Implicit Submanifolds

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Let M^n be a topological submanifold of R^{n+k} . Through the years, authors have introduced a variety of ‘niceness’ conditions on the way that M^n has been embedded, so that they can prove some reasonably strong results about M^n , *e.g.* M^n is a combinatorial submanifold, M^n has a transverse field, M^n is locally flat, etc. We are going to introduce an analytic condition on M^n which has some pleasing properties: we require that M^n be embedded so that the charts of M^n can be defined by implicit relations. We obtain the following insight into the class of submanifolds which can be smoothed locally by the Cairns–Whitehead technique ([2] and [3]): M^n has a transverse k -plane at each point if and only if the charts of M^n can be defined by Lipschitz implicit relations.

We next turn to the case where the manifold has a global implicit description. We show that such a manifold has a product normal microbundle neighborhood defined by a field of complementary k -planes, which may not be transverse, but which do have simple intersection with the manifold at the base point (we shall say that such complementary planes are *topologically transverse*). It then follows from the methods of [5; §4] that the manifold can be smoothed by arbitrarily small isotopic deformations. This theorem was proved by Whitney [3] under the additional hypothesis that the defining function is Lipschitzian. We conclude the paper with a discussion of submanifolds with codimension one which can be described locally by implicit relations where there is no transverse differentiability, only transverse monotonicity.

1. The implicit function theorem. The implicit function theorem was first proved by Cauchy for analytic functions. In this case, the regular level surfaces of the defining function are analytic submanifolds. Dini then showed that the theorem holds for C^r functions ($r > 0$) and that in this case, the regular level surfaces are C^r submanifolds. These results are treated in standard advanced calculus texts. Young [6] then investigated some cases where the defining functions are not even C^1 , and obtained some sufficient conditions for the level surfaces to be topological manifolds. We shall state two such theorems here. A

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proof of the first theorem can be found in [1: page 270]. The second theorem can be proved by elementary methods.

I. Implicit function theorem: transverse differentiability. Let $f : P^{n+k} \rightarrow R^k$ be a continuous function which has the following properties at a point $(x_0, y_0) \in R^n \times R^k$:

1. The component functions f_i ($i = 1, \dots, k$) of f are continuously differentiable with respect to the variables $y = (y^1, \dots, y^k)$ on some neighborhood N of (x_0, y_0) .
2. $f(x_0, y_0) = 0$.
3. $\partial f / \partial y(x_0, y_0)$ is a nonsingular matrix.

Then there are neighborhoods N_x of x_0 in R^n and N_y of y_0 in R^k and there is a continuous function $g : N_x \rightarrow N_y$ which satisfies:

1. $g(x_0) = y_0$
2. A point $(x_1, y_1) \in N_x \times N_y$ satisfies the equation $f(x, y) = 0$ if and only if $y_1 = g(x_1)$
3. g satisfies any differentiability condition in x that f satisfies.

II. Implicit function theorem: transverse monotonicity. Let $f : R^{n+1} \rightarrow R^1$ be a continuous function which has the following properties at a point $(x_0, y_0) \in R^n \times R^1$:

1. There is a neighborhood N of (x_0, y_0) in which $f(x, \cdot)$ is a monotone function of y for each fixed x
2. $f(x_0, y_0) = 0$

Then there are neighborhoods N_x of x in R^n and N_y of y in R^1 and there is a continuous function $g : N_x \rightarrow N_y$ which satisfies:

1. $g(x_0) = y_0$
2. A point $(x_1, y_1) \in N_x \times N_y$ satisfies the equation $f(x, y) = 0$ if and only if $y_1 = g(x_1)$.

We shall need to sharpen the third conclusion of the first theorem and provide an analogue for the second theorem. It is probably well known that g inherits Lipschitz properties as well as differentiability properties from f , but we have been able to find no mention of this fact in the literature; so we shall include a proof. Actually, we shall prove a slightly sharper statement.

We say that a function $f : R^n \rightarrow R^m$ satisfies a *Lipschitz condition* if for every compact set $A \subset R^n$, there is a constant K such that for all $x, x' \in A$, $\|f(x) - f(x')\| < K \|x - x'\|$. We say that a homeomorphism h between subsets of R^n is a *regular Lipschitz homeomorphism* if h and h^{-1} both satisfy Lipschitz conditions.

