Classification of Space-Time Curvature Tensor

V Ia. Complex Basic Characteristics

*of the Types I, II, IV**

VÁCLAV HLAVATÝ

I

1. Introduction and prerequisites	237
2. Auxiliary results I	241
3. Auxiliary results II	244
4. U-permutations	250

II

5. Characteristic $[2 \ 2 \ \bar{2}]$	254
6. Principal matrix $((R_{AB}))$	257
7. Characteristic $[2 \ \bar{2} \ 1 \ 1]$	259
8. Characteristic $[2 \ \bar{2} \ 1 \ \bar{1}]$	262
9. Characteristic $[2 \ 2 \ 1 \ \bar{1}]$	266
10. Digression	268
11. Principal matrices $((R_{AB}))$	269
12. Characteristic $[3 \ \bar{3}]$	271
13. Auxiliary lemmas	274
14. Principal matrix $((R_{AB}))$	276
15. Other canonical frames	277
16.*g-equivalence of the results	278

I.

1. Introduction and prerequisites. (1) This is another paper in the series of papers concerning the classification of the space-time curvature tensor ([1], [2], [3], [4], [5]). It is based on the results of the papers [4], [5] which will be denoted by I and II. In particular, the results and methods displayed in II will serve here as a guide. However, in the present paper we are dealing *exclusively* with *basic complex* characteristics, while in II we displayed only *real basic characteristics*. This necessitates a new step in our investigation which was

* This paper was prepared with the support of the Aerospace Research Laboratory of the Office of Aerospace Research USAF.

omitted in II as obvious. Indeed we did not have any difficulty in II to see that a *real* canonical frame has the least possible number of non zero components. This is not *a priori* evident in the case of a *complex* canonical frame and therefore it must be proved in every particular case. To facilitate the formulation of statements, we shall refer to non zero elements of a matrix under consideration as *Substantials*.

The quotation "Theorem (I(5.2))" indicates the second Theorem of the fifth section in I. All results of I and II needed here will be quoted in a similar way. For the benefit of the reader we shall repeat here some results from I and II as prerequisites.

(2) Upper case (lower case) Latin indices, with the exception of X, Y, Z (of x, y, z), have the range i, ii, $\dots$, vi (1, 2, $\dots$, 6). The range of the indices X, Y, Z, x, y, z will always be explicitly stated. *These indices do not follow the summation convention.* Upper case Greek indices run from I to VI. (The range of upper case Latin indices is different from the one used in I and II. This change was necessitated by technical reasons. No confusion with results from I or II will arise.)

(3) We shall very often use the word "vectors", meaning K -vectors (*i.e.* bi-vectors). Similarly we refer to K -tensors as tensors. Any ambiguity is excluded because we do not use here tensors and vectors of the space-time under consideration, with the exception of the "curvature tensor" which appears here as a quadratic symmetric K -tensor.

(4) The characteristic of a space-time curvature tensor is of the form

$$(1.1)a \quad [q \ r \ s \ 1 \ \dots \ 1] \quad (1 \text{ repeated } p\text{-times}),$$

where the numerals q, r, s —if different from zero—are greater than one and

$$(1.1)b \quad q + r + s + p = 6.$$

There are eleven types I, $\dots$, XI of these characteristics. They are displayed in the following table (1.2). The column headed by N indicates the number of non zero numerals q, r, s , the last column informs us about the number of possible basic characteristics belonging to the type under consideration.

	N	q	r	s	p	
I	3	2	2	2	0	2
II	2	2	2	0	2	4
III	2	3	2	0	1	1
IV	2	3	3	0	0	2
V	2	4	2	0	0	1
VI	1	2	0	0	4	3
VII	1	3	0	0	3	2
VIII	1	4	0	0	2	2
IX	1	5	0	0	1	1
X	1	6	0	0	0	1
XI	0	0	0	0	6	4

It is clear that only the types

$$(1.3) \quad \text{I II IV VI VII VIII XI}$$

have complex basic characteristics. These characteristics are our "point de départ". In the first part (this present paper) we shall consider the first three characteristics (1.3), the second part (to be published later) deals with the remaining characteristics (1.3).

(5) As in II, the curvature tensor $R_{\alpha\lambda}$ will be described by

$$(1.4)a \quad R_{\alpha\lambda} = R_{AB} P_{\alpha}{}^{A|} P_{\lambda}{}^{B|}.$$

Here $P_{\alpha}{}^{A|}$ are the vectors of a fundamental K -frame (Definition (I(3.1))). R_{AB} are orthogonal components of $R_{\alpha\lambda}$ with respect to the fundamental K -frame. They constitute the matrix

$$(1.4)b \quad ((R_{AB})) = \left(\left(\frac{M}{N^t} \middle| \frac{N}{S} \right) \right)$$

where M, N, S are 3×3 matrices (to be found) and N^t is the transpose of N . It is known (cf. Theorems (II(2.1), (2.2))) that the traces of M, N, N^t, S satisfy the relations

$$(1.4) \quad \begin{array}{ll} \text{c} & \text{Trace } N = \text{Trace } N^t = \frac{1}{2}(\xi_1 + \cdots + \xi_6) \\ \text{d} & \text{Trace } M = \text{Trace } S \end{array}$$

($\xi_1, \dots, \xi_6$ are eigenvalues of $R_{\alpha\lambda}$ with respect to $g_{\alpha\lambda}$).

Our classification of the curvature tensor holds in a generic point P of the space-time. In P the metric K -tensor has the orthogonal components

$$g_{AB} = g_{\alpha\lambda} P_{A|}{}^{\alpha} P_{B|}{}^{\lambda}$$

satisfying the condition

$$(1.5) \quad ((g_{AB})) = ((g^{AB})) = \left[\begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix} \right].$$

This accounts for the fact that the curvature tensor of g_{AB} is not uniquely defined by (1.4)a. Indeed R_{AB} are orthogonal components of all metrically congruent curvature tensors with respect to all different and metrically congruent fundamental K -frames (cf. Theorem (I(7.2))), equations (2.19c) and Remark (16.2).)

To complete necessary information we mention that the orthogonal components of the quadratic K -tensor (proportional to Levi-Civita four indices tensor) are given by

$$(1.6) \quad ((e_{AB})) = ((\check{\epsilon}^{AB})) = \text{Diag } (1 \ 1 \ 1 \ -1 \ -1 \ -1)$$

(cf. (I(3.1)b)).

(6) The eigenvalues of $R_{\Omega\Lambda}$ with respect to $g_{\Omega\Lambda}$ will be denoted sometimes by $\xi_1, \dots, \xi_6$, more often by

$$(1.7)a \quad \alpha, \beta, \gamma, \lambda, \mu, \nu.$$

If two eigenvalues coincide, say $\gamma = \lambda$, we write instead of (1.7)a

$$(1.7)b \quad \alpha, \beta, \gamma, \gamma, \mu, \nu.$$

Similarly for other multiple eigenvalues. If two eigenvalues, say γ, λ , are complex conjugate, $\lambda = \bar{\gamma}$, we write

$$(1.7)c \quad \alpha, \beta, \gamma, \bar{\gamma}, \mu, \nu.$$

Similarly for other conjugate eigenvalues.

(7) The “canonical deltas” δ_A^a, δ_a^A (Theorem (I(6.4)) and Definition (I(6.1))) will be very often denoted simply by δ^a, δ_a , displaying only the ordinal numbers of the vectors δ_A^a, δ_a^A . This gives rise to another abbreviation, $R_{ab}\delta^{ab}$ (see (1.10)), which stands for $R_{ab}\delta_A^a\delta_B^b$. The product of ordinary deltas δ_b^a will be always written with one kernel letter only, for instance

$$(1.8) \quad \delta_{bd}^{ac} \stackrel{\text{def}}{=} \delta_b^a \delta_d^c.$$

(8) Let $\mathbf{V}$ and $\mathbf{W}$ be two vectors. The operation “.” and “ $\times$ ” are defined as follows

$$(1.9) \quad \begin{aligned} a) \quad \mathbf{V} \cdot \mathbf{W} &\stackrel{\text{def}}{=} V^A W^B g_{AB} = V_A W_B \check{g}^{AB} \\ b) \quad \mathbf{V} \times \mathbf{W} &\stackrel{\text{def}}{=} V^A W^B e_{AB}. \end{aligned}$$

While the operation “.” is $\mathcal{G}$ -invariant, the operation “ $\times$ ” is not.

(9) Let $((R_a^b))$ be the canonical matrix of the curvature tensor so that R_a^b are its components with respect to the canonical frame δ^a, δ_b . If g_{ab} are the components of g_{AB} with respect to the same frame, then

$$(1.10) \quad R_{ac} = R_a^b g_{bc}.$$

It has been proved (Theorem (II(2.2))) that (1.4)d is equivalent to

$$(1.11) \quad R_{ab}\check{\epsilon}^{ab} = 0$$

where $\check{\epsilon}^{ab}$ are components of $\check{\epsilon}^{AB}$ with respect to δ^c . In Section (II.2) we introduced and justified the Agreement (2.2) concerning (1.11) and the (Ch, δ)-equivalence subclass. This Agreement can be formulated also in the following way.

Agreement (1.1). The condition (1.11) is identically satisfied for all admissible eigenvalues and all admissible canonical matrices $((g_{ab}))$.

The word “admissible” means “compatible with the characteristic” which gives rise to the structure of the matrix $((R_a^b))$.

Remark (1.1) Cfr. Remark (II(2.6)) and Remark (II(3.3)).

2. Auxiliary results I. In this section as well as in the next one we shall derive some results needed later.

Lemma (2.1). Let two complex vectors $\mathbf{V}$ and $\mathbf{W}$ satisfy the relation

$$(2.1) \quad \mathbf{V} \cdot \mathbf{W} = a + bi \neq 0.$$

Put

$$(2.2)a \quad \varphi \stackrel{\text{def}}{=} [(b - ai)\epsilon]^{1/2}, \quad \epsilon = \pm 1$$

$$(2.2)b \quad {}^*\mathbf{V} = \mathbf{V}/\varphi, \quad {}^*\mathbf{W} = \mathbf{W}/\varphi.$$

Then

$${}^*\mathbf{V} \cdot {}^*\mathbf{W} = \epsilon i.$$

We leave the easy proof to the reader.

Remark (2.1). This Lemma will be always used in “the second step” of the proof of every main Theorem *without being mentioned explicitly*.

Lemma (2.2). Let $\mathbf{V}, \bar{\mathbf{V}}$ be complex conjugate vectors satisfying the relations

$$(2.3) \quad \text{a) } \mathbf{V} \cdot \bar{\mathbf{V}} = 0 \quad \text{b) } \mathbf{V} \cdot \mathbf{V} \neq 0, \quad \text{c.c.}$$

(c.c. stands for complex conjugate relation). Then $i\mathbf{V}$ and $\bar{\mathbf{V}}$ span a hyperbolic K -plane. There is such a factor φ that

$$(2.4)a \quad {}^*\mathbf{V} = \mathbf{V}/\varphi \quad \text{c.c.}$$

satisfies the relation

$$(2.4)b \quad {}^*\mathbf{V} \cdot {}^*\mathbf{V} = \epsilon i \quad \epsilon = \pm 1, \quad \text{c.c.}$$

Proof. There are two real vectors $\mathbf{R}$ and $\mathbf{I}$ such that

$$(2.5) \quad \mathbf{V} = \mathbf{R} + i\mathbf{I} \quad \text{c.c.}$$

Substituting from (2.5) into (2.3), we get

$$(2.6) \quad \begin{aligned} \text{a) } & \mathbf{R} \cdot \mathbf{R} + \mathbf{I} \cdot \mathbf{I} = 0 \\ \text{b) } & \mathbf{R} \cdot \mathbf{R} - \mathbf{I} \cdot \mathbf{I} + 2i\mathbf{R} \cdot \mathbf{I} \neq 0, \quad \text{c.c.} \end{aligned}$$

The equation (2.6)a shows that the K -plane spanned by the K -vectors $\mathbf{R}, \mathbf{I}$ (or which is the same the K -plane spanned by $i\mathbf{V}$ and $\bar{\mathbf{V}}$) cannot be elliptic. Assume

that it is parabolic *i.e.* that it has either only one null direction or that every direction in this plane is a null direction. In the first case, the condition (2.3)a tells us that this null direction must be either $\bar{\mathbf{V}}$ or $\mathbf{V}$ which contradicts (2.3)b. In the second case we must have

$$\mathbf{R} \cdot \mathbf{R} = \mathbf{I} \cdot \mathbf{I} = \mathbf{R} \cdot \mathbf{I} = 0.$$

These conditions are incompatible with (2.6)b. Hence the plane spanned by $i\mathbf{V}$ and $\bar{\mathbf{V}}$ is neither elliptic nor parabolic and therefore it must be hyperbolic. The factor φ mentioned in the second part of the Theorem is obviously

$$\varphi = [-\epsilon i \mathbf{V} \cdot \mathbf{V}]^{1/2}.$$

Lemma (2.3). *Let $\mathbf{V}$ and $\bar{\mathbf{V}}$ be complex conjugate vectors given by (2.5) and satisfying the conditions*

$$(2.7) \quad \text{a) } \mathbf{V} \cdot \mathbf{V} = 0, \quad \text{c.c.} \quad \text{b) } \mathbf{V} \cdot \bar{\mathbf{V}} = 0.$$

There is always such a fundamental K -frame with respect to which the covariant components of $\mathbf{V}$ and $\bar{\mathbf{V}}$ are given by

$$(2.8) \quad \begin{aligned} \text{a) } V_A &= \delta_A^X + i \delta_A^Y, & \text{c.c.} & \quad \text{b) } V_A = i^{1/2} \delta_A^X + (-i)^{1/2} \delta_A^Y, & \text{c.c.} \\ \text{c) } V_A &= i^{1/2} \delta_A^X + \frac{1}{2^{1/2}} \delta_A^Y, & \text{c.c.} & \quad \text{d) } V_A = i^{1/2} \delta_A^X + \frac{i}{2^{1/2}} \delta_A^Y, & \text{c.c.} \end{aligned}$$

for

$$(2.9) \quad \begin{aligned} \text{a) } (\mathbf{I} \times \mathbf{I})(\mathbf{R} \times \mathbf{R}) &\neq 0 & \text{b) } \mathbf{I} \times \mathbf{I} &= \mathbf{R} \times \mathbf{R} = 0 \\ \text{c) } \mathbf{I} \times \mathbf{I} &\neq 0 = \mathbf{R} \times \mathbf{R} & \text{d) } \mathbf{I} \times \mathbf{I} &= 0 \neq \mathbf{R} \times \mathbf{R}. \end{aligned}$$

The range of the indices is

$$(2.8) \quad \text{e) } X, Y = \text{i, ii, iii} \quad \text{or} \quad X, Y = \text{iv, v, vi}$$

or

$$(2.8) \quad \text{f) } X = \text{i, ii, iii}, \quad Y = \text{iv, v, vi}, \quad Y - X \neq \text{iii}$$

for (2.8)a. For (2.8)bcd the range is (2.8)f.

Proof. Substituting into (2.7) from (2.5) we obtain

$$(2.10) \quad \mathbf{I} \cdot \mathbf{I} = \mathbf{R} \cdot \mathbf{R} = \mathbf{R} \cdot \mathbf{I} = 0.$$

Assume first that the given vectors $\mathbf{R}$, $\mathbf{I}$ satisfy (2.9)a. If $\mathbf{P}^{A1}$ is an arbitrary fundamental K -frame, we have for (2.8)ef

$$(2.11)\text{a} \quad \mathbf{P}^{X1} \cdot \mathbf{P}^{X1} = \mathbf{P}^{Y1} \cdot \mathbf{P}^{Y1} = \mathbf{P}^{X1} \cdot \mathbf{P}^{Y1} = 0.$$

(Cf. Theorem (I(3.1)) and equations (1.5) (1.6)). Moreover, using the information just mentioned in parenthesis, we find easily that there is such a fundamental K -frame which satisfies for (2.8)ef besides (2.11)a also

$$(2.11)b \quad 0 \neq \mathbf{P}^{x|} \times \mathbf{P}^{x|} = \mathbf{R} \times \mathbf{R}, \quad 0 \neq \mathbf{P}^{y|} \times \mathbf{P}^{y|} = \mathbf{I} \times \mathbf{I}.$$

An easy inspection shows that by virtue of (2.10) and (2.9)a the equations (2.11) are satisfied by

$$(2.12) \quad \mathbf{P}^{x|} \stackrel{\text{def}}{=} \mathbf{R}, \quad \mathbf{P}^{y|} \stackrel{\text{def}}{=} \mathbf{I}.$$

These equations together with

$$(2.13) \quad V_A = (\mathbf{R} + i\mathbf{I}) \cdot \mathbf{P}_{A|}$$

lead at once to (2.8)a. Assume now that the given vectors $\mathbf{R}$, $\mathbf{I}$ satisfy (2.9)b. We can derive easily from Theorem (I(3.1)) and from the equations (1.5) and (1.6) that there are fundamental K -frames $\mathbf{P}^{A|}$ which satisfy for (2.8)f the condition (2.11)a and

$$(2.14) \quad \begin{aligned} &a) \quad \mathbf{P}^{x|} \times \mathbf{P}^{x|} + \mathbf{P}^{y|} \times \mathbf{P}^{y|} = 0 \\ &b) \quad \mathbf{P}^{x|} \times \mathbf{P}^{y|} = 0. \end{aligned}$$

Hence, taking into consideration (2.9)b and (2.10), we see that (2.14) are satisfied by

$$(2.15)a \quad 2^{1/2}\mathbf{P}^x \stackrel{\text{def}}{=} \mathbf{R} + \mathbf{I}, \quad 2^{1/2}\mathbf{P}^y \stackrel{\text{def}}{=} \mathbf{R} - \mathbf{I}.$$

These equations yield

$$(2.15)b \quad \mathbf{R} = 2^{-1/2}(\mathbf{P}^{x|} + \mathbf{P}^{y|}), \quad \mathbf{I} = 2^{-1/2}(\mathbf{P}^{x|} - \mathbf{P}^{y|}).$$

The equations (2.13) and (2.15)b lead to (2.8)b. The remaining equations (2.8)cd may be proved in a similar way.

