Interpolation by Ritz Approximation

BERNIE L. HULME

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1. Introduction. We obtain below interpolation properties and sharpened error bounds for approximate piecewise polynomial solutions to

$$\begin{aligned} (1) \quad & u^{(2n)}(x) = r(x), \quad a \leq x \leq b, \quad n \geq 1, \\ (2) \quad & u^{(j)}(a) = 0, \quad u^{(j)}(b) = 0, \quad j = 0, \dots, n-1, \end{aligned}$$

obtained by variational methods. It is classic that the exact solution of (1) – (2), $\bar{u}(x)$, minimizes

$$(3) \quad J[u] = \frac{1}{2} \int_a^b [u^{(n)}(x)]^2 dx - (-1)^n \int_a^b r(x)u(x) dx.$$

over $K_0^{n,2}[a, b]$, the linear space of all real-valued functions $u(x)$ defined on $[a, b]$, such that $u^{(n-1)}(x)$ is absolutely continuous on $[a, b]$, $u^{(n)}(x) \in L^2[a, b]$, and $u(x)$ satisfies the homogeneous conditions (2).

Let $\pi: a = x_0 < x_1 < \dots < x_N = b$ be any partition of $[a, b]$ with $\bar{h} = \max \Delta x_i$. For fixed n and π consider a two-parameter family of spaces $F(\pi, p, 2m)$ consisting of functions $y \in C^{p-1}[a, b]$ which are polynomials of degree $2m - 1$ on each subinterval $[x_i, x_{i+1}]$, where $m \geq n$ and $n \leq p \leq 2m - 1$. Finally let $F_0(\pi, p, 2m) \subset F(\pi, p, 2m)$ denote the subspaces of functions which satisfy (2). We shall minimize J over each $F_0(\pi, p, 2m)$.

Using the notation

$$y^{(-j)}(x) \equiv \int_a^x \int_a^{t_1} \dots \int_a^{t_{j-1}} y(t_j) dt_j \dots dt_1, \quad j \geq 1,$$

we state in the following theorem our main result, which is an extension of the spline interpolation results of de Boor and Lynch [1].

Theorem 1. *The minimum of J on $F_0(\pi, p, 2m)$, where $m = n + q$, $q \geq 0$, and $n \leq p \leq 2m - 1$, is given by $\bar{y} \in F_0(\pi, p, 2m)$ which interpolates $\bar{u}^{(j)}(x_i)$ for all $x_i \in \pi$ and $j = -2q, \dots, 2n - p - 1$, as well as $\bar{u}^{(j)}(a)$ and $\bar{u}^{(j)}(b)$ for $j = -2q, \dots, n - 1$.*

2. Proof of Theorem 1 and error bounds. The proof of Theorem 1 requires

Lemma 1. Let $v \in C^{n-1}[a, b]$ satisfy $v^{(i)}(x_i) = 0$ for all $x_i \in \pi$ and $j = -2q, \dots, 2n - p - 1$, as well as $v^{(i)}(a) = v^{(i)}(b) = 0$ for $j = -2q, \dots, n - 1$, and let $w \in F(\pi, p, 2m)$. Then

$$(4) \quad \int_a^b v^{(n)}(x) w^{(n)}(x) dx = 0.$$

Proof. Integration by parts $n + 2q$ times yields

$$\begin{aligned} \int_a^b v^{(n)}(x) w^{(n)}(x) dx &= \left[\sum_{k=0}^{p-n-1} (-1)^k v^{(n-k-1)} w^{(n+k)} \right]_a^b \\ &\quad + \sum_{i=0}^{N-1} \left[\sum_{k=p-n}^{n+2q-1} (-1)^k v^{(n-k-1)} w^{(n+k)} \right]_{x_i}^{x_{i+1}} \\ &\quad + (-1)^{n+2q} \int_a^b v^{(-2q)}(x) w^{(2m)}(x) dx = 0. \end{aligned}$$

The equation holds because the sums vanish due to the hypothesis on v while the integral vanishes since w is a piecewise polynomial of degree $2m - 1$. Q. E. D.

For any $y \in F_0(\pi, p, 2m)$ consider the error function

$$e(x) = y(x) - \bar{u}(x) \in K_0^{n,2}[a, b]$$

and the excess functional

$$(5) \quad I[e] = J[y] - J[\bar{u}] = \frac{1}{2} \int_a^b [e^{(n)}(x)]^2 dx.$$

Because functions in $K_0^{n,2}[a, b]$ satisfy homogeneous boundary conditions (2), the functional in (5) is a norm for this space. Writing $y = \bar{y} + \Delta y$ and $\bar{e} = \bar{y} - \bar{u}$, we have $e = \bar{e} + \Delta y$. Setting $v = \bar{e}$ and $w = \Delta y$ in Lemma 1, we see that $\bar{e}$ is orthogonal to the increment Δy in the inner product associated with (5), i.e.

$$(6) \quad \int_a^b \bar{e}^{(n)}(x) \Delta y^{(n)}(x) dx = 0.$$

As a result $I[e] = I[\bar{e}] + I[\Delta y] \geq I[\bar{e}]$, and hence $J[y] \geq J[\bar{y}]$.

