < k_1 < x_2 - \bar{x}(A) < k_2 < 1 - \bar{x}(A).$$

Thus, we can write

$$(3.22) \quad \begin{aligned} I_1 &= k_1^{\lambda-1} \int_{\bar{x}(A)}^1 |x - \bar{x}(A)| [A(x) - A^*(x)] dx \\ &\quad + (k_2^{\lambda-1} - k_1^{\lambda-1}) \int_{x_2}^1 |x - \bar{x}(A)| [A(x) - A^*(x)] dx \end{aligned}$$

and conclude from (3.17), (3.21) and (3.22) that

$$(3.23) \quad I_1 > 0.$$

In a completely similar way, one shows that

$$(3.24) \quad I_2 \equiv \int_0^{\bar{x}(A)} |x - \bar{x}(A)|^\lambda [A(x) - A^*(x)] dx > 0,$$

where we have set $A^*(x) = 0$ for $x \leq x_0$. Now, the inequality (3.16) follows directly from (3.23) and (3.24) and the inequality (1.7) follows from (3.16), (3.9) and (3.8). The proof of the theorem is now complete.

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Date communicated: FEBRUARY 2, 1968