To evaluate the consequences of Lemma (2.3) we need another Lemma which deals with congruent vectors. Denote by V_A the components of the vector (2.5) with respect to a fundamental K -frame $\mathbf{P}^{A|}$

$$(2.16) \quad a) \quad \mathbf{V} = V_A \mathbf{P}^{A|}.$$

If $\mathbf{P}^{A|}$ is another fundamental K -frame, then the vector (2.16)a and

$$(2.16) \quad b) \quad {}_*\mathbf{V} = V_A {}_*\mathbf{P}^{A|}$$

are *congruent* in the classical sense. We shall use this concept in the next Lemma.

Lemma (2.4). *Let $\mathbf{V}$ be a vector given by (2.5). There is always such a vector*

$$(2.17) \quad {}_*\mathbf{V} = {}_*\mathbf{R} + i {}_*\mathbf{I}$$

congruent (in the classical sense) to the vector $\mathbf{V}$ for which

$$(2.18) \quad ({}_*\mathbf{R} \times {}_*\mathbf{R})({}_*\mathbf{I} \times {}_*\mathbf{I}) \neq 0.$$

Proof. The relationship between the congruent vectors

$$(2.19) \quad a) \quad \mathbf{V} = V_A \mathbf{P}^{A|} \quad b) \quad {}_*\mathbf{V} = {}_*V_A {}_*\mathbf{P}^{A|}$$

is established by means of a transformation $\check{U}_A{}^B$ of the group ${}^*\mathcal{G}$ (Theorem (I(4.2)))

$$(2.20) \quad {}^*V_A = \check{U}_A{}^B V_B.$$

Indeed $\mathbf{P}^{A|}$ and

$$\mathbf{P}^{B|} = \check{U}_A{}^B \mathbf{P}^{A|}$$

are two fundamental K -frames and therefore congruent (Lemma (I(4.1))). Hence

$$(2.19)c \quad {}^*\mathbf{V} = {}^*V_B \mathbf{P}^{B|} = \check{U}_B{}^A V_A \mathbf{P}^{B|} = V_A \mathbf{P}^{A|}.$$

The equations (2.19)ac show that $\mathbf{V}$ and ${}^*\mathbf{V}$ are indeed congruent. On the other hand, the operation “ $\times$ ” is not ${}^*\mathcal{G}$ -invariant. Hence it is always possible to use a transformation ${}^*\mathbf{V} \rightarrow \mathbf{V}$ of ${}^*\mathcal{G}$ which maps $\mathbf{V}$ on a congruent vector ${}^*\mathbf{V}$ satisfying the condition (2.18).

Remark (2.2). Comparing the relation (2.8)a–(2.8)d, we see that (2.8)a is the simplest one. It corresponds to the case (2.9)a. According to Lemma (2.4), it is always possible to interpret V_A as components of two congruent vectors (2.5) and (2.17) such that (2.18) holds. It is obvious that we shall look always for the simplest cases without restricting the generality. In view of this consideration one understands easily the existence of the following Agreement.

Agreement (2.1). *Unless stated explicitly otherwise, we shall tacitly assume that the constituent vectors $\mathbf{R}$, $\mathbf{I}$ of $\mathbf{V}$ which satisfies (2.7)ab are subject to the requirement (2.9)a.*

Remark (2.3). It is obvious that this Agreement does not restrict the generality of classification. Indeed performing it we deal with components like V_A and each set of these components represents, if necessary, all vectors (2.19)c, congruent with (2.19)a. Among them there is at least one satisfying the requirement (2.9)a.

Remark (2.4). It is evident that a vector given by (2.8) satisfies (2.7). The inverse is not *a priori* evident. Therefore it was necessary to prove that (2.8) is not a consequence of a special position of the vector under consideration in the K -space and can be achieved by a proper choice of a fundamental K -frame. Similar considerations apply also to all Lemmas of the next Section. The case of the real basic characteristic—dealt with in II—did not require this kind of special precaution. Indeed for *real* vectors the Lemmas similar to Lemma (2.3) are almost trivial.

3. Auxiliary results II. In this Section we shall display other Lemmas needed later on.

Lemma (3.1). *Let $\mathbf{V}$, $\bar{\mathbf{V}}$ be complex conjugate vectors subject to conditions*

$$\begin{aligned}
 (3.1) \quad & \text{a) } \mathbf{V} \cdot \mathbf{V} = \epsilon' i \quad \text{c.c.} \quad (\epsilon' = \pm 1) \\
 & \text{b) } \mathbf{V} \cdot \bar{\mathbf{V}} = 0 \\
 & \text{c) } \mathbf{V} \times \mathbf{V} = A + iB \neq 0 \quad \text{c.c.}
 \end{aligned}$$

where A, B are real constants. Then there is such a fundamental K -frame, with respect to which the vectors $\mathbf{V}, \bar{\mathbf{V}}$ have the components

$$\begin{aligned}
 (3.2) \quad & \text{a) } 2^{1/2} V_A = \delta_A^X + i \delta_A^{X+iii} \epsilon' \quad \text{c.c.} \\
 & \text{b) } X = i, ii, iii.
 \end{aligned}$$

Proof. The vectors $\mathbf{V}, \bar{\mathbf{V}}$ may be always assumed to be of the form (2.5). Then (3.1)ab are equivalent to

$$(3.3) \quad \text{a) } \mathbf{R} \cdot \mathbf{R} = 0 \quad \text{b) } \mathbf{I} \cdot \mathbf{I} = 0 \quad \text{c) } \mathbf{R} \cdot \mathbf{I} = \epsilon'/2$$

while (3.1)c is equivalent to

$$(3.3) \quad \text{d) } \mathbf{R} \times \mathbf{R} - \mathbf{I} \times \mathbf{I} = A \quad \text{e) } \mathbf{R} \times \mathbf{I} = B/2.$$

We know (Theorem (I(3.1))) that the elements of any fundamental K -frame $\mathbf{P}^{A\mathbf{I}}$ satisfy the conditions

$$(3.4) \quad \mathbf{P}^{X\mathbf{I}} \cdot \mathbf{P}^{X\mathbf{I}} = \mathbf{P}^{X+iii\mathbf{I}} \cdot \mathbf{P}^{X+iii\mathbf{I}} = 0, \quad \mathbf{P}^{X\mathbf{I}} \cdot \mathbf{P}^{X+iii\mathbf{I}} = 1, \quad X = i, ii, iii.$$

On the other hand, we derive easily from (1.5), (1.6) and Theorem (I(3.1)) that there are fundamental K -frames the elements of which satisfy the requirements

$$\begin{aligned}
 (3.5)\text{a) } \quad & \mathbf{P}^{X\mathbf{I}} \times \mathbf{P}^{X\mathbf{I}} - \mathbf{P}^{X+iii\mathbf{I}} \times \mathbf{P}^{X+iii\mathbf{I}} = 2A \\
 & \mathbf{P}^{X\mathbf{I}} \times \mathbf{P}^{X+iii\mathbf{I}} = B\epsilon'.
 \end{aligned}$$

Comparing (3.4), (3.5)a and (3.3)a-e, we see that we may put

$$(3.5)\text{b) } \mathbf{P}^{X\mathbf{I}} \stackrel{\text{def}}{=} 2^{1/2} \mathbf{R} \quad \mathbf{P}^{X+iii\mathbf{I}} = \epsilon' 2^{1/2} \mathbf{I}.$$

These equations, together with (2.13), lead to (3.2).

In the next Lemma we shall introduce the concept of *Substantials*, i.e. of non-zero elements of a matrix or non-zero components of a vector. (Cf. Section I, Paragraph 1)

Lemma (3.2)a. *If the complex conjugate vectors $\mathbf{V}$ and $\bar{\mathbf{V}}$ satisfy (3.1)ab and*

$$(3.1)\text{d) } \mathbf{V} \times \mathbf{V} = 0 \quad \text{c.c.}$$

then it is impossible to find a fundamental K -frame with respect to which $\mathbf{V}$ has only two Substantials and which satisfies (1.5) and (1.6).

Proof. Assume that $\mathbf{V}$ ($\bar{\mathbf{V}}$) has only two Substantials. Then we see from (3.1)a that it must be of the structure described by (3.2), i.e.

$$(3.6) \quad V_A = (a + bi) \delta_A^X + (c + di) \delta_A^{X+iii} \quad \text{c.c.} \quad (a, b, c, d \text{ real}).$$

Then according to (3.1)d and (1.6)

$$\mathbf{V} \times \mathbf{V} = (a + bi)^2 - (c + di)^2 = 0$$

i.e.

$$(3.7) \quad \text{a) } ab - dc = 0 \quad \text{b) } a^2 + d^2 = b^2 + c^2$$

while-if (1.5) holds-(3.1)b is equivalent to

$$(3.7) \quad \text{c) } ac + db = 0.$$

The equations (3.7)ac may be looked upon as two homogeneous equations in two unknowns, a, d , with the determinant $\Delta = b^2 + c^2$. Assume first

$$\Delta = 0, \quad \text{i.e. } b = c = 0$$

Then we obtain from (3.7)b also $a = d = 0$. Assume now

$$\Delta \neq 0, \quad \text{i.e. } a = d = 0$$

Then we obtain from (3.7)b also $b = c = 0$. Hence our assumption concerning the existence of two Substantials only has been incorrect.

The previous Lemma raises the question: what is the minimum number of Substantials of vectors $\mathbf{V}, \bar{\mathbf{V}}$ satisfying (3.1)abd? In order to answer this question we need an auxiliary result stated in the following Lemma.

Lemma (3.2)b. *If the vectors $\mathbf{V}, \bar{\mathbf{V}}$ given by (2.5) satisfy besides (3.1)abd also the condition*

$$(3.1)e \quad \mathbf{R} \times \mathbf{R} \neq 0,$$

then there is a real vector $\mathbf{M}$ satisfying the equations

$$\mathbf{M} \cdot \mathbf{V} = 0, \quad \text{c.c.}; \quad \mathbf{M} \times \mathbf{V} = 2\epsilon \mathbf{R} \times \mathbf{R}, \quad \text{c.c.}$$

$$\mathbf{M} \cdot \mathbf{M} = 0; \quad \mathbf{M} \times \mathbf{M} = 2\mathbf{R} \times \mathbf{R} \quad (\epsilon = \pm 1).$$

Proof. Let the range of three different indices X, Y, Z be i, ii, iii. While the conditions (3.1)ab are equivalent to (3.3)abc, the conditions (3.1)de are equivalent to

$$(3.1)f \quad \mathbf{R} \times \mathbf{R} = \mathbf{I} \times \mathbf{I} \neq 0, \quad \mathbf{R} \times \mathbf{I} = 0.$$

Starting with the given vector $\mathbf{R}$ we see from (3.3)a and (3.1)e that we may put

$$\mathbf{P}^{YI} = \alpha \mathbf{R} \quad (Y = \text{i, or ii, or iii})$$

where $\mathbf{P}^{YI}$ is an element of a fundamental K -frame which satisfies besides (1.5) also the condition (1.6) (*cf.* Agreement (3.1)) and $\alpha \neq 0$ is a factor to be found. The remaining elements $\mathbf{P}^{AI}$ may be so chosen that the given vector $\mathbf{I}$ is perpendicular to $\mathbf{P}^{XI}$

$$\mathbf{I} = \omega_A \mathbf{P}^{AI}, \quad \omega_{X+iii} = 0, \quad \forall \omega_X \neq 0, \quad (X \neq Y).$$

The unknown vector $\mathbf{M}$ will be expressed as follows

$$\mathbf{M} = \mu_A \mathbf{P}^{A|}$$

The conditions imposed on $\mathbf{M}$ yield first

$$(3.8) \quad a) \quad \mathbf{M} \cdot \mathbf{R} = \mu_{Y+iii}/\alpha = 0 \quad \mathbf{M} \times \mathbf{R} = \mu_{Y/\alpha} = 0$$

so that the remaining conditions reduce to

$$(3.8)b) \quad \begin{aligned} \mathbf{M} \cdot \mathbf{I} &= \mu_{X+ii}\omega_X + \mu_Z\omega_{Z+ii} + \mu_{Z+ii}\omega_Z = 0 \\ \frac{1}{2}\mathbf{M} \cdot \mathbf{M} &= \mu_X\mu_{X+ii} + \mu_Z\mu_{Z+ii} = 0 \\ \mathbf{M} \times \mathbf{I} &= \mu_X\omega_X + \mu_Z\omega_Z - \mu_{Z+ii}\omega_{Z+ii} = 2\epsilon/\alpha^2 \\ \mathbf{M} \times \mathbf{M} &= \mu_X^2 + \mu_Z^2 - \mu_{X+ii}^2 - \mu_{Z+ii}^2 = 2/\alpha^2. \end{aligned}$$

These equations are satisfied by

$$(3.8)c) \quad \begin{aligned} \mu_X &= \epsilon\omega_X, \quad \mu_T = 0 \quad \text{for } T \neq X \\ \alpha^2 &= 2/\omega_X^2. \end{aligned}$$

Hence the vector $\mathbf{M}$ satisfying the conditions of Lemma (3.2)b is a real null vector.

Remark (3.1). This result is compatible with the conditions (3.3)bc and (3.1)f not used in the Lemma just proved.

Remark (3.2). Lemma (3.2)b, but not necessarily the last of (3.8)c, holds also for a fundamental K -frame $\mathbf{P}^{A|}$ which represents a slight generalization of the fundamental K -frame which leads to equations (1.6). What we have in mind is a fundamental K -frame satisfying the conditions

$$(3.9) \quad \begin{aligned} \mathbf{P}^{X|} \times \mathbf{P}^{Y+iii|} &= \mathbf{P}^{Y|} \times \mathbf{P}^{X|} = \mathbf{P}^{Y|} \times \mathbf{P}^{Y+iii|} = 0 \\ \mathbf{P}^{Y|} \times \mathbf{P}^{Y|} &= \mathbf{P}^{X|} \times \mathbf{P}^{X|} \neq 0 \\ \mathbf{P}^{X+iii|} \times \mathbf{P}^{X+iii|} &= -\mathbf{P}^{X|} \times \mathbf{P}^{X|} \neq 0 \\ \mathbf{P}^{Y+iii|} \times \mathbf{P}^{Y+iii|} &= -\mathbf{P}^{Y|} \times \mathbf{P}^{Y|} \neq 0. \end{aligned}$$

Remark (3.3). If in Lemma (3.2)b the condition (3.1)e is replaced by $\mathbf{R} \times \mathbf{R} = 0$, then it can be proved that $\mathbf{V}$ (as well as $\bar{\mathbf{V}}$) has *not less than four* Substantials. The proof follows the pattern similar to the proof of Lemma (3.2)a.

Lemma (3.2)c. *There is such a fundamental K -frame satisfying (3.9) that the vectors $\mathbf{V}$, $\bar{\mathbf{V}}$ of Lemma (3.2)b have three Substantials only and in particular either*

$$(3.10)a) \quad V_A = i\epsilon \delta_A^X + \frac{\epsilon'}{2^{1/2}} \delta_A^Y + \frac{i}{2^{1/2}} \delta_A^{Y+iii} \quad c.c.$$

or

$$(3.10)b) \quad V_A = \epsilon \delta_A^X + \frac{i}{2^{1/2}} \delta_A^Y + \frac{\epsilon'}{2^{1/2}} \delta_A^{Y+iii} \quad c.c.$$

Here

$$(3.10)c \quad X, Y = \text{i, ii, iii} \quad X \neq Y.$$

Proof. Denote by $\mathbf{P}^{41}$ a fundamental K -frame satisfying (3.9). (A simple inspection shows that these conditions may be always met by a fundamental K -frame properly chosen). Put tentatively

$$(3.11)a \quad \mathbf{P}^{Y1} = 2^{1/2} \epsilon' \mathbf{R}, \quad \mathbf{P}^{Y+iii} = 2^{1/2} (\mathbf{I} - \epsilon \mathbf{P}^{X1})$$

so that

$$(3.11)b \quad \mathbf{R} = \epsilon' \mathbf{P}^{Y1} / 2^{1/2}, \quad \mathbf{I} = \epsilon \mathbf{P}^{X1} + \mathbf{P}^{Y+iii} / 2^{1/2}.$$

Substituting from (3.11)a into (3.9), we see according to (1.5) that $\mathbf{P}^{41}$ is a fundamental K -frame if the following conditions are satisfied

$$(3.11)c \quad \begin{aligned} \mathbf{P}^{X1} \cdot \mathbf{R} &= \mathbf{P}^{X1} \times \mathbf{R} = 0 \\ \mathbf{P}^{X1} \cdot \mathbf{I} &= \mathbf{P}^{X1} \cdot \mathbf{P}^{X1} = 0 \\ \mathbf{P}^{X1} \times \mathbf{I} &= i\epsilon\alpha^{-2}, \quad \mathbf{P}^{X1} \times \mathbf{P}^{X1} = 2\alpha^{-2}, \end{aligned}$$

where

$$2\alpha^{-2} = \mathbf{P}^{Y1} \times \mathbf{P}^{Y1}.$$

Comparing these conditions with (3.8) and remembering Remark (3.2), we see that (3.11)c admits a real null vector $\mathbf{P}^{X1}$ so that $\mathbf{P}^{41}$ is a fundamental K -frame and (3.11) are meaningful. The equations (3.11)b together with (2.13) lead immediately to (3.10)ac. The equations (3.10)bc may be proved in a similar way.