Q.E.D.

Now $\bar{y}^{(-2q)} \in C^{2q+p-1}[a, b]$ is a piecewise polynomial of degree $2m + 2q - 1$ which interpolates $\bar{u}^{(-2q+k)}(x_i)$, $k = 0, \dots, 2m - p - 1$, for all $x_i \in \pi$ as well as $\bar{u}^{(-2q+k)}(a)$ and $\bar{u}^{(-2q+k)}(b)$ for $k = 0, \dots, m + q - 1$. Thus $\bar{y}^{(-2q)}$ is the L -spline interpolate of $\bar{u}^{(-2q)}$ of Type 1 with $L = D^{m+q}$, $z_1 = \dots = z_{N-1} = 2m - p$, and $z_0 = z_N = m + q$ (cf. [3]). From [3, Thm. 10] we find that for $\bar{u} \in C^{2m}[a, b]$ the interpolation error bounds are

$$\|\bar{e}^{(-2q+k)}\|_\infty \leq M \bar{h}^{2m+2q-k-1/2}, \quad k = 0, \dots, 2m + 2q - 1.$$

Replacing k by $k + 2q$, we obtain

Corollary 1. If $\bar{u} \in C^{2m}[a, b]$ where $m = n + q$, $q \geq 0$, then the error $\bar{e} = \bar{y} - \bar{u}$

of Ritz approximation over $F_0(\pi, p, 2m)$, $n \leq p \leq 2m - 1$, satisfies

$$(7) \quad \|\bar{e}^{(k)}\|_\infty \leq M_1 \bar{h}^{2m-k-1/2}, \quad k = 0, \dots, 2m - 1.$$

Corollary 1 improves the result $\|\bar{e}\|_\infty = O(\bar{h}^{2m-n})$ given in [3, Thm. 24] for the cases $m \leq p \leq 2m - 1$ (where $F_0(\pi, p, 2m)$ is an L -spline space with $L = D^m$, $z_1 = \dots = z_{N-1} = 2m - p$, and $z_0 = z_N = n$) and the same result given in [2, Thm. 12] for $n \leq p \leq m$ (where $F_0(\pi, p, 2m)$ is a Hermite space).

In general, we can see that increasing the degree of the polynomials from $2n - 1$ to $2m - 1$, where $m = n + q$, reduces the error by a factor of $O(\bar{h}^{2q})$. However the power of $\bar{h}$ in the error is independent of the continuity parameter p . Increasing p simply results in the interpolation of fewer derivatives of $\bar{u}$.

3. Special subspaces. Sharper error estimates for lower derivatives can be given when $p = n$ because in this case $\bar{y}^{(-2q)} \in C^{m+q-1}[a, b]$ is the piecewise Hermite interpolate of $\bar{u}^{(-2q+k)}(x_i)$ for all $x_i \in \pi$ and $k = 0, \dots, m + q - 1$. From Ciarlet, Schultz and Varga [2, Thm. 9] we find that for $\bar{u} \in C^{2m}[a, b]$ the interpolation error satisfies

$$\|\bar{e}^{(-2q+k)}\|_\infty \leq \frac{\|\bar{u}^{(2m)}\|_\infty \bar{h}^{2m+2q-k}}{2^{2m+2q-2k} k! (2m + 2q - 2k)!}, \quad k = 0, \dots, m + q.$$

(Even sharper estimates can be obtained from Birkhoff and Priver [6] for cubic and quintic interpolation, i.e., $m + q = 2, 3$.) Replacing k by $k + 2q$, we obtain

Corollary 2. *The minimum of J over the Hermite space $H_0(\pi, n, 2m) = F_0(\pi, n, 2m)$, where $m = n + q$, $q \geq 0$, is given by $\bar{y} \in H_0(\pi, n, 2m)$ which interpolates $\bar{u}^{(j)}(x_i)$ for all $x_i \in \pi$ and $j = -2q, \dots, n - 1$. If $\bar{u} \in C^{2m}[a, b]$, then the error $\bar{e} = \bar{y} - \bar{u}$ satisfies*

$$(8) \quad \|\bar{e}^{(k)}\|_\infty \leq \frac{\|\bar{u}^{(2m)}\|_\infty \bar{h}^{2m-k}}{2^{2n-2k} (2q + k)! (2n - 2k)!}, \quad k = 0, \dots, n.$$

Corollary 2 also improves the result $\|\bar{e}\|_\infty = O(\bar{h}^{2m-n})$ given in [2, Thm. 12].