In the next Lemma we shall consider two complex vectors $\mathbf{V}, \mathbf{W}$ which are not complex conjugate.

Lemma (3.3)a. *Let $\mathbf{V}, \mathbf{W}$ be two complex vectors subject to the conditions*

$$(3.12) \quad \begin{aligned} \text{a)} \quad \mathbf{V} \cdot \mathbf{V} &= 0 \quad c.c. & \text{c)} \quad \mathbf{W} \cdot \mathbf{W} &= 0 \quad c.c. \\ \text{b)} \quad \mathbf{V} \cdot \bar{\mathbf{V}} &= 0 & \text{d)} \quad \mathbf{W} \cdot \bar{\mathbf{W}} &= 0 \\ \text{e)} \quad \mathbf{V} \cdot \mathbf{W} &= \epsilon' i \quad c.c. & \text{f)} \quad \mathbf{V} \cdot \bar{\mathbf{W}} &= 0 \quad c.c. \end{aligned}$$

Then there are exactly two real vectors $\mathbf{M}$ satisfying the conditions

$$\begin{aligned} \mathbf{V} \cdot \mathbf{M} &= 0 \quad c.c.; & \mathbf{W} \cdot \mathbf{M} &= 0 \quad c.c. \\ \mathbf{M} \cdot \mathbf{M} &= 0. \end{aligned}$$

Proof. Denote by C the cone of the null vectors of g_{AB} . According to (3.12)a, c, the vectors $\mathbf{V}, \bar{\mathbf{V}}, \mathbf{W}, \bar{\mathbf{W}}$ are rulings of C . Assume that they are linearly dependent. Then there are four scalars $\omega, \rho, \sigma, \varphi$ not all equal to zero such that

$$\omega \mathbf{V} + \rho \bar{\mathbf{V}} + \sigma \mathbf{W} + \varphi \bar{\mathbf{W}} = 0$$

Multiplying this equation by $\mathbf{V}$, $\bar{\mathbf{V}}$, $\mathbf{W}$, $\bar{\mathbf{W}}$ and using (3.12) we find $\omega = \rho = \sigma = \varphi = 0$. Hence the vectors under consideration are linearly independent and define a four dimensional space, say S . A similar method based on (3.12) shows that S is not tangent to C (and its matrix has the signature $+-+-$). Therefore there is only one plane, say E , perpendicular to S , intersecting S only in the vertex of C . Every vector $\mathbf{M} \subset E$ satisfies the first two equations of the Lemma. Because the metric of the K -space has the signature $+-+-$, the signature of the metric of E must be $+-$. Hence the plane E is a hyperbolic one and therefore there are two real vectors $\mathbf{M} \subset E$, satisfying all equations of the Lemma, namely the null vectors of E .

This result will be used in the second half of the following Lemma.

Lemma (3.3)b. *There is such a fundamental K -frame with respect to which the vectors $\mathbf{V}$, $\bar{\mathbf{V}}$, $\mathbf{W}$, $\bar{\mathbf{W}}$ of the previous Lemma have the components*

$$(3.13) \quad \text{a) } V_A = \epsilon' 2^{-1/2} (\delta_A^X + i \delta_A^Y), \quad \text{c.c.} \quad \text{b) } W_A = 2^{-1/2} (i \delta_A^{X+iii} + \delta_A^{Y+iii}), \quad \text{c.c.} \\ X, Y = \text{i, ii, iii}, \quad X \neq Y.$$

Moreover, there is such a fundamental K -frame with respect to which the components of $\mathbf{V}$, $\bar{\mathbf{V}}$, $\mathbf{W}$, $\bar{\mathbf{W}}$ are given by (3.13)a and

$$(3.13)\text{c} \quad W_A = 2^{-1/2} (-i \delta_A^Z + i \delta_A^{X+iii} + \delta_A^{Y+iii}) \quad \text{c.c.} \\ X, Y, Z = \text{i, ii, iii}, \quad XYZ \neq.$$

Proof. In order to prove the first part of our statement, we put

$$(3.14) \quad \mathbf{V} = \mathbf{R}_1 + i\mathbf{I}_1, \quad \text{c.c.} \quad \mathbf{W} = \mathbf{R}_2 + i\mathbf{I}_2, \quad \text{c.c.}$$

Then the equations (3.12)a-f are equivalent to

$$(3.15) \quad \text{a) } \mathbf{R}_x \cdot \mathbf{R}_y = 0 \quad \text{b) } \mathbf{I}_x \cdot \mathbf{I}_y = 0 \quad \text{c) } \mathbf{R}_x \cdot \mathbf{I}_x = 0 \\ \text{d) } \mathbf{I}_1 \cdot \mathbf{R}_2 = \mathbf{I}_2 \cdot \mathbf{R}_1 = \epsilon'/2, \quad x, y = 1, 2.$$

Hence we can concentrate on (3.15). Let us define four vectors $\mathbf{P}^{x|}$, $\mathbf{P}^{y|}$, $\mathbf{P}^{x+iii|}$ and $\mathbf{P}^{y+iii|}$ by

$$(3.16) \quad \text{a) } \mathbf{P}^{x|} \stackrel{\text{def}}{=} 2^{1/2} \mathbf{R}_1 \epsilon' \quad \text{c) } \mathbf{P}^{x+iii|} \stackrel{\text{def}}{=} 2^{1/2} \mathbf{I}_2 \\ \text{b) } \mathbf{P}^{y|} \stackrel{\text{def}}{=} 2^{1/2} \mathbf{I}_1 \epsilon' \quad \text{d) } \mathbf{P}^{y+iii|} \stackrel{\text{def}}{=} 2^{1/2} \mathbf{R}_2.$$

An easy inspection based on (3.15) shows that these vectors may be thought of as elements of a fundamental K -frame. Using (3.14) and (3.16) we find

$$(3.17) \quad \text{a) } \mathbf{V} = \frac{\epsilon'}{2^{1/2}} (\mathbf{P}^{x|} + i\mathbf{P}^{y|}) \quad \text{c.c.} \\ \text{b) } \mathbf{W} = \frac{1}{2^{1/2}} (i\mathbf{P}^{x+iii|} + \mathbf{P}^{y+iii|}) \quad \text{c.c.}$$

These equations lead in the usual way to (3.13)ab.

In order to prove the second part of our statement, we denote by $\mathbf{P}^{z^1} = \mathbf{M}$ one of the real null vectors, solution of the equations of Lemma (3.3)a. Hence these equations may be written

$$(3.18) \quad \mathbf{P}^{z^1} \cdot \mathbf{R}_x = \mathbf{P}^{z^1} \cdot \mathbf{I}_x = \mathbf{P}^{z^1} \cdot \mathbf{P}^{z^1} = 0, \quad x = 1, 2.$$

Then we define $\mathbf{P}^{x^1}$, $\mathbf{P}^{y^1}$, $\mathbf{P}^{x+iii^1}$ by (3.16)abd and put

$$(3.16)e \quad P^{x+iii^1} \stackrel{\text{def}}{=} 2^{1/2} \mathbf{I}_2 + \mathbf{P}^{z^1}.$$

It is not difficult to prove that (3.16)abde and (3.18) define the elements of a fundamental K -frame. Moreover, the equations (3.16)ab lead again to (3.13)a, while (3.16)de lead to

$$(3.17)c \quad \mathbf{W} = 2^{-1/2} \{ (-\mathbf{P}^{z^1} + \mathbf{P}^{x+iii^1})i + P^{y+iii^1} \}$$

$$X, Y, Z = i, ii, iii, \quad XYZ \neq.$$

The equations (3.17)ac lead in the usual way to (3.13)ac.

Remark (3.4). In Sections 2 and 3 we have shown that the components of certain complex vectors, subject to some conditions, can be of a very simple structure *without* these vectors having a *special position* in the K -space. We achieved these results by a special choice of the fundamental K -frame. For such a fundamental K -frame the components

$$(3.19) \quad \xi^{AB} = P_{\alpha}^{A1} P_{\alpha}^{B1} \xi^{\alpha\alpha}$$

do not always satisfy the condition (1.6). After having shown the existence of very simple structure of the components of complex vectors under consideration we shall adopt a simpler point of view concerning the fundamental K -frame underlying our considerations.

Agreement (3.1). From now on when not stated explicitly otherwise, the vector components introduced in Sections 2 and 3 will be thought of as components with respect to a fundamental K -frame chosen a priori in such a way that the conditions (3.19) and (1.6) hold.

Remark (3.5). This agreement will enable us to evaluate the conditions into which (1.11) splits by virtue of Agreement (1.1).

4. U-permutations. In I we introduced the basic group ${}^*\mathcal{G}$ of transformations which preserve the metric tensor g_{AB} of the K -space

$$(4.1)a \quad g_{AB} = U_A^c U_B^D g_{CD} = \check{U}_A^c \check{U}_B^D g_{CD}.$$

(Cf. Agreement (I(4.1)) and equation (I(3.9))). In II we have shown that the one of the algebraic identities which was not considered in I reduces the choice of the transformations of ${}^*\mathcal{G}$ by the condition

$$(4.1)b \quad R_{CD} \check{U}_A^c \check{U}_B^D \xi^{AB} = 0.$$

(Cf. Theorem (II(3.1))). Using (1.6) (cf. Agreement (3.1)) one can rewrite (4.1)b in the form

$$(4.1)c \quad \sum_{X=i}^{X=iii} R_{AB} \check{U}_X^A \check{U}_X^B = \sum_{Y=i}^{Y=iii} R_{AB} \check{U}_Y^A \check{U}_Y^B.$$

These results will be used in the text which follows where we shall consider a very special kind of transformation of ${}_*\mathcal{G}$.

Definition (4.1). If a transformation U_A^B of ${}_*\mathcal{G}$ is only a permutation of tensor indices, then this transformation will be termed a *U-permutation*.

Remark (4.1). A *U-permutation* must satisfy (4.1) and in particular (4.1)c. This latter condition can be rewritten in another form if we take account of the following interplay of indices.

a) To every index $X (= i, \dots, vi)$ there is *associate* only one index X^* such that $g_{XX^*} = 1$ (cf. (1.5)).

b) To every index $X (= i, \dots, vi)$ there is *adjoint* only one index $*X$ by a *U-permutation*. Indeed every *U-permutation* U_A^B satisfies the conditions

$$(4.2)a \quad U_X^{*X} = \check{U}_{*X}^X = 1, \quad U_X^Y = \check{U}_Y^X = 0, \quad Y \neq *X.$$

In order to rewrite (4.1)c in a form suitable for our purposes, we need the following Lemma.

Lemma (4.1)a. We have for an arbitrary index $X (= i, \dots, vi)$

$$(4.2)b \quad *X^* \stackrel{\text{def}}{=} (*X)^* = *(X^*).$$

Proof. The index associate to X is X^* and the index associate to $*X$ is $(*X)^*$

$$g_{XX^*} = 1, \quad g_{*X(*X)^*} = 1.$$

On the other hand, applying to the first of these equations a *U-permutation*, we obtain

$$g_{*X * (*X)^*} = 1.$$

Comparing the last two equations, we arrive at (4.2)b.

This Lemma enables us to rewrite (4.1)c in the form

$$(4.1)d \quad \sum_{Y=i}^{Y=iii} R_{*Y * Y} = \sum_{Y=i}^{Y=iii} R_{*Y^* * Y^*}$$

which we shall use later.

Remark (4.2). The condition (4.1)d may be looked upon as a result of the application of a *U-permutation* to (1.4)d. This raises the question whether the application of a *U-permutation* to (1.4)c does not yield another condition imposed on R_{AB} . The answer to this question is no. Indeed we have

$$\text{Trace } N = \text{Trace } N^t = \sum_{X=i}^{X=iii} R_{XX^*},$$

where X^* is associate to X (i.e. $X^* = X + \text{iii}$ for $X = \text{i, ii, iii}$) so that the last sum is invariant under a U -permutation.

The indices

$$(4.3)\text{a} \quad X, \quad *X \neq X$$

define uniquely the indices

$$(4.3)\text{b} \quad X^*, \quad *X^* \neq X^*.$$

In the next Lemma we denote by Z, T the remaining two indices so that the indices (4.3) and Z, T describe the whole range $\text{i, } \dots, \text{vi}$.

Lemma (4.1)b. *Assume that the indices (4.3)a have the range i, ii, iii or iv, v, vi . There is such a U -permutation which permutes the indices according to the scheme*

$$(4.4) \quad \begin{bmatrix} X & *X & Z & X^* & *X^* & T \\ *X & X & Z & *X^* & X^* & T \end{bmatrix}, \quad (T = Z^*).$$

Proof. From (4.4) and (1.5) we obtain

$$g_{XX^*} = g_{*X *X^*} \quad (=1).$$

On the other hand, the equation

$$g_{XY} = 0, \quad Y \neq X^*$$

is invariant under the permutation (4.4). Hence (4.1)a is satisfied by (4.4). Moreover, if we denote by (AB) the component R_{AB} of the curvature tensor, we see that (4.4) permutes these components according to the scheme

$$(4.5) \quad \begin{bmatrix} (XX) & (*X *X) & (ZZ) & (X^* X^*) & (*X^* *X^*) & (TT) \\ (*X *X) & (XX) & (ZZ) & (*X^* *X^*) & (X^* X^*) & (TT) \end{bmatrix}$$

so that either one of the sums

$$(4.6) \quad (XX) + (*X *X) + (ZZ), \quad (X^* X^*) + (*X^* *X^*) + (TT)$$

is invariant under (4.4). According to our assumption, the indices $X, *X$ belong to the same range i, ii, iii (or iv, v, vi). We may assume without loss of generality the Z belongs to the same range. Then the sums (4.6) are nothing else than Trace M and Trace S (cf. (1.4)b) which satisfy (1.4)d. We obtain the same equation (1.4)d if we put in (4.1)d resp. $Y = X, *X, Z$ (instead of $Y = \text{i, ii, iii}$) and apply (4.4). Hence (4.1)d is satisfied by (4.4) and so is (4.1)a. Therefore (4.4) is a U -permutation.

Remark (4.3). Any U -permutation satisfies (4.1)a and therefore the product $V \cdot W$ of two vectors is invariant under any U -permutation as it was to be expected.

In the previous Lemma we considered the indices (4.3)a to be of the same range i, ii, iii or iv, v, vi . In the next Lemma we shall deal with the remaining case.

Lemma (4.2). *Let X be an index of the range i, ii, iii (or iv, v, vi) and $*X \neq X^*$ of the remaining range iv, v, vi (or i, ii, iii). A necessary and sufficient condition for (4.4) to be a U -permutation is the fulfillment of one of the mutually equivalent conditions*

$$(4.7) \quad \begin{aligned} &\text{a) } (ZZ) = (TT) \\ &\text{b) } (XX) + (*X^* *X^*) = (X^* X^*) + (*X^* X^*). \end{aligned}$$

Proof. The proof that (4.4) satisfies (4.1)a is obviously independent of the mutual position of the indices (4.3)a and therefore it holds also here. Because X^* and $*X \neq X^*$ belong to the different range than does X , these indices belong to the same range and therefore, the same is true of indices $*X^*$ and $X = *(X)$. Hence the condition (1.4)d may be written

$$(4.8)\text{a} \quad (XX) + (*X^* *X^*) + ZZ = (*X^* X^*) + (X^* X^*) + (TT).$$

Assume now that (4.4) is a U -permutation, so that (4.1)d is satisfied. This condition can also be obtained by application of (4.4) to (4.8)a so that it can be written in the form

$$(4.8)\text{b} \quad (*X^* X^*) + (X^* X^*) + (ZZ) = (XX) + (*X^* *X^*) + (TT).$$

Comparing (4.8)ab, one obtains at once the mutually equivalent conditions (4.7)a, (4.7)b. Assume now that one of the equations (4.7)ab is satisfied. Then the equation (1.4)d in the form (4.8)a yields at once the remaining equation (4.7). According to either one of these equations, and by virtue of (4.8)a, the condition (4.8)b is satisfied. It is nothing else but (4.1)d rewritten by means of (4.4). Hence this permutation satisfying (4.1)a and (4.1)d is a U -permutation.

In the previous two Lemmas we assumed $*X \neq X^*$. In the next Lemma we shall consider the case where for one index X we have

$$(4.9)\text{a} \quad *X = X^*.$$

Lemma (4.3). *If the result of a permutation of the indices is*

$$(4.9)\text{b} \quad *X^* = X$$

then (4.9)a holds. If the remaining indices remain unchanged, then the permutation is a U -permutation if and only if

$$(4.10) \quad (XX) = (X^* X^*).$$

Proof. Applying to (4.9)b either one of the operators $*$ (), () $*$ and remembering Lemma (4.1), we get (4.9)a. It is obvious from the formulation of our Lemma and from (4.9)a that the components g_{AB} are invariants of our permutation. Applying it to (1.4)d, we obtain an equation of the form

$$(4.11) \quad \text{Trace } *M = \text{Trace } *S.$$

The equations (4.11) and (1.4)d lead immediately to (4.10). The remaining part of the proof is almost trivial.

Remark (4.4). There are also other U -permutations not considered here, for instance, a U -permutation which acts *on all* indices. The corresponding results may be obtained by using the pattern of the method displayed in this Section.

II.

CHARACTERISTIC OF THE TYPE I

5. Characteristic $[2 \ 2 \ \bar{2}]$. One sees from the table (1.2) that there is substantially only one basic complex characteristic of the type I, namely

$$(5.1) \quad [2 \ 2 \ \bar{2}].$$

(By renumbering the eigenvalues properly one arrives at $[2 \ \bar{2} \ 2]$ or $[\bar{2} \ 2 \ 2]$. These characteristics are not considered to be substantially different from (5.1) and therefore we limit ourselves to the characteristic (5.1).)

The characteristic under consideration leads to three double eigenvalues, two of them complex conjugate

$$(5.2) \quad \alpha, \quad \gamma = \gamma_1 + i\gamma_2, \quad \bar{\gamma} = \gamma_1 - i\gamma_2.$$

The corresponding curvature tensor $R_A{}^B$ is of the structure given by

$$(5.3) \quad R_A{}^B = \alpha(\delta_A^1 \delta_1^B + \delta_A^2 \delta_2^B) + \delta_A^2 \delta_1^B \\ + \gamma(\delta_A^3 \delta_3^B + \delta_A^4 \delta_4^B) + \delta_A^4 \delta_3^B \\ + \bar{\gamma}(\delta_A^5 \delta_5^B + \delta_A^6 \delta_6^B) + \delta_A^6 \delta_5^B.$$

The vectors

$$(5.4)a \quad \begin{array}{ccc} \delta_B^2, & \delta_B^4, & \delta_B^6 \\ \delta_1^A, & \delta_3^A, & \delta_5^A \end{array}$$

are obviously eigenvectors belonging to the eigenvalues (5.2), while

$$(5.4)b \quad \begin{array}{ccc} \delta_B^1, & \delta_B^3, & \delta_B^5 \\ \delta_2^A, & \delta_4^A, & \delta_6^A \end{array}$$

are auxiliary vectors. In the next Lemma we shall consider the relationship of the vectors (5.4).

Lemma (5.1). *Every pair out of the four pairs of vectors*

$$(5.5) \quad \begin{array}{ll} \text{a)} & \delta^4, \ \delta^6; \quad \delta_3, \ \delta_5 \\ \text{b)} & \delta^3, \ \delta^5; \quad \delta_4, \ \delta_6 \end{array}$$

consists of complex conjugate vectors

$$(5.6) \quad \begin{array}{ll} \text{a)} & \delta^6 = \bar{\delta}^4; \quad \delta_5 = \bar{\delta}_3 \\ \text{b)} & \delta^5 = \bar{\delta}^3; \quad \delta_6 = \bar{\delta}_4. \end{array}$$

Proof. Every pair (5.5)a consists of eigenvectors belonging to complex conjugate eigenvalues. Hence the vectors of every pair (5.5)a must be complex conjugate, *i.e.* they must satisfy (5.6)a. The sum of the second and third line in (5.3) must be real. This can happen if and only if the following sums are complex conjugate (see (5.6)a).

$$(5.7) \quad \begin{aligned} \text{a)} \quad & \delta_A^3 \delta_3^B + \delta_A^4 \delta_4^B \quad \text{and} \quad \delta_A^5 \delta_5^B + \delta_A^6 \delta_6^B = \delta_A^5 \bar{\delta}_3^B + \bar{\delta}_A^4 \delta_6^B \\ \text{b)} \quad & \delta_A^4 \delta_3^B \quad \quad \quad \text{and} \quad \delta_A^6 \delta_5^B = \bar{\delta}_A^4 \bar{\delta}_3^B. \end{aligned}$$

This condition is obviously satisfied by (5.7)b, while the sum of the expressions (5.7)a is real if and only if (5.6)b is satisfied.

Theorem (5.1). *Put*

$$\epsilon, \epsilon' \stackrel{\text{def}}{=} \pm 1.$$

The curvature tensor belonging to the characteristic (5.1) is given by (1.4)ab, where

$$(5.8) \quad \begin{aligned} \text{a)} \quad ((M)) &= \begin{bmatrix} \epsilon & \epsilon' & \alpha - \gamma_1 \\ \epsilon' & \cdot & \gamma_2 \\ \alpha - \gamma_1 & \gamma_2 & \cdot \end{bmatrix}, \quad \text{b)} \quad ((N)) = \begin{bmatrix} \gamma_1 & \gamma_2 & \epsilon \\ -\gamma_2 & \gamma_1 & \cdot \\ \cdot & \cdot & \alpha \end{bmatrix}, \\ \text{c)} \quad ((S)) &= \text{Diag} (\cdot \cdot \epsilon). \end{aligned}$$

In order to prove this Theorem, we shall make use of the method displayed in II adapted for complex characteristics. Hence the proof will consist of four steps.

First step. The canonical matrix $((R_a^b))$ of the curvature tensor belonging to (5.1) is given by the elements

$$(5.9) \quad \begin{aligned} R_a^b &= \alpha(\delta_a^1{}^b + \delta_a^2{}^b) + \delta_a^2{}^b \\ &+ \gamma(\delta_a^3{}^b + \delta_a^4{}^b) + \delta_a^4{}^b \\ &+ \bar{\gamma}(\delta_a^5{}^b + \delta_a^6{}^b) + \delta_a^6{}^b. \end{aligned}$$

Second step with inserted arithmetic change leads by virtue of Lemma (5.1) to

$$(5.10) \quad \begin{aligned} \text{a)} \quad g_{ab} &= 2\epsilon \delta_{(ab)}^{12} + 2\epsilon' i(\delta_{(ab)}^{56} - \delta_{(ab)}^{34}) \\ \text{b)} \quad g^{ab} &= 2\epsilon \delta_{12}^{(ab)} + 2\epsilon' i(\delta_{34}^{(ab)} - \delta_{56}^{(ab)}). \end{aligned}$$

Using (5.9) and (5.10), we arrive at

$$(5.11) \quad \begin{aligned} R_{ab} &= 2\epsilon\alpha \delta_{(ab)}^{12} + 2\epsilon' i(\bar{\gamma} \delta_{(ab)}^{56} - \gamma \delta_{(ab)}^{34}) \\ &+ \epsilon \delta_{ab}^{22} + \epsilon' i(\delta_{ab}^{66} - \delta_{ab}^{44}). \end{aligned}$$

Third step. According to Agreement (1.1), we must have by virtue of (5.11)

$$(5.12) \quad \begin{aligned} \text{a)} \quad \check{\epsilon}^{12} &= 0 & \text{b)} \quad \check{\epsilon}^{34} &= 0 & \text{c)} \quad \check{\epsilon}^{56} &= 0 & \text{d)} \quad \check{\epsilon}^{22} &= 0 & \text{e)} \quad \check{\epsilon}^{44} &= \check{\epsilon}^{66}. \end{aligned}$$

(These conditions resulting from Agreement (1.1) restrict the number of parameters of the canonical frame to 10). On the other hand, if we put

$$(5.13) \quad \begin{array}{ll} \delta^3 \stackrel{\text{def}}{=} \mathbf{W} & \delta^4 \stackrel{\text{def}}{=} \mathbf{V} \\ \delta^5 \stackrel{\text{def}}{=} \overline{\mathbf{W}} & \delta^6 \stackrel{\text{def}}{=} \overline{\mathbf{V}} \end{array}$$

the conditions (3.12) reduce to

$$(5.14) \quad \begin{array}{ll} \text{a) } g^{44} = 0, & g^{66} = 0 \\ \text{c) } g^{33} = 0, & g^{55} = 0 \\ \text{e) } g^{34} = -g^{56} = \epsilon' i & \text{f) } g^{36} = 0, \quad g^{45} = 0. \end{array} \quad \begin{array}{l} \text{b) } g^{46} = 0 \\ \text{d) } g^{35} = 0 \end{array}$$

They are satisfied because of (5.10). Hence we may try to put according to Lemma (3.3)b (for $X = \text{i}$, $Y = \text{ii}$, $Z = \text{iii}$)

$$(5.15) \quad ((\delta_A^x)) = 2^{-1/2} \left[\begin{array}{cccccc} \cdot & \cdot & -i & i & 1 & \cdot \\ \epsilon' & i\epsilon' & \cdot & \cdot & \cdot & \cdot \end{array} \right], \quad x = 3, 4,$$

while δ^5 and δ^6 are given by the first equations (5.6)ab. If we take into account Agreement (3.1), we see easily that the vectors $\delta^3, \delta^4, \delta^5, \delta^6$ satisfy (5.12)bce. The remaining conditions (5.12) and (5.10) lead to δ^1 and δ^2 so that the canonical frame is given by the matrix

$$(5.16) \quad ((\delta_A^a)) = 2^{-1/2} \left[\begin{array}{cccccc} \cdot & \cdot & \epsilon 2^{1/2} & \cdot & \cdot & \cdot \\ 2^{1/2} & \cdot & \cdot & \cdot & \cdot & 2^{1/2} \\ \cdot & \cdot & -i & i & 1 & \cdot \\ \epsilon' & \epsilon' i & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & i & -i & 1 & \cdot \\ \epsilon' & -\epsilon' i & \cdot & \cdot & \cdot & \cdot \end{array} \right].$$

Fourth step. Using (5.11) we obtain

$$(5.17) \quad \begin{aligned} R_{AB} &= R_{ab} \delta^{ab} \\ &= 2\epsilon\alpha \delta^{(12)} + 2\epsilon' i (\bar{\gamma} \delta^{(56)} - \gamma \delta^{(34)}) \\ &\quad + \epsilon \delta^{22} + \epsilon' i (\delta^{66} - \delta^{44}). \end{aligned}$$

(The symbol $R_{ab} \delta^{ab}$ has been explained in the seventh paragraph of Section 1). Substituting into (5.17) from (5.16), we obtain (5.8).

Remark (5.1). In the fourth Section we introduced the concept of U -permutations. They are handy if by some reason one needs to change the number of Substantials in some lines of the matrix $((R_{AB}))$. Thus, for instance, we see that the condition (4.10) is satisfied by $X = \text{ii}$ in R_{AB} . Hence we may apply the U -permutation of Lemma (4.3) (for $X = \text{ii}$) and obtain

$$\begin{aligned}
 (5.18) \quad \text{a) } ((M)) &= \begin{bmatrix} \epsilon & \gamma_2 & \alpha - \gamma_1 \\ \gamma_2 & \cdot & \cdot \\ \alpha - \gamma_1 & \cdot & \cdot \end{bmatrix} & \text{b) } ((N)) &= \begin{bmatrix} \gamma_1 & \epsilon' & \epsilon \\ \cdot & \gamma_1 & \cdot \\ \cdot & \gamma_2 & \alpha \end{bmatrix} \\
 \text{c) } ((S)) &= \begin{bmatrix} \cdot & -\gamma_2 & \cdot \\ -\gamma_2 & \cdot & \cdot \\ \cdot & \cdot & \epsilon \end{bmatrix}.
 \end{aligned}$$

Similar remarks hold also for all classification of this paper. However, it should be stressed that the results of Section 4 will be handy also in other instances as shows the text of the next Section.

6. Principal matrix $((R_{AB}))$. In the case of a real characteristic it was very easy to recognize that the *real* canonical frame (*i.e.* consisting only *real* vectors) has the minimal number of Substantials. This was the reason for describing the frame of vectors δ_A^a as *canonical*. The situation is more complicated if one considers complex characteristics. (*Cf.* Lemmas (3.1), (3.2)_{ac} and Remark (3.3)). Although we use the same name “canonical frame” for the *complex* frame δ_A^a in the case of a complex characteristic it is not *a priori* evident that this canonical frame has a minimal number of Substantials. In order to deal with this situation, we need the following Definition, justified by the above mentioned complications in the case of complex characteristic.

Definition (6.1). A complex canonical frame with the minimal number of Substantials will be referred to as a *principal canonical frame*. The matrix $((R_{AB}))$ built up by means of the principal canonical frame will be termed the *principal matrix* $((R_{AB}))$.

Remark (6.1). In order to prove that $((R_{AB}))$ is a principal matrix, it is sufficient to prove that the underlying canonical frame is a principal canonical frame. In this paper we shall always limit ourselves to this part of the proof only.

Theorem (6.1). The matrix $((R_{AB}))$ given by (1.4)_b and (5.8) is a *principal matrix*.

Proof. There are two possible approaches to the problem of the reduction of Substantials in (5.16). In the first one we consider the influence of the distribution of the two Substantials in δ_A^4 . In the second one we shall consider the possibility to reduce the number of Substantials in (5.16) by changing the number of Substantials of the real vectors.

The first approach is justified by two facts. The number of Substantials in δ^1 and δ^2 is already the minimal one (*cf.* (5.12)_d). The number of Substantials in δ^4 and δ^6 cannot be reduced because these vectors are complex vectors. On the other hand, comparing the structure of δ^3 and δ^5 as given by (5.16) with the

statement of Lemma (3.3)b, we see that the possibility of reducing the number of Substantials in δ^3 and δ^5 must not be *a priori* excluded. Start first with such a fundamental K -frame with respect to which $\delta^4 = \mathbf{V}$ and $\delta^3 = \mathbf{W}$ have the components given by (3.13)a and (3.13)b (and not by (3.13)a and (3.13)c used in the “third step” of the proof of Theorem (5.1)). Then the vectors δ^1, δ^2 must have the components

$$\delta_A^y = c_y \delta_A^Z + f_y \delta_A^{Z+iii}, \quad y = 1, 2, \quad Z \neq X, Y.$$

By an appropriate choice of c_y and f_y we may always satisfy the conditions

$$(6.1) \quad g^{11} = g^{22} = 0 \neq g^{12}$$

and (5.12)a, but never (6.1) and (5.12)d. Hence the above mentioned assumption is not admissible. Assume now that there is such a fundamental K -frame with respect to which $\delta^4 = \mathbf{V}$ has the components (3.13)a *where X belongs to the range i, ii, iii (or iv, v, vi) and Y belongs to the range iv, v, vi (or i, ii, iii)*. Because of $g^{44} = 0$ we must have $Y - X \neq iii$ in the first and $X - Y \neq iii$ in the second case. Using the method displayed in the third step of the proof of Theorem (5.1) one finds easily that the corresponding canonical frame has at least as many Substantials as the canonical frame (5.16). Hence it is impossible to reduce the number of Substantials in (5.16) by this kind of approach. Let us mention parenthetically that the same result is valid if we take for the coordinates of $((\delta_A^4))$ the last row of the matrix (5.16).

Let us now consider the second approach. If we start with δ^1 and δ^2 and choose *one more* Substantial for these vectors, the possibility of reducing *by two* the number of Substantials of the remaining vectors cannot be *a priori* excluded. The vectors δ^1 and δ^2 must satisfy the conditions

$$(6.3)b \quad g^{11} = g^{22} = 0, \quad g^{12} = \epsilon, \quad \check{e}^{12} = \check{e}^{22} = 0.$$

In order to consider all possible distributions of two Substantials satisfying (6.3)b, let us start first with δ^1 given by

$$(6.4)a \quad \delta_A^1 = \delta_A^X + \delta_A^Y,$$

where X, Y are from the same range i, ii, iii or iv, v, vi. It is not difficult to prove that there exist U -permutations of Lemmas (4.1)b and (4.3) which bring X, Y to *different* ranges i, ii, iii and iv, v, vi, for instance

$$(6.4)b \quad ((\delta_A^1)) = ((\cdot \cdot 1 1 \cdot \cdot)).$$

The conditions (6.3)b do not allow us other possibilities than

$$(6.4) \quad \begin{array}{ll} \text{c)} & ((\delta_A^2)) = \frac{1}{2}((\epsilon \cdot \cdot \cdot \cdot \epsilon)) \\ \text{d)} & ((\delta_A^2)) = ((\epsilon \cdot \cdot \cdot \cdot \pm \epsilon \cdot)) \\ \text{e)} & ((\delta_A^2)) = ((\cdot \pm \epsilon \cdot \cdot \cdot \cdot \epsilon)). \end{array}$$

In the cases (6.4)bc, (6.4)bd, (6.4)be, the remaining vectors of the canonical frame have the structure of the components.

$$\begin{aligned}
 & \text{bc) } (a \ b \ c \ -c \ e \ -a), \quad 2ac = be \\
 (6.4) \quad & \text{bd) } (a \ b \ c \ \mp b \ e \ -a), \quad a(\pm b + c) = be \\
 & \text{be) } (a \ b \ c \ d \ \mp c \ -a), \quad c(\pm b + a) = ad.
 \end{aligned}$$

These conditions cannot be met by *four linearly independent complex vectors* having altogether $M < 9$ Substantials. Hence this approach does not reduce the number of Substantials of a canonical frame to $N < 13$.