Two other cases of special interest arise for $p = m$ and $p = 2m - 1$. These cases are treated in Corollaries 3 and 4.

Corollary 3. *The minimum of J over the smooth Hermite space $H_0^m(\pi) = F_0(\pi, m, 2m)$, where $m = n + q$, $q \geq 0$, is given by $\bar{y} \in H_0^m(\pi)$ which interpolates $\bar{u}^{(j)}(x_i)$ for all $x_i \in \pi$ and $j = -2q, \dots, n - q - 1$, as well as $\bar{u}^{(j)}(a)$ and $\bar{u}^{(j)}(b)$ for $j = -2q, \dots, n - 1$. The error is governed by Corollary 1.*

Corollary 4. *The minimum of J over the natural spline space $S_0^m(\pi) = F_0(\pi, 2m - 1, 2m)$, where $m = n + q$, $q \geq 0$, is given by $\bar{y} \in S_0^m(\pi)$ which interpolates $\bar{u}^{(-2q)}(x_i)$ for all $x_i \in \pi$ as well as $\bar{u}^{(j)}(a)$ and $\bar{u}^{(j)}(b)$ for $j = -2q, \dots, n - 1$. The error is governed by Corollary 1.*

Notice that Corollaries 2 and 3 coincide when $q = 0$ and $m = n$. The result of these corollaries for the case $m = n = 1$ was first proved by Rose [4], and an alternate proof is given in [2, p. 427].

Let us consider for example the beam bending problem $u^{(iv)}(x) = r(x)$ with $r \in C^2[a, b]$. The results of Ritz approximation over several different subspaces are summarized in Table 1. The table gives the functions $\bar{u}^{(i)}$ which are matched exactly at the nodes of any mesh and the consequent order of error $\bar{e} = \bar{y} - \bar{u}$ between the nodes.

We see that the higher degree polynomial approximations are globally more accurate. Also for polynomials of fixed degree a unit increase of the continuity parameter p causes one less derivative of $\bar{u}$ to be interpolated.

All the above results are still valid if one allows the boundary conditions (2) to be inhomogeneous, i.e., $u^{(i)}(a) = \alpha_i$ and $u^{(i)}(b) = \beta_i$. In this situation instead of $K_0^{n,2}[a, b]$ and $F_0(\pi, p, 2m)$, one defines the sets of functions $K_1^{n,2}[a, b]$ and $F_1(\pi, p, 2m)$ which satisfy the *inhomogeneous conditions* (2). The fact that these sets are not linear spaces is of no consequence since all functions $e = y - \bar{u}$ and $\Delta y = y - \bar{y}$ still belong to the linear space $K_0^{n,2}[a, b]$ of functions which satisfy *homogeneous conditions* (2).

Also the above results are still valid if some "natural" boundary conditions, e.g., $u^{(2n-1-i)}(b) = \gamma_i$, are substituted in (2) for corresponding "essential" conditions, i.e., $u^{(i)}(b) = \beta_i$, as long as there are enough "essential" conditions to determine a unique solution $\bar{u}$. It is only necessary to modify J by adding terms of the form $(-1)^{n-i}\gamma_i u^{(i)}(b)$ and to define $K_1^{n,2}[a, b]$ and $F_1(\pi, p, 2m)$ as those functions which satisfy the "essential" boundary conditions. In this case all $e = y - \bar{u}$ and $\Delta y = y - \bar{y}$ belong to the linear space $K_0^{n,2}[a, b]$ of functions which satisfy homogeneous "essential" conditions and the functional in (5) is a norm for this space.

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TABLE 1

Piecewise Polynomial Spaces		Indices				Cor. No.	Functions $\bar{u}^{(i)}$ Interpolated	Order of Error $\bar{e}$
Name	Continuity	m	n	p	q			
Cubic Hermite	C^1	2	2	2	0	2	$\bar{u}, \bar{u}'$	$\bar{h}^4$
Quintic Hermite	C^1	3	2	2	1	2	$\bar{u}^{(-2)}, \bar{u}^{(-1)}, \bar{u}, \bar{u}'$	$\bar{h}^6$
Quintic Hermite	C^2	3	2	3	1	3	$\bar{u}^{(-2)}, \bar{u}^{(-1)}, \bar{u}$	$\bar{h}^{11/2}$
Cubic Spline	C^2	2	2	3	0	4	$\bar{u}$	$\bar{h}^{4*}$
Quintic Spline	C^4	3	2	5	1	4	$\bar{u}^{(-2)}$	$\bar{h}^{11/2}$

* Note: Although Corollary 4 indicates $O(\bar{h}^{7/2})$ accuracy in the cubic spline case, we know from Birkhoff and de Boor [5] that cubic spline interpolation error is actually $O(\bar{h}^4)$.

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Harvard University

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