Assume now that δ^1 has only one Substantial while δ^2 has three Substantials. Without restriction of generality, we can assume that the Substantial of δ^1 is its third component. Then according to (6.3)b we have for $x = 1, 2$

$$\begin{aligned}
 & \text{a) } ((\delta_A^x)) = \left[\begin{pmatrix} \cdot & \cdot & \epsilon & \cdot & \cdot & \cdot \\ q & \cdot & \cdot & \cdot & p & 1 \end{pmatrix} \right], \quad q^2 = p^2 + 1 \\
 (6.5) \quad & \text{b) } ((\delta_A^x)) = \left[\begin{pmatrix} \cdot & \cdot & \epsilon & \cdot & \cdot & \cdot \\ \cdot & q & \cdot & p & \cdot & 1 \end{pmatrix} \right], \quad q^2 = p^2 + 1 \\
 & \text{c) } ((\delta_A^x)) = \left[\begin{pmatrix} \cdot & \cdot & \epsilon & \cdot & \cdot & \cdot \\ q & p & \cdot & \cdot & \cdot & 1 \end{pmatrix} \right], \quad q^2 + p^2 = 1.
 \end{aligned}$$

The components of the remaining vectors have the structure

$$(6.6) \quad (a \ b \ c \ d \ e \ \cdot)$$

where

$$\text{a) } c = -(dq + bp) \quad (6.5)\text{a}$$

$$(6.7) \quad \text{b) } c = -(eq + ap) \quad \text{for } (6.5)\text{b}$$

$$\text{c) } c = -(dq + ep) \quad (6.5)\text{c}.$$

It is impossible to find four linearly independent complex vectors with two Substantials each which have the structure prescribed by (6.6) and one of the conditions (6.7)abc, and satisfy (5.10) and (5.12).

Hence neither one of the possible approaches leads to the reduction of Substantials in a canonical frame to $N < 13$. Therefore (5.16) is a principal canonical frame.

CHARACTERISTICS OF THE TYPE II

7. Characteristic [2 $\bar{2}$ 1 1]. One sees from the table (1.2) that there are three complex basic characteristics of type II. We shall consider first the characteristic

$$(7.1) \quad [2 \ \bar{2} \ 1 \ 1].$$

The characteristic (7.1) leads to two complex conjugate double eigenvalues and two single real eigenvalues

$$(7.2) \quad \alpha = \alpha_1 + i\alpha_2, \quad \bar{\alpha} = \alpha_1 - i\alpha_2, \quad \mu, \quad \nu.$$

The corresponding curvature tensor is of the structure

$$(7.3) \quad R_A{}^B = \alpha(\delta_A^1 \delta_1^B + \delta_A^2 \delta_2^B) + \delta_A^2 \delta_1^B \\ + \bar{\alpha}(\delta_A^2 \delta_3^B + \delta_A^4 \delta_4^B) + \delta_A^4 \delta_3^B \\ + \mu \delta_A^5 \delta_5^B + \nu \delta_A^6 \delta_6^B.$$

The vectors

$$(7.4)a \quad \delta_B^2, \quad \delta_B^4, \quad \delta_B^5, \quad \delta_B^6 \\ \delta_1^A, \quad \delta_3^A, \quad \delta_5^A, \quad \delta_6^A$$

are obviously eigenvectors belonging to the eigenvalues (7.2), the remaining vectors

$$(7.4)b \quad \delta_B^1, \quad \delta_B^3 \\ \delta_2^A, \quad \delta_4^A$$

are auxiliary vectors. In the next Lemma we shall consider the relationship of the vectors (7.4)ab.

Lemma (7.1). *Every pair out of the four pairs of vectors*

$$(7.5) \quad a) \quad \delta^2, \quad \delta^4; \quad \delta_1, \quad \delta_3 \\ b) \quad \delta^1, \quad \delta^3; \quad \delta_2, \quad \delta_4$$

consists of complex conjugate vectors

$$(7.6) \quad a) \quad \delta^4 = \bar{\delta}_2; \quad \delta_3 = \bar{\delta}_1 \\ b) \quad \delta^3 = \bar{\delta}^1; \quad \delta_4 = \bar{\delta}_2.$$

Proof. Every pair (7.5)a consists of eigenvectors belonging to complex conjugate eigenvalues. Hence the vectors of either one of these two pairs must be complex conjugate, *i.e.* they satisfy (7.6)a. The remaining part of the proof follows the corresponding part of the proof of Lemma (5.1).

Theorem (7.1). *Put*

$$\epsilon, \epsilon' \stackrel{\text{def}}{=} \pm 1.$$

Then the curvature tensor belonging to the characteristic (7.1) is given by (1.4)ab where

$$\begin{aligned}
 (7.7) \quad & a) \quad ((M)) = \text{Diag.} \left(\begin{array}{ccc} \cdot & \cdot & \epsilon'(\mu - \nu)/2 \end{array} \right) \\
 & b) \quad ((N)) = \left[\begin{array}{ccc} \alpha_1 & \alpha_2 & \cdot \\ \alpha_2 & \alpha_1 & \cdot \\ \cdot & \cdot & (\mu + \nu)/2 \end{array} \right] \\
 & c) \quad ((S)) = \left[\begin{array}{ccc} \cdot & \epsilon & \cdot \\ \epsilon & \cdot & \cdot \\ \cdot & \cdot & \epsilon'(\mu - \nu)/2 \end{array} \right].
 \end{aligned}$$

The proof consists of four steps.

First step. The canonical matrix $((R_a^b))$ of the curvature tensor belonging to (7.1) is given by the elements

$$(7.8) \quad R_a^b = \alpha(\delta_a^1{}^b + \delta_a^2{}^b) + \delta_a^2{}^b + \bar{\alpha}(\delta_a^3{}^b + \delta_a^4{}^b) + \delta_a^4{}^b + \mu \delta_a^5{}^b + \nu \delta_a^6{}^b.$$

Second step with inserted arithmetic change. We must have by virtue of Lemma (7.1)

$$\begin{aligned}
 (7.9) \quad g_{ab} &= 2\epsilon_1 i(\delta_{(ab)}^{34} - \delta_{(ab)}^{12}) + \epsilon_2 \delta_{ab}^{55} + \epsilon_3 \delta_{ab}^{66} \\
 &\quad \epsilon_1, \epsilon_2, \epsilon_3 \stackrel{\text{def}}{=} \pm 1.
 \end{aligned}$$

Because of (7.6)b and

$$g^{11} = g^{33} = g^{13} = 0$$

the plane $i \delta^{11} \delta^{31}$ is a real parabolic plane so that every vector incident with it is a null vector. The same holds for the real plane $i \delta^{12} \delta^{41}$. These two real parabolic planes define a four-dimensional real space in which there are always at least two real hyperbolic planes. Hence the metric of this space is of the signature $+-+-$ and because the signature of the K -space is $+-+-+-$ we must have $\epsilon_2 = -\epsilon_3$ in (7.9). If we put

$$\epsilon_1 \stackrel{\text{def}}{=} \epsilon, \quad \epsilon_2 = -\epsilon_3 \stackrel{\text{def}}{=} \epsilon'$$

we may write for (7.9)

$$(7.10) \quad a) \quad g_{ab} = 2\epsilon i(\delta_{(ab)}^{34} - \delta_{(ab)}^{12}) + \epsilon'(\delta_{ab}^{55} - \delta_{ab}^{66})$$

so that

$$(7.10) \quad b) \quad g^{ab} = 2\epsilon i(\delta_{12}^{(ab)} - \delta_{34}^{(ab)}) + \epsilon'(\delta_{55}^{ab} - \delta_{66}^{ab}).$$

Using (7.8) and (7.10)a, we arrive at

$$(7.11) \quad R_{ab} = 2\epsilon i(\bar{\alpha} \delta_{(ab)}^{34} - \alpha \delta_{(ab)}^{12}) + \epsilon i(\delta_{ab}^{44} - \delta_{ab}^{22}) + \epsilon' \mu \delta_{ab}^{55} - \epsilon' \nu \delta_{ab}^{66}.$$

Third step. According to Agreement (1.1), we must have by virtue of (7.11)

$$(7.12) \quad a) \quad \check{\epsilon}^{12} = 0; \quad b) \quad \check{\epsilon}^{34} = 0; \quad c) \quad \check{\epsilon}^{55} = \check{\epsilon}^{66} = 0; \quad d) \quad \check{\epsilon}^{22} = \check{\epsilon}^{44}.$$

On the other hand, if we put

$$(7.13) \quad \delta^1 = \mathbf{V}, \quad \delta^3 = \bar{\mathbf{V}}, \quad \delta^2 = \mathbf{W}, \quad \delta^4 = \bar{\mathbf{W}}$$

we see easily that the equations (3.12) (with an appropriate sign in (3.12)e) constitute a special set of conditions (7.10)b so that Lemma (3.3)b applies and we are entitled to put

$$(7.14) \quad ((\delta_A^x)) = 2^{-1/2} \left[\begin{pmatrix} 1 & i & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \epsilon i & \epsilon & \cdot \end{pmatrix} \right], \quad x = 1, 2,$$

while δ^3, δ^4 are given by the first equations (7.6)ab. An easy inspection shows that as far as the vectors $\delta^1, \dots, \delta^4$ are concerned, the conditions (7.10) and (7.12) are satisfied. Taking into consideration the remaining conditions (7.10), (7.12) we arrive at δ^5, δ^6 and the canonical frame is then given by the matrix

$$(7.15) \quad ((\delta_A^x)) = 2^{-1/2} \left[\begin{pmatrix} 1 & i & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \epsilon i' & \epsilon & \cdot \\ 1 & -i & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -\epsilon i & \epsilon & \cdot \\ \cdot & \cdot & \epsilon' & \cdot & \cdot & 1 \\ \cdot & \cdot & -\epsilon' & \cdot & \cdot & 1 \end{pmatrix} \right].$$

Fourth step. Using (7.11) we arrive at

$$(7.16) \quad R_{AB} = R_{ab} \delta^{ab} \\ = 2\epsilon i (\bar{\alpha} \delta^{(34)} - \alpha \delta^{(12)}) + \epsilon i (\delta^{44} - \delta^{22}) + \epsilon' \mu \delta^{55} - \epsilon' \nu \delta^{66}.$$

Substituting into (7.16) from (7.15), we arrive at (7.7).

Theorem (7.2). *The matrix $((R_{AB}))$ given by (1.4)b and (7.7) is a principal matrix.*

Proof. The least number of Substantials of a complex vector is obviously 2. Hence the number of Substantials in the first four rows of the matrix (7.15) cannot be reduced. On the other hand, one sees from (1.6) that the least number of Substantials of a real vector $\mathbf{V}$ satisfying the condition $\mathbf{V} \times \mathbf{V} = 0$ is 2. Hence the number of Substantials in the last two rows of (7.15) cannot be reduced. It follows from these results that the canonical frame given by (7.15) is a principal canonical frame.

8. Characteristic $[2 \bar{2} 1 \bar{1}]$. The second complex basic characteristic of the type II to be considered is

$$(8.1) \quad [2 \bar{2} 1 \bar{1}].$$

This characteristic leads to two complex conjugate double eigenvalues and to

complex conjugate single eigenvalues

$$(8.2) \quad \alpha = \alpha_1 + i\alpha_2, \quad \bar{\alpha} = \alpha_1 - i\alpha_2, \quad \mu = \mu_1 + i\mu_2, \quad \bar{\mu} = \mu_1 - i\mu_2.$$

The corresponding curvature tensor is of the structure

$$(8.3) \quad R_A{}^B = \alpha(\delta_A^1 \delta_1^B + \delta_A^2 \delta_2^B) + \delta_A^2 \delta_1^B \\ + \bar{\alpha}(\delta_A^3 \delta_3^B + \delta_A^4 \delta_4^B) + \delta_A^4 \delta_3^B \\ + \mu \delta_A^5 \delta_5^B + \bar{\mu} \delta_A^6 \delta_6^B.$$

The vectors

$$(8.4)a \quad \delta_B^2, \quad \delta_B^4, \quad \delta_B^5, \quad \delta_B^6 \\ \delta_1^4, \quad \delta_3^4, \quad \delta_5^4, \quad \delta_6^4$$

are obviously eigenvectors, belonging to the eigenvalues (8.2); the remaining vectors

$$(8.4)b \quad \delta_B^1, \quad \delta_B^3 \\ \delta_2^4, \quad \delta_4^4$$

are auxiliary vectors. In the next Lemma we shall consider the relationship of vectors (8.4)ab.

Lemma (8.1). *Every pair out of six pairs of vectors*

$$(8.5) \quad \begin{aligned} & \text{a) } \delta_1, \delta_3; \quad \delta^2, \delta^4; \quad \delta^5, \delta^6; \quad \delta_5, \delta_6 \\ & \text{b) } \delta^1, \delta^3; \quad \delta_2, \delta_4 \end{aligned}$$

consists of complex conjugate vectors

$$(8.6) \quad \begin{aligned} & \text{a) } \delta_3 = \bar{\delta}_1; \quad \delta^4 = \bar{\delta}^2, \quad \delta^6 = \bar{\delta}^5, \quad \delta_6 = \bar{\delta}_5 \\ & \text{b) } \delta^3 = \bar{\delta}^1, \quad \delta_4 = \bar{\delta}_2. \end{aligned}$$

The proof follows the pattern of the proofs of Lemmas (5.1) and (7.1).

Theorem (8.1). *Put*

$$\epsilon, \epsilon' \stackrel{\text{def}}{=} \pm 1.$$

Then the curvature tensor, belonging to (8.1), is given by (1.4)ab where

$$(8.7) \quad \begin{aligned} & \text{a) } ((M)) = \left[\begin{pmatrix} -2\epsilon'\mu_2 & \epsilon & \epsilon 2^{1/2}(\mu_1 - \alpha_1) \\ \epsilon & \cdot & \epsilon 2^{1/2}\alpha_2 \\ \epsilon 2^{1/2}(\mu_1 - \alpha_1) & \epsilon 2^{1/2}\alpha_2 & \epsilon'\mu_2 \end{pmatrix} \right] \\ & \text{b) } ((N)) = \left[\begin{pmatrix} \alpha_1 & \alpha_2 & -\epsilon\epsilon' 2^{1/2}\mu_2 \\ -\epsilon\alpha_2 & \alpha_1 & \cdot \\ \cdot & \cdot & \mu_1 \end{pmatrix} \right] \\ & \text{c) } ((S)) = \text{Diag}(\cdot \quad \cdot \quad -\epsilon'\mu_2). \end{aligned}$$

The proof will be accomplished in four steps.

First step. The canonical matrix $((R_a^b))$ of the curvature tensor belonging to (8.1) is given by the elements

$$(8.8) \quad R_a^b = \alpha(\delta_a^1{}^b{}_1 + \delta_a^2{}^b{}_2) + \delta_a^2{}^b{}_1 + \bar{\alpha}(\delta_a^3{}^b{}_3 + \delta_a^4{}^b{}_4) + \delta_a^4{}^b{}_3 + \mu \delta_a^5{}^b{}_5 + \bar{\mu} \delta_a^6{}^b{}_6.$$

Second step with inserted arithmetic change leads to

$$(8.9) \quad \begin{aligned} \text{a)} \quad g_{ab} &= 2\epsilon i(\delta_{(ab)}^{34} - \delta_{(ab)}^{12}) + \epsilon' i(\delta_{ab}^{66} - \delta_{ab}^{55}) \\ \text{b)} \quad g^{ab} &= 2\epsilon i(\delta_{12}^{(ab)} - \delta_{34}^{(ab)}) + \epsilon' i(\delta_{ab}^{55} - \delta_{ab}^{66}). \end{aligned}$$

Using (8.8) and (8.9)a we arrive at

$$(8.10) \quad R_{ab} = 2\epsilon i(\bar{\alpha} \delta_{(ab)}^{34} - \alpha \delta_{(ab)}^{12}) + \epsilon i(\delta_{ab}^{44} - \delta_{ab}^{22}) + \epsilon' i(\bar{\mu} \delta_{ab}^{66} - \mu \delta_{ab}^{55}).$$

Third Step. According to Agreement (1.1) we must have by virtue of (8.10)

$$(8.11) \quad \text{a)} \quad \check{\epsilon}^{12} = 0, \quad \text{b)} \quad \check{\epsilon}^{34} = 0, \quad \text{c)} \quad \check{\epsilon}^{55} = \check{\epsilon}^{66} = 0, \quad \text{d)} \quad \check{\epsilon}^{22} = \check{\epsilon}^{44}.$$

On the other hand, if we put

$$\delta^5 = \mathbf{V}, \quad \delta^6 = \bar{\mathbf{V}}$$

in Lemma (3.2)c, we may write by virtue of (3.10)a for $X = i$, $Y = iii$,

$$(8.12) \quad ((\delta_A^x)) = 2^{-1/2} \left[\begin{pmatrix} i2^{1/2}\epsilon & \cdot & \epsilon' & \cdot & \cdot & i \\ -i2^{1/2}\epsilon & \cdot & \epsilon' & \cdot & \cdot & -i \end{pmatrix} \right], \quad x = 5, 6.$$

Substituting into Lemma (2.3) δ^2 and δ^4 for $\mathbf{V}$ and $\bar{\mathbf{V}}$ and taking into account Agreement (2.1) we are encouraged to put

$$(8.13) \quad ((\delta_A^y)) = 2^{-1/2} \left[\begin{pmatrix} 1 & i & \cdot & \cdot & \cdot & \cdot \\ 1 & -i & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \right], \quad y = 2, 4.$$

The vectors (8.12) and (8.13) satisfy the corresponding equations (8.9), (8.11). These vectors and the remaining equations (8.9) and (8.11) define uniquely the vectors δ^1 , δ^3 so that the canonical frame is given by

$$(8.14) \quad ((\delta_A^a)) = 2^{-1/2} \left[\begin{pmatrix} \cdot & \cdot & -i2^{1/2} & i\epsilon & \epsilon & \cdot \\ 1 & i & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & i2^{1/2} & -i\epsilon & \epsilon & \cdot \\ 1 & -i & \cdot & \cdot & \cdot & \cdot \\ i\epsilon 2^{1/2} & \cdot & \epsilon' & \cdot & \cdot & i \\ -i\epsilon 2^{1/2} & \cdot & \epsilon' & \cdot & \cdot & -i \end{pmatrix} \right].$$

Fourth step. Using (8.10), we obtain

$$(8.15) \quad R_{AB} = R_{ab} \delta^{ab} \\ = 2\epsilon i(\bar{\alpha} \delta^{34} - \alpha \delta^{12}) + \epsilon i(\delta^{44} - \delta^{22}) + \epsilon' i(\bar{\mu} \delta^{66} - \mu \delta^{55}).$$

Substituting into (8.15) from (8.14) we arrive at (8.7).

Theorem (8.2). *The matrix $((R_{AB}))$ given by (1.4)b and (8.7) is a principal matrix.*

Proof. Lemmas (3.2)ac show that δ^5 and δ^6 cannot have less than three Substantials. The equations (8.12) are nothing else than a special case of (3.10)a. By means of U -permutations displayed in Lemma (4.1)b and applied to (8.12), one can get all six possibilities condensed in (3.10)ac (cf. Lemma (11.1)a). Hence the number of Substantials in a canonical frame is independent of the choice of one of the six possible distributions (3.10)ac in δ^5 and δ^6 so that we may start with (8.12) without restriction of generality. On the other hand, if we denote by W_A the vector denoted in (3.10)b by V_A , we see that the distribution of real and complex Substantials in the vectors $\bar{W}_A i$ and (3.10)a is the same (cf. also Lemma (11.1)b). This shows that it is irrelevant for the number of Substantials in a canonical frame if we start with (3.10)ac or (3.10)bc. The number of Substantials in the vectors δ^2 and δ^4 cannot be more reduced because these vectors are complex. Hence the only possibility of reducing the number of Substantials in a canonical frame lies in another configuration of 10 Substantials of the vectors $\delta^2, \delta^4, \delta^5, \delta^6$ which could reduce the number of Substantials of δ^1, δ^3 . Therefore we must consider all possible distributions of Substantials in δ^2, δ^4 with respect to δ^5, δ^6 which we can take without the restriction of generality in the form (8.12). Put

$$(8.16) \quad ((\bar{\delta}_A^4)) = ((\delta_A^2)) = ((m \ n \ p \ q \ r \ s))$$

with the understanding that δ^2 has only two Substantials. Because of $g^{22} = 0$, we cannot have Substantials p, s . Because of $g^{52} = g^{62} = 0$, we must have

$$s = 0, p + \epsilon 2^{1/2} q = 0.$$

The assumption that δ^2 is a complex vector with two Substantials only leads immediately to $s = p = q = 0$. This leaves us with three pairs of Substantials, namely

$$(8.17) \quad mn, \quad nr, \quad mr.$$

The first pair has been already considered (cf. (8.13)), the second pair is not admissible because of $g^{22} = 0$. The last pair leads by virtue of (8.11)d to

$$(8.18) \quad ((\bar{\delta}_A^4)) = ((\delta_A^2)) = 2^{-1/2}((1 \ \cdot \ \cdot \ \cdot \ iv \ \cdot)), \quad v \neq 0,$$

so that according to (8.9)b we get

$$(8.19) \quad ((\delta_A^1)) = 2^{-1/2}((a \ \epsilon/v \ -i2^{1/2} \ \epsilon i \ -aiv \ \cdot)),$$

where a is arbitrary. Hence even if $a = 0$, we do not have less than three Sub-

stantials in δ^1 (and in $\delta^3 = \bar{\delta}^1$). Hence no distribution of two Substantials in δ^2 and $\delta^4 = \bar{\delta}^2$ can reduce the number of Substantials in δ^1 and δ^3 to $M < 3$. Therefore it follows easily that the number of Substantials in a canonical frame cannot be reduced to a number $N < 16$ and consequently (8.14) defines a principal canonical frame.

9. Characteristic [2 2 1 $\bar{1}$]. The last characteristic of the type II to be considered is

$$(9.1) \quad [2 \ 2 \ 1 \ \bar{1}].$$

The complex conjugate eigenvectors belonging to complex conjugate eigenvalues

$$\mu = \mu_1 + i\mu_2, \quad \bar{\mu} = \mu_1 - i\mu_2$$

will be denoted by

$$\begin{aligned} \delta^5, \quad \delta^6 &= \bar{\delta}^5 \\ \delta_5, \quad \delta_6 &= \bar{\delta}_5. \end{aligned}$$

Theorem (9.1). Put

$$\epsilon, \epsilon' = \pm 1.$$

Then the curvature tensor belonging to (9.1) is given by (1.4)ab, where

$$(9.2) \quad ((N)) = \begin{bmatrix} \gamma & \cdot & -\epsilon' 2^{1/2} \mu_2 \\ \cdot & \alpha & \cdot \\ \cdot & \cdot & \mu_1 \end{bmatrix}$$

and either

$$\begin{aligned} (9.3)a \quad ((M)) &= \begin{bmatrix} \epsilon - 2\epsilon' \mu_2 & \cdot & 2^{1/2}(\mu_1 - \gamma) \\ \cdot & \cdot & \cdot \\ 2^{1/2}(\mu_1 - \gamma) & \cdot & \epsilon' \mu_2 \end{bmatrix} \\ ((S)) &= \text{Diag } ((\cdot \quad \epsilon \quad -\epsilon' \mu_2)) \end{aligned}$$

or

$$\begin{aligned} (9.3)b \quad ((M)) &= \begin{bmatrix} -\epsilon - 2\epsilon' \mu_2 & \cdot & 2^{1/2}(\mu_1 - \gamma) \\ \cdot & \epsilon & \cdot \\ 2^{1/2}(\mu_1 - \gamma) & \cdot & \epsilon' \mu_2 \end{bmatrix} \\ ((S)) &= \text{Diag } ((\cdot \quad \cdot \quad -\epsilon' \mu_2)). \end{aligned}$$

The proof will be accomplished in four steps.

First step. The canonical matrix $((R_a^b))$ of the curvature tensor belonging to (9.1) is given by the elements

$$(9.4) \quad R_a{}^b = \alpha(\delta_a^1{}^b + \delta_a^2{}^b) + \delta_a^2{}^b + \gamma(\delta_a^3{}^b + \delta_a^4{}^b) + \delta_a^4{}^b + \mu \delta_a^5{}^b + \bar{\mu} \delta_a^6{}^b.$$

Second step with inserted arithmetic change leads to

$$(9.5) \quad \begin{aligned} \text{a)} \quad g_{ab} &= 2\epsilon_1 \delta_{(ab)}^{12} + 2\epsilon_2 \delta_{(ab)}^{34} + \epsilon' i(\delta_{ab}^{66} - \delta_{ab}^{55}) \\ \text{b)} \quad g^{ab} &= 2\epsilon_1 \delta_{12}^{(ab)} + 2\epsilon_2 \delta_{34}^{(ab)} + \epsilon' i(\delta_{55}^{ab} - \delta_{66}^{ab}). \end{aligned}$$

Using (9.4) and (9.5)a, we arrive at

$$(9.6) \quad R_{ab} = 2\alpha\epsilon_1 \delta_{(ab)}^{12} + \epsilon_1 \delta_{(ab)}^{22} + 2\gamma\epsilon_2 \delta_{(ab)}^{34} + \epsilon_2 \delta_{ab}^{44} + \epsilon' i(\bar{\mu} \delta_{ab}^{66} - \mu \delta_{ab}^{55}).$$

Third step. By virtue of (9.6) and Agreement (1.1), we must have

$$(9.7) \quad \text{a)} \quad \check{\epsilon}^{12} = 0 \quad \text{b)} \quad \check{\epsilon}^{34} = 0 \quad \text{c)} \quad \check{\epsilon}^{55} = \check{\epsilon}^{66} = 0$$

and

$$(9.8) \quad \epsilon_1 \check{\epsilon}^{22} + \epsilon_2 \check{\epsilon}^{44} = 0.$$

Taking into account (9.7)c and (9.5)b for $ab = 55, 66$ we may put according to Lemmas (3.2)ac

$$(9.9) \quad ((\delta_A^x)) = 2^{-1/2} \left[\begin{pmatrix} i2^{1/2} & \cdot & \epsilon' & \cdot & \cdot & i \\ -i2^{1/2} & \cdot & \epsilon' & \cdot & \cdot & -i \end{pmatrix} \right], \quad x = 5, 6.$$

From now on, we have to distinguish two different cases. Either

$$(9.10) \quad \text{a)} \quad \epsilon_1 = \epsilon_2 = \epsilon$$

or

$$(9.10) \quad \text{b)} \quad \epsilon_1 = -\epsilon_2 = \epsilon$$

so that (9.8) splits into

$$(9.11) \quad \begin{aligned} \text{a)} \quad \check{\epsilon}^{22} + \check{\epsilon}^{44} &= 0 & \text{for (9.10)a} \\ \text{b)} \quad \check{\epsilon}^{22} - \check{\epsilon}^{44} &= 0 & \text{(9.10)b} \end{aligned}$$

These conditions, as well as (9.5)b for $a, b = 2, 4, 5, 6$ are met by the real vectors

$$(9.12) \quad ((\delta_A^y)) = \left[\begin{pmatrix} \cdot & \rho & \cdot & \cdot & \sigma & \cdot \\ -\epsilon & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \right], \quad y = 2, 4$$

where

$$(9.13) \quad \begin{aligned} \text{a)} \quad \rho &= 0 \quad \sigma = \epsilon & \text{for (9.10)a} \\ \text{b)} \quad \rho &= \epsilon \quad \sigma = 0 & \text{(9.10)b} \end{aligned}$$

The vectors δ^1 and δ^3 result from (9.5)b, (9.7), (9.9) and (9.12)

$$\begin{aligned}
 (9.14) \quad a) \quad ((\delta_A^z)) &= \left[\begin{pmatrix} a & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2^{1/2} & -1 & -a & \cdot \end{pmatrix} \right] & (9.10)a \\
 b) \quad ((\delta_A^z)) &= \left[\begin{pmatrix} a & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & -a & -2^{1/2} & 1 & \cdot & \cdot \end{pmatrix} \right] & (9.10)b \\
 & z = 1, 3
 \end{aligned}$$

where a is an arbitrary constant not defined by the above mentioned conditions. Putting $a = 0$ we arrive at the canonical frame

$$\begin{aligned}
 (9.15) \quad a) \quad ((\delta_A^a)) &= \left[\begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \epsilon & \cdot \\ \cdot & \cdot & 2^{1/2} & -1 & \cdot & \cdot \\ -\epsilon & \cdot & \cdot & \cdot & \cdot & \cdot \\ i & \cdot & \epsilon'/2^{1/2} & \cdot & \cdot & i/2^{1/2} \\ -i & \cdot & \epsilon'/2^{1/2} & \cdot & \cdot & -i/2^{1/2} \end{pmatrix} \right] & (9.10)a \\
 b) \quad ((\delta_A^a)) &= \left[\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \epsilon & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -2^{1/2} & 1 & \cdot & \cdot \\ -\epsilon & \cdot & \cdot & \cdot & \cdot & \cdot \\ i & \cdot & \epsilon'/2^{1/2} & \cdot & \cdot & i/2^{1/2} \\ -i & \cdot & \epsilon'/2^{1/2} & \cdot & \cdot & -i/2^{1/2} \end{pmatrix} \right] & (9.10)b.
 \end{aligned}$$

Fourth step. Using (9.6), we arrive at

(9.16)

$$R_{AB} = R_{ab} \delta^{ab} = 2\epsilon_1 \alpha \delta^{(12)} + \epsilon_1 \delta^{22} + 2\gamma \epsilon_2 \delta^{(34)} + \epsilon_2 \delta^{44} + \epsilon' i (\bar{\mu} \delta^{66} - \mu \delta^{55}).$$

Substituting into this equation from (9.10)a and (9.15)a (from (9.10)b and (9.15)b), we arrive at (9.2), (9.3)a (at (9.2), (9.3)b).

10. Digression. Theorem (9.1) yields two curvature tensors belonging to the same characteristic (9.1). The question arises whether one of these tensors is not redundant in the sense that it can be deduced from the other by a transformation of the group ${}^*\mathcal{G}$. We remind the reader that two tensors $T_{AB\dots M}$ and ${}^*T_{AB\dots M}$ are called ${}^*\mathcal{G}$ -equivalent if they can be transformed into each other by a transformation of the group ${}^*\mathcal{G}$ (cf. Definition (I(6.2))).

Theorem (10.1). *The curvature tensors given by (1.4)ab and*

$$(10.1) \quad \begin{array}{l} \text{a) } (9.2), \quad (9.3)\text{a} \quad \text{or} \\ \text{b) } (9.2), \quad (9.3)\text{b} \end{array}$$

are not ${}^*\mathcal{G}$ -equivalent.

Proof. In Theorem (I(6.3)) we introduced the group $\mathcal{H}$. The concept of $\mathcal{H}$ -equivalence has been introduced by Definition (I(6.2)). According to this Definition, the tensors

$$(10.2) \quad \begin{array}{l} \text{a) } (9.5), \quad (9.10)\text{a} \quad \text{and} \\ \text{b) } (9.5), \quad (9.10)\text{b} \end{array}$$

are not $\mathcal{H}$ -equivalent. On the other hand, the groups ${}^*\mathcal{G}$ and $\mathcal{H}$ are “similar” in the sense defined in Theorem (I(6.3)). Hence the tensor (10.1)a, based on (10.2)a, and the tensor (10.1)b, based on (10.2)b, are not ${}^*\mathcal{G}$ -equivalent.

11. Principal matrices ((R_{AB})). In the eighth Section we anticipated results of some Lemmas which we shall prove now because we shall make extensive use of them. The first Lemma is concerned with the distribution of Substantials in the vectors given by (3.10)ac.

Lemma (11.1)a. *Any two arbitrary vectors out of the six vectors given by (3.10)ac are related by a U -permutation (or by a sequence of U -permutations).*

Proof. Denoting by u, v, w the Substantials of the vectors (3.10)ac, we can describe the distribution of these Substantials by the six rows of the following self explanatory table.

$$(11.2) \quad \begin{array}{cccccc} u & v & \cdot & \cdot & w & \cdot \\ u & \cdot & v & \cdot & \cdot & w \\ v & u & \cdot & w & \cdot & \cdot \\ v & \cdot & u & w & \cdot & \cdot \\ \cdot & u & v & \cdot & \cdot & w \\ \cdot & v & u & \cdot & w & \cdot \end{array}$$

Two arbitrary out of these six rows are related by a U -permutation (or by a sequence of U -permutations) described in Lemma (4.1)b. Indeed every such U -permutation preserves not only g_{AB} but also either one of the sums

$$(11.3) \quad \sum_{Z=1}^{Z=iii} (ZZ), \quad \sum_{Z=iv}^{Z=vi} (ZZ).$$

In the next Lemma we shall introduce the concept of “elementary change”. By it we understand a construction of a new fundamental K -frame by multiplying the elements of an old fundamental K -frame by an appropriate number.

Lemma (11.1)b. Denote by W_A the complex vector (3.10)b. The components W_A may be obtained from the components $i\bar{V}_A$ (V_A being given by (3.10)a) by the elementary change

$$(11.4) \quad \mathbf{P}^{Y1} \rightarrow \epsilon' \mathbf{P}^{Y1}, \quad \mathbf{P}^{Y+iii1} \rightarrow \epsilon' \mathbf{P}^{Y+iii1}.$$

Proof. We have from (3.10)ab

$$(11.5) \quad \begin{aligned} i\bar{V}_A &= 2^{-1/2}(2^{1/2}\epsilon \delta_A^X + i\epsilon' \delta_A^Y + \delta_A^{Y+iii}), \quad \text{c.c.} \\ W_A &= 2^{-1/2}(2^{1/2}\epsilon \delta_A^X + i \delta_A^Y + \epsilon' \delta_A^{Y+iii}) \end{aligned}$$

The elementary change (11.4) follows immediately from (11.5).

Remark (11.1). The elementary change (11.4) yields a new fundamental K -frame which satisfies also the requirements displayed in the Agreement (3.1).

In the next Lemma we denote Substantials by the symbol $*$.

Lemma (11.2). Let the vectors δ^5 and δ^6 be given by (9.9). Let each of the vectors δ^2 , δ^4 have only one Substantial. Then there are only two possible distributions of these Substantials, namely

$$(11.6) \quad \begin{aligned} \text{a)}_1 \quad ((\delta_A^y)) &= \left(\begin{bmatrix} * & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & * & \cdot \end{bmatrix} \right) \\ \text{a)}_2 \quad ((\delta_A^y)) &= \left(\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & * & \cdot \\ * & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \right) \end{aligned} \quad \text{for (9.11)a,} \quad y = 2, 4,$$

and

$$(11.6) \quad \begin{aligned} \text{b)}_1 \quad ((\delta_A^y)) &= \left(\begin{bmatrix} * & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & * & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \right) \\ \text{b)}_2 \quad ((\delta_A^y)) &= \left(\begin{bmatrix} \cdot & * & \cdot & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \right) \end{aligned} \quad \text{for (9.11)b,} \quad y = 2, 4.$$

In either one of these cases, the Substantials under consideration have the same absolute value.

Proof. The vectors δ^2 , δ^4 being perpendicular to δ^5 , δ^6 we must have according to (9.9)

$$((\delta_A^y)) = ((a_y \quad b_y \quad -2^{1/2} d_y \quad d_y \quad e_y \quad \cdot)), \quad y = 2, 4.$$

The condition that each of these two vectors has only one Substantial requires $d = 0$ so that

$$(11.7) \quad ((\delta_A^y)) = ((a_y \quad b_y \quad \cdot \quad \cdot \quad e_y \quad \cdot)), \quad y = 2, 4.$$

The conditions

$$(11.8) \quad g^{22} = g^{44} = g^{24} = 0$$

combined with (9.11)a (with (9.11)b) lead immediately to (11.6)_{a₁a₂} (to (11.6)_{b₁b₂}). The conditions (9.11) show that the absolute values of the Substantials under consideration must be equal.

The results of previous Lemmas enable us to prove the following Theorem.

Theorem (11.1). *The matrix $((R_{AB}))$ given by (1.4)b and (10.1)a ((1.4)b and (10.1)b) is a principal matrix.*

Proof. According to Lemmas (3.2)ac, the number of Substantials in δ^5 and δ^6 cannot be less than 3. According to Lemma (11.1)a, the number of Substantials in the canonical frame is independent of the admissible distributions of Substantials in δ^5 and δ^6 as it is displayed in (3.10)ac. Hence, taking into account Lemmas (11.1)ab, we may stick without restriction of generality to (9.9). From (9.12) we know that the least number of Substantials in δ^2 (in δ^4) is one. Lemma (11.2) gives us all possible distribution of these Substantials. The distribution (11.6)_{a₂} (the distribution (11.6)_{b₂}) has been previously considered (cf. (9.12)). The distribution (11.6)_{a₁} (the distribution (11.6)_{b₁}) leads obviously to the same minimal number 3 of Substantials of vectors δ^1 and δ^3 as did the distribution (9.12). Hence we arrive again at the number 11 of Substantials in a canonical frame. This proves that (9.15) is a principal canonical frame.

CHARACTERISTICS OF THE TYPE IV

12. Characteristic [3 $\bar{3}$]. According to the table (1.2), there is only one basic complex characteristic of the type IV, namely

$$(12.1) \quad [3 \quad \bar{3}].$$

This characteristic leads to two complex conjugate triple eigenvalues

$$(12.2) \quad \alpha = \alpha_1 + i\alpha_2, \quad \bar{\alpha} = \alpha_1 - i\alpha_2.$$

The corresponding curvature tensor is of the structure

$$(12.3) \quad R_A{}^B = \alpha(\delta_A^1 \delta_1^B + \delta_A^2 \delta_2^B + \delta_A^3 \delta_3^B) \\ + \bar{\alpha}(\delta_A^4 \delta_4^B + \delta_A^5 \delta_5^B + \delta_A^6 \delta_6^B) \\ + \delta_A^2 \delta_1^B + \delta_A^3 \delta_2^B \\ + \delta_A^5 \delta_4^B + \delta_A^6 \delta_5^B.$$

The vectors

$$(12.4) \quad \delta_1^A, \quad \delta_4^A \\ \delta_B^3, \quad \delta_B^6$$

are obviously the eigenvectors belonging to the eigenvalues (12.2). The remaining vectors involved in (12.3) are auxiliary vectors. In the next lemma we shall consider the relationships of these vectors.

Lemma (12.1). *The following vectors are complex conjugate*

$$(12.5) \quad \begin{array}{ll} \text{a)} & \delta_1 \text{ and } \delta_4 ; \quad \delta^3 \text{ and } \delta^6 \\ \text{b)} & \delta_2 \text{ and } \delta_5 ; \quad \delta^2 \text{ and } \delta^5 \\ \text{c)} & \delta_3 \text{ and } \delta_6 ; \quad \delta^1 \text{ and } \delta^4. \end{array}$$

Proof. The statement (a) follows immediately from the fact that the vectors under consideration are eigenvectors belonging to the complex conjugate eigenvalues. This statement enables us to rewrite the last two rows in (12.3) as follows

$$\delta_A^2 \delta_1^B + \delta_A^5 \bar{\delta}_1^B + \delta_A^3 \delta_2^B + \bar{\delta}_A^3 \delta_5^B.$$

This sum must be real, which is possible if and only if the statement (12.5)b holds. The statements (12.5)ab enable us to rewrite the first two rows in (12.3) as follows

$$\begin{aligned} & \alpha(\delta_A^1 \delta_1^B + \delta_A^2 \delta_2^B + \delta_A^3 \delta_3^B) \\ & \bar{\alpha}(\delta_A^4 \bar{\delta}_1^B + \bar{\delta}_A^2 \bar{\delta}_2^B + \bar{\delta}_A^3 \bar{\delta}_6^B). \end{aligned}$$

This sum must be real, which is possible if and only if the statements (12.5)c holds.

Theorem (12.1). *Put*

$$\epsilon = \pm 1.$$

Then under a certain assumption (cf. (12.11)) the curvature tensor belonging to (12.1) is given by (1.4)ab where

$$(12.6) \quad \begin{array}{ll} \text{a)} & ((M)) = \left[\begin{pmatrix} \epsilon\alpha_2 & \cdot & \epsilon/2^{1/2} \\ \cdot & \epsilon\alpha_2 & \cdot \\ \epsilon/2^{1/2} & \cdot & -\epsilon\alpha_2 \end{pmatrix} \right] \\ \text{b)} & ((N)) = \left[\begin{pmatrix} \alpha_1 & \alpha_2 & \cdot \\ -\alpha_2 & \alpha_1 & -2^{-1/2} \\ \cdot & 2^{1/2} & \alpha_1 \end{pmatrix} \right] \\ \text{c)} & ((S)) = \left[\begin{pmatrix} \cdot & \cdot & \epsilon 2^{1/2} \\ \cdot & \cdot & \cdot \\ \epsilon 2^{1/2} & \cdot & \epsilon\alpha_2 \end{pmatrix} \right]. \end{array}$$

The proof will be accomplished in four steps.

First step. The canonical matrix $((R_a^b))$ of the curvature tensor belonging to (12.1) is given by the elements

$$(12.7) \quad R_a{}^b = \alpha(\delta_a^1{}_1{}^b + \delta_a^2{}_2{}^b + \delta_a^3{}_3{}^b) + \delta_a^2{}_1{}^b + \delta_a^3{}_2{}^b \\ + \bar{\alpha}(\delta_a^4{}_4{}^b + \delta_a^5{}_5{}^b + \delta_a^6{}_6{}^b) + \delta_a^5{}_4{}^b + \delta_a^6{}_5{}^b.$$

Second step with inserted arithmetic change leads to

$$(12.8) \quad \begin{aligned} \text{a)} \quad g_{ab} &= -\epsilon i(2 \delta_{(ab)}^{13} + \delta_{ab}^{22}) + \epsilon i(2 \delta_{(ab)}^{46} + \delta_{ab}^{55}) \\ \text{b)} \quad g^{ab} &= \epsilon i(2 \delta_{13}^{(ab)} + \delta_{22}^{ab}) - \epsilon i(2 \delta_{46}^{(ab)} + \delta_{55}^{ab}). \end{aligned}$$

Using (12.7) and (12.8)a, we obtain

$$(12.9) \quad R_{ab} = -\epsilon i \alpha(2 \delta_{(ab)}^{13} + \delta_{ab}^{22}) + \epsilon i \bar{\alpha}(2 \delta_{(ab)}^{46} + \delta_{ab}^{55}) - 2\epsilon i(\delta_{(ab)}^{23} - \delta_{(ab)}^{56}).$$

Third step. Whatever the sign ϵ is, we must have by virtue of (12.9) and Agreement (1.1)

$$(12.10) \quad \text{a)} \quad 2\check{\epsilon}^{13} + \check{\epsilon}^{22} = 0 \quad \text{b)} \quad 2\check{\epsilon}^{46} + \check{\epsilon}^{55} = 0 \quad \text{c)} \quad \check{\epsilon}^{23} = \check{\epsilon}^{56}.$$

According to (12.8)b, the vectors δ^2 and δ^5 are not null vectors of g_{ab} . The conditions (12.10) do not say anything about their behaviour with respect to e^{AB} . *We shall assume for the time being that*

$$(12.11) \quad \delta^2 \times \delta^2 \neq 0, \quad c.c.$$

According to this assumption and by virtue of Lemma (3.1), the least number of Substantials of each one of these vectors is two, and we may put according to (12.8)b and (3.2)

$$(12.12) \quad ((\bar{\delta}_A^5)) = ((\delta_A^2)) = 2^{-1/2}((\cdot \quad \cdot \quad i \quad \cdot \quad \cdot \quad \epsilon)).$$

The remaining vectors being perpendicular to δ^2 , δ^5 , we must have

$$(12.13)\text{a} \quad ((\bar{\delta}_A^{x+3})) = ((\delta_A^x)) = \frac{1}{2}((a_x \quad b_x \quad \cdot \quad d_x \quad e_x \quad \cdot)), \quad x = 1, 3,$$

with the condition

$$(12.13)\text{b} \quad 2g^{xx} = a_x d_x + b_x e_x = 0.$$

This condition is in particular satisfied by

$$(12.14) \quad ((\bar{\delta}_A^4)) = ((\delta_A^1)) = \frac{1}{2}((1 \quad i \quad \cdot \quad \cdot \quad \cdot \quad \cdot)).$$

The conditions

$$(12.15)\text{a} \quad g^{13} = \epsilon i, \quad g^{43} = 0$$

yield

$$(12.15)\text{b} \quad e_3 = 2\epsilon, \quad d_3 = 2\epsilon i$$

so that

$$(12.15)\text{c} \quad g^{33} = \epsilon(a_3 i + b_3) = 0.$$

On the other hand, (12.14), (12.12) and (12.10)a lead to

$$(12.15)d \quad a_3 + ib_3 = 2$$

so that (12.15)ed yield

$$(12.15)e \quad a_3 = 1, \quad b_3 = -i.$$

The equations (12.13)a, (12.15)be lead directly to the components of δ^3 and $\delta^9 = \bar{\delta}^3$ so that the canonical frame is given by

$$(12.16) \quad ((\delta_A^a)) = \frac{1}{2} \begin{pmatrix} 1 & i & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & i2^{1/2} & \cdot & \cdot & \epsilon 2^{1/2} \\ 1 & -i & \cdot & 2i\epsilon & 2\epsilon & \cdot \\ 1 & -i & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -i2^{1/2} & \cdot & \cdot & \epsilon 2^{1/2} \\ 1 & i & \cdot & -2i\epsilon & 2\epsilon & \cdot \end{pmatrix}.$$

Fourth step. Using (12.9) we obtain

$$(12.17) \quad R_{AB} = R_{ab} \delta^{ab} \\ = -\epsilon i \alpha (2 \delta^{(13)} + \delta^{22}) + \epsilon i \bar{\alpha} (2 \delta^{(46)} + \delta^{55}) - 2\epsilon i (\delta^{(23)} - \delta^{(56)}).$$

Substituting into (12.17) from (12.16), one arrives at (12.6).

13. Auxiliary lemmas. In this section we shall derive some auxiliary results needed in the next section.

Lemma (13.1). *Any two arbitrary vectors out of the three vectors given by (3.2) are related by a U -permutation (or by a sequence of U -permutations).*

Proof. Taking into account (3.2)b, we see that the indices of the three vectors (3.2)a may be arranged according to the scheme

$$(13.1) \quad \begin{array}{cccccc} X & \cdot & \cdot & X + iii & \cdot & \cdot \\ i & ii & iii & iv & v & vi \\ ii & i & iii & v & iv & vi \\ iii & i & ii & vi & iv & v \end{array}$$

Two arbitrary out of three rows of this table are related by a U -permutation (or by a sequence of U -permutations). The proof of this statement follows the pattern of the proof of Lemma (4.1)b.

Remark (13.1). If $\mathbf{V}$ is a vector with two Substantials, f.i.

$$(13.2)a \quad ((V_A)) = ((\epsilon \ i \ \cdot \ \cdot \ \cdot \ \cdot)), \quad \epsilon = \pm 1,$$

then

$$(13.2)b \quad \epsilon i ((\bar{V}_A)) = ((i \ \epsilon \ \cdot \ \cdot \ \cdot \ \cdot)).$$

If $\mathbf{V}$ is an element of a canonical frame, so is also $\bar{\mathbf{V}}$. On the other hand, an element of a canonical frame may be multiplied by a non zero factor. Hence $\epsilon i \bar{\mathbf{V}}$ may be regarded also as an element of a canonical frame. Hence from the point of view of the construction of a canonical frame it is irrelevant if we start with (13.2)a or (13.2)b. In the following lemmas which will be proven simultaneously, we shall consider only one of these possibilities.

Lemma (13.2)a. *There are only three possible types of the distribution of the Substantials of the vector (2.8)a, namely*

$$(13.3) \quad \begin{array}{ll} \text{a)} & ((1 \quad i \quad \cdot \quad \cdot \quad \cdot \quad \cdot)) \\ \text{b)} & ((1 \quad \cdot \quad i \quad \cdot \quad \cdot \quad \cdot)) \\ \text{c)} & ((\cdot \quad 1 \quad i \quad \cdot \quad \cdot \quad \cdot)) \end{array}$$

$$(13.4) \quad \begin{array}{ll} \text{a)} & ((\cdot \quad \cdot \quad \cdot \quad 1 \quad i \quad \cdot)) \\ \text{b)} & ((\cdot \quad \cdot \quad \cdot \quad 1 \quad \cdot \quad i)) \\ \text{c)} & ((\cdot \quad \cdot \quad \cdot \quad \cdot \quad 1 \quad i)) \end{array}$$

and

$$(13.5) \quad \begin{array}{ll} \text{a)} & ((1 \quad \cdot \quad \cdot \quad \cdot \quad i \quad \cdot)) \\ \text{b)} & ((1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad i)) \\ \text{c)} & ((\cdot \quad 1 \quad \cdot \quad \cdot \quad \cdot \quad i)) \\ \text{d)} & ((\cdot \quad 1 \quad \cdot \quad i \quad \cdot \quad \cdot)) \\ \text{e)} & ((\cdot \quad \cdot \quad 1 \quad i \quad \cdot \quad \cdot)) \\ \text{f)} & ((\cdot \quad \cdot \quad 1 \quad \cdot \quad i \quad \cdot)) \end{array}$$

Lemma (13.2)b. *Every two arbitrary matrices of (13.3) or (13.4) or (13.5) are related by a U-permutation introduced in Lemma (4.1)b or by a sequence of such U-permutations.*

Proof. Lemma (13.2)a follows immediately from (2.8)aef. The proof of Lemma (13.2)b follows the pattern of the proof of Lemma (13.1).

Lemma (13.3)a. *Let $\delta^2 = \bar{\delta}^5$ be given by (12.12) and $\delta^1 = \mathbf{V} = \bar{\delta}^4$ by (2.8)a. Then the following statements hold.*

(a) *The distribution of Substantials in δ^1 is given by either one of the matrices (13.3)a, (13.4)a, (13.5)ad.*

(b) *The vector $\delta^3 = \bar{\delta}^6$ has exactly four Substantials.*

(c) *In the case of (13.3)a or (13.4)a, the vectors δ_A^* satisfy (12.8)b and (12.10).*

(d) *In the cases (13.5)ad, the conditions (12.8)b and (12.10) are incompatible.*

Proof. The statement (a) follows immediately from (12.13)a and Lemma (13.2)a. The statements (b) and (c) have been proved for (13.3)a in the third step

of the proof of Theorem (12.1). Similar procedure proves these statements also for (13.4)a. In the cases (13.5)ad, the same procedure leads to incompatible conditions

$$\begin{aligned} 2g^{33} = \epsilon(a_3i + e_3) &= 0, & 2\check{e}^{13} = \tfrac{1}{2}(a_3 - ie_3) &= 1, \\ 2g^{33} = \epsilon(d_3 + ib_3) &= 0, & 2\check{e}^{13} = \tfrac{1}{2}(b_3 - id_3) &= 1. \end{aligned}$$

Lemma (13.3)b. *Lemma (13.3)a holds also for any distribution of Substantials which one obtains by applying to the matrices*

$$(13.6) \quad (12.12), \quad (13.3)a, \quad (13, 4)a, \quad (13.5)ad$$

the same U -permutation (or a sequence of U -permutations) introduced in Lemma (4.1)b.

Proof. A U -permutation is a transformation of the group ${}_*\mathcal{G}$ which transforms the canonical frame into another canonical frame of the same number of Substantials. Obviously the same U -permutation (or the same sequence of U -permutations) must be applied to the matrices (13.6) in order to obtain a meaningful result.

Remark (13.2). To avoid any misunderstanding, let us stress that if one of the matrices (13.3)bc, (13.4)bc, (13.5)bcef represents a possible distribution of Substantials in δ^1 , the vector δ^2 cannot be given by (12.12). However, a U -permutation (a sequence of U -permutations) which permutes for instance (13.4)a into (13.4)c permutes (12.12) into another matrix which represents the components of δ^2 . Similar remarks hold for the other matrices (13.3)–(13.5).

14. Principal matrix $((R_{AB}))$. Previous Lemmas enable us to prove the following important Theorem.

Theorem (14.1). *Under the assumption (12.11) the matrix $((R_{AB}))$ given by (1.4)b and (12.6) is a principal matrix.*

Proof. According to (12.11), the vector $\bar{\delta}^2 = \delta^5$ is not a null vector of e^{AB} . According to (12.8)b it is neither a null vector of g^{AB} . Hence Lemma (3.1) applies so that the least number of the Substantials of δ^2 is two and one of their distributions is (12.12). Let us start *for the time being* with the distribution (12.12). Then the vector $\delta^1 = \bar{\delta}^4$ must satisfy the equation (12.13)a (for $x = 1$). The least number of the Substantials of the complex vector is obviously two. Let us assume *for the time being* that their distribution is given by (13.3)a or (13.4)a. Then Lemma (13.3)a applies. It follows that both canonical frames constituted by the vectors δ_A^a mentioned in this Lemma have the same number of Substantials, namely 16 (cf. (12.16)). According to Lemma (13.3)b, *this number is independent of the possible distribution of the Substantials in $\delta^1, \delta^2, \delta^4, \delta^5$.* It is also obvious that we get the same number of Substantials of the canonical

frame if we replace the numbers 1, i , in (13.3) (13.4) by numbers $a + ib$, $c + id$. Hence the number 16 is the least number of Substantials in a canonical frame. Therefore each one of the six canonical frames constructed in the usual way by means of Lemmas (13.3)ab is a principal canonical frame. This statement holds in particular also for the canonical frame (12.16). Hence this frame is a principal canonical frame.

15. Other canonical frames. Previous considerations were based on the assumption (12.11). In this section we shall assume

$$(15.1) \quad \delta^2 \times \delta^2 = 0, \quad \text{c.c.},$$

so that according to Lemmas (3.2)ac we may put

$$(15.2) \quad ((\bar{\delta}^5)) = ((\delta^2)) = ((\epsilon \quad \cdot \quad i/2^{1/2} \quad \cdot \quad \cdot \quad \epsilon/2^{1/2})).$$

The consideration applied to the matrix of the structure (13.2)a holds also for the matrix

$$(15.3) \quad ((V_A')) = ((1 \quad \cdot \quad \cdot \quad \cdot \quad i \quad \cdot))$$

We shall apply it in the next Lemma.

Lemma (15.1). *If $\delta^1 = \bar{\delta}^4$ has only two Substantials proportional to 1, i , then their distribution is either*

$$(15.4) \quad \text{a) } ((\delta^1)) = \frac{1}{2}((1 \quad i \quad \cdot \quad \cdot \quad \cdot \quad \cdot))$$

or

$$(15.4) \quad \text{b) } ((\delta^1)) = \frac{1}{2}((1 \quad \cdot \quad \cdot \quad \cdot \quad i \quad \cdot)).$$

Proof. The null vector $\delta^1 = \bar{\delta}^4$ being perpendicular to δ^2 and δ^5 we must have

$$(15.5) \quad \begin{aligned} \text{a) } ((\delta^4)) &= ((\delta^1)) = \frac{1}{2}(a_1 \quad b_1 \quad -2^{1/2}d_1 \quad d_1 \quad e_1 \quad \cdot) \\ \text{b) } a_1 d_1 + b_1 e_1 &= 0. \end{aligned}$$

Our statement follows immediately from (15.5).

Remark (15.1). Because of

$$\delta^1 \cdot \delta^1 = \delta^1 \cdot \delta^4 = \delta^1 \cdot \bar{\delta}^1 = \delta^4 \cdot \delta^4 = 0$$

the requirement that δ^1 has only two Substantials and that these Substantials are proportional to 1, i is compatible with Lemma (2.3) and Agreement (2.1).

Lemma (15.2). *The vectors (15.2) and (15.4)a lead in the usual way to the canonical frame*

$$(15.6)a \quad ((\delta_A^a)) = \frac{1}{2} \begin{bmatrix} 1 & i & \cdot & \cdot & \cdot & \cdot \\ 2\epsilon & \cdot & i2^{1/2} & \cdot & \cdot & \epsilon 2^{1/2} \\ \cdot & \cdot & -i\epsilon 8^{1/2} & 2i\epsilon & 2\epsilon & \cdot \\ 1 & -i & \cdot & \cdot & \cdot & \cdot \\ 2\epsilon & \cdot & -i2^{1/2} & \cdot & \cdot & \epsilon 2^{1/2} \\ \cdot & \cdot & i\epsilon 8^{1/2} & -2i\epsilon & 2\epsilon & \cdot \end{bmatrix}.$$

The vectors (15.2) and (15.4)b lead in the usual way to the canonical frame

$$(15.6)b \quad ((\delta_A^a)) = \frac{1}{2} \begin{bmatrix} 1 & \cdot & \cdot & \cdot & i & \cdot \\ 2\epsilon & \cdot & i2^{1/2} & \cdot & \cdot & \epsilon 2^{1/2} \\ \cdot & 2\epsilon & -i\epsilon 8^{1/2} & 2i\epsilon & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & -i & \cdot \\ 2\epsilon & \cdot & -i2^{1/2} & \cdot & \cdot & \epsilon 2^{1/2} \\ \cdot & 2\epsilon & i\epsilon 8^{1/2} & -2i\epsilon & \cdot & \cdot \end{bmatrix}.$$

Proof. Using the vectors (15.2) and (15.4)a (the vectors (15.2) and (15.4)b) and applying to them the conditions (12.15)ac and (12.10), we obtain the components of the remaining two vectors δ^3 and δ^6 . In this way we get the canonical frame (15.6)a (the canonical frame (15.6)b).

Lemma (15.3). *Either one of the canonical frames (15.6)ab is a principal one.*

Proof. In both canonical frames the vectors δ^1 , δ^2 , δ^4 , δ^5 have the minimal number of Substantials and the vectors δ^3 and δ^6 have three Substantials. Hence in order to prove our statement, it is sufficient to prove that the number three of Substantials in either one of the vectors δ^3 , δ^6 is independent of the distribution of the Substantials in the remaining vectors of the frame. If we apply the U -permutations mentioned in Lemma (4.1)b to (15.2) and (15.4)a (to (15.2) and (15.4)b), we must obtain again the number three of Substantials in the vectors δ^3 , δ^6 . Moreover, an easy inspection shows that this number cannot be lowered if we replace in (15.4) the number 1, i by $a' + ia''$, $b' + ib''$ or choose for δ^2 one of the vectors (3.10)a.

Remark (15.2). While under the assumption (12.11), we arrived at the distributions (13.3), (13.4) and (13.5) of Substantials in δ^1 , the assumption (15.1) admits only the distributions (15.4). This discrepancy is caused by the different structures (12.13)a (for $x = 1$) and (15.5).

16. $\ast\mathfrak{g}$ -equivalence of the results. In [I] we denoted the coefficients of transformation of $\ast\mathfrak{g}$ and $\mathfrak{H}$ by U , $\check{U}$ and X , $\check{X}$. They act on δ_A^a in the following way

$$\begin{aligned}
 (16.1) \quad & \text{a) } {}_*\delta_A^a = \check{U}_A{}^B \delta_B^a, \quad \check{U}_A{}^B U_C{}^A = \delta_C{}^B \\
 & \text{b) } {}'\delta_A^a = X_b{}^a \delta_A^b \\
 & \text{c) } {}'\delta_A^a = X_b{}^a \delta_B^b \check{U}_A{}^B.
 \end{aligned}$$

(Cf. Agreement (I(4.1), Theorem (I(6.3)) and Theorem (I(6.5))). The transformations of ${}_*\mathcal{G}$ and of $\mathcal{H}$ preserve g_{AB} , g^{AB} and g_{ab} , g^{ab} . They are “similar” in the sense that

$$\begin{aligned}
 (16.2) \quad & \text{a) } X_a{}^b = \delta_a^A \check{U}_A{}^B \delta_B^b \\
 & \text{b) } \check{U}_A{}^B = \delta_A^a X_a{}^b \delta_b^B.
 \end{aligned}$$

In the next Theorem and its proof we shall use the concept of ${}_*\mathcal{G}$ -equivalence, of $\mathcal{H}$ -equivalence and of $({}_*\mathcal{G}, \mathcal{H})$ -equivalence (cf. Definition (I(6.2))).

Theorem (16.1). *Arbitrary two out of three canonical frames (12.16), (15.6)a and (15.6)b are $({}_*\mathcal{G}, \mathcal{H})$ -equivalent.*

Proof. Either one of the three canonical frames under consideration represents 6 linearly independent vectors in the six dimensional K -space. Hence they must be connected by transformations of the type

$$\begin{aligned}
 (16.3) \quad & \text{a) } {}'\delta_A^a = Y_b{}^a \delta_A^b \\
 & \text{b) } {}_*\delta_A^a = \check{V}_A{}^B \delta_B^a.
 \end{aligned}$$

All three canonical frames satisfy (12.8) and therefore (16.3)a leads immediately to

$$\begin{aligned}
 g^{ab} &= Y_c{}^a Y_d{}^b g^{cd} \\
 g_{ab} &= Y_a{}^c Y_b{}^d g_{cd}.
 \end{aligned}$$

Hence $Y_b{}^a$ and its inverse $\check{Y}_a{}^c$ represents a transformation of $\mathcal{H}$. From (16.3)ab we derive via

$${}'\delta_A^a = {}_*\delta_A^a$$

the relationship

$$(16.3)c \quad \check{V}_A{}^B = \delta_A^a Y_a{}^b \delta_b^B$$

which leads directly to

$$g_{CD} = g_{AB} \check{V}_C{}^A \check{V}_D{}^B.$$

Hence $\check{V}_C{}^A$ (and its inverse $V_A{}^B$) represents a transformation of ${}_*\mathcal{G}$. Using U and X instead of V and Y , we may condense (16.3)ab into (16.1)c, which proves our statement.

Remark (16.1). If we denote the elements of the canonical frame (12.16) by ${}_*\delta_A^a$ and those of (15.6)a by δ_A^a , then the transformation ${}_*\delta_A^a \rightarrow \delta_A^a$ is a transformation of ${}_*\mathcal{G}$

$$(16.4)\text{a} \quad {}_*\delta_A^a = \check{U}_A{}^B \delta_B^a$$

where

$$(16.4)\text{b} \quad ((\check{U}_A{}^B)) = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \epsilon/2 & -2^{1/2} \\ \cdot & 1 & \cdot & -\epsilon/2 & \cdot & \cdot \\ \cdot & \cdot & 1 & 2^{1/2} & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}.$$

(The index A [the index B] refers to the rows [to the columns] of the matrix.) An easy inspection shows that (16.4)b satisfies the condition

$$g_{AB} = \check{U}_A{}^C \check{U}_B{}^D g_{CD}$$

as it was to be expected.

Remark (16.2). The orthogonal components of two ${}_*\mathcal{G}$ -equivalent tensors

$$(16.5) \quad \begin{aligned} \text{a)} \quad R_{\Omega\Lambda} &= R_{AB} P_{\Omega}{}^A P_{\Lambda}{}^B \\ \text{b)} \quad {}_*R_{\Omega\Lambda} &= {}_*R_{AB} P_{\Omega}{}^A P_{\Lambda}{}^B \end{aligned}$$

are related by a transformation of ${}_*\mathcal{G}$

$$(16.6) \quad \text{a)} \quad {}_*R_{AB} = \check{U}_A{}^C \check{U}_B{}^D R_{CD}.$$

We know from Section (I.4) that

$$\backslash P_{\Omega}{}^C \stackrel{\text{def}}{=} P_{\Omega}{}^A \check{U}_A{}^C$$

is *also* a fundamental K -frame so that R_{AB} are orthogonal components of $R_{\Omega\Lambda}$ and ${}_*R_{\Omega\Lambda}$ with respect to two different fundamental K -frames. Indeed we have besides (16.5)a also

$${}_*R_{\Omega\Lambda} = R_{AB} \backslash P_{\Omega}{}^A \backslash P_{\Lambda}{}^B.$$

Hence $R_{\Omega\Lambda}$ and ${}_*R_{\Omega\Lambda}$ are *metrically congruent in the classical sense*. (They have the same components with respect to two metrically congruent K -frames; cf. also the fifth paragraph in Section I.) We remind the reader of this geometrical significance of the ${}_*\mathcal{G}$ -equivalence because it could be applied to the result of the next Theorem.

Denote by ${}^{(1)}R_{\Omega\Lambda} = R_{\Omega\Lambda}$ the curvature tensor given by (1.4)ab and (12.6) and by ${}^{(2)}R_{\Omega\Lambda}$ (by ${}^{(3)}R_{\Omega\Lambda}$) the curvature tensor obtained via (1.4)ab by substituting into (12.17) from (15.6)a (from (15.6)b).

Theorem (16.2). *Arbitrary two out of the three curvature tensors*

$$(16.7)\text{a} \quad {}^{(1)}R_{\Omega\Lambda}, \quad {}^{(2)}R_{\Omega\Lambda}, \quad {}^{(3)}R_{\Omega\Lambda}$$

are ${}_*\mathcal{G}$ -equivalent.

Proof. It is sufficient to prove that the orthogonal components of the arbitrary two out of the three curvature tensors (16.7)a are related by a transformation of the type (16.6)a. The proof of this statement follows immediately from the application of Theorem (16.1) to the formula (12.17).

Remark (16.3). We know from Theorem (14.1) that $((^{(1)}R_{AB}))$ is a principal matrix (under the assumption (12.11)) while Lemma (15.3) asserts that either one of the matrices $((^{(2)}R_{AB}))$, $((^{(3)}R_{AB}))$ is a principal matrix (under the assumption (15.1)). Theorem (16.2) tells us that the corresponding curvature tensors are metrically congruent *regardless whether we use the assumption* (12.11) *or* (15.1).

In the next paper we shall consider the curvature tensors belonging to the complex basic characteristics of the types

VI, VII, VIII, XI.

BIBLIOGRAPHY

- [1] V. HLAVATÝ & R. S. MISHRA, Classification of space-time curvature tensor I. Introduction, *Tensor* (New Series), **14** (1963) 138–168.
- [2] V. HLAVATÝ & R. S. MISHRA, Classification of space-time curvature tensor II. General theory of h-classification, *Rend. Circolo Mat. Palermo, Serie II*, **11** (1962) 1–32.
- [3] V. HLAVATÝ & R. S. MISHRA, Classification of space-time curvature tensor III. Projective classification, *Rend. Circolo Mat. Palermo, Serie II*, **13** (1964) 1–47.
- [4] V. HLAVATÝ, Classification of space-time curvature tensor IV. General theory of metric classification, *J. Math. Mech.*, **16** (1967) 1291–1320.
- [5] V. HLAVATÝ, Classification of space-time curvature tensor V. Basic real characteristics, *J. Math. Mech.*, **17** (1968) 921–952.

Indiana University

Date communicated: APRIL 11, 1968