Proposition 1.1. Let $f : R^{n+k} \rightarrow R^n$ be a continuous function whose level surface $f^{-1}(0)$ is described by an implicit function g in a neighborhood $N_x \times N_y$ of a point

$(x_0, y_0) \in f^{-1}(0)$. Suppose that there are positive constants α, β, K, L such that

1. f satisfies a Hölder condition of order α with constant K on $N_x \times N_y$,
2. For each fixed $x \in N_x$ and every $y, y' \in N_y$, f satisfies the growth condition

$$|f(x, y) - f(x, y')| > L \|y - y'\|^\beta$$

Then the implicit function g satisfies a Hölder condition of order α/β .

Proof.

$$\begin{aligned} \|g(x) - g(x')\| &< \frac{1}{L} \|f(x, g(x)) - f(x, g(x'))\|^{1/\beta} \\ &< \frac{1}{L} \|f(x', g(x')) - f(x, g(x'))\|^{1/\beta} \\ &< \frac{K^{1/\beta}}{L} \|x' - x\|^{\alpha/\beta} \end{aligned}$$

Corollary 1.2. If f satisfies the hypotheses of the transverse differentiability implicit function theorem, and if g is the implicit function defined by $f(x, y) = 0$ near (x_0, y_0) , then g satisfies a Lipschitz condition whenever f does.

Proof. Take $L = \min \{ \|\partial f / \partial y(x, y)\| \mid (x, y) \in N' \}$ where N' is a compact neighborhood of (x_0, y_0) on which $\partial f / \partial y$ remains nonsingular. Then f satisfies a growth condition for this L and $\beta = 1$.

The following is an example of a function which satisfies the hypothesis of the transverse monotonicity implicit function theorem, but not the transverse differentiability. The defining function satisfies a Lipschitz condition, but does not satisfy a growth condition with $\beta \leq 1$, and its implicit function fails to satisfy a Lipschitz condition.

$$f(x, y) = \begin{cases} y^2 - |x| & \text{if } y \geq |x|^{\frac{1}{2}} \\ -|x| (1 - y/|x|^{\frac{1}{2}}) & \text{if } 0 \leq y \leq |x|^{\frac{1}{2}} \\ -(x^2 + y^2)^{\frac{1}{2}} & \text{if } y \leq 0 \end{cases}$$

The level surface $f^{-1}(0)$ is described by the implicit function $y = g(x) = |x|^{\frac{1}{2}}$.

2. Implicit coordinate systems. Let M^n be a submanifold of R^{n+k} . We shall call M^n an *implicit submanifold* if at each point p of M^n , there is a function $f_p : R^{n+k} \rightarrow R^k$ which satisfies the hypotheses of the implicit function theorem, (I), and which has its level surface $f_p^{-1}(0)$ coincident with M^n on some neighborhood of p .

Proposition 2.1. The related implicit function defines a system of local coordinates for M^n near p . If f satisfies a Lipschitz condition, then the coordinate mapping is a regular Lipschitz homeomorphism.

Proof. We can factor $R^{n+k} = R^n \times R^k$ so that $g_p : R^n \rightarrow R^k$. Assume that $p = (0, 0)$ and let $\pi : R^{n+k} \rightarrow R^n$ be the projection mapping. Near p , $M^n = \{x, g(x)\}$, and $\pi \mid M^n$, the projection onto the first component, is a coordinate homeomorphism. Now $\pi \mid M^n$ and $(\pi \mid M^n)^{-1}(x) = (x, g(x))$ satisfy a Lipschitz condition whenever g does. By 1.2, it is sufficient that f satisfy a Lipschitz condition.

Thus if the local implicit descriptions of M^n are by Lipschitz implicit relations, then the coordinate transformations are regular Lipschitz homeomorphisms, and so M^n has the structure of a Lipschitz manifold. We shall call such a submanifold a *Lipschitz implicit submanifold*. Not every regular Lipschitz embedding of a Lipschitz manifold into R^m is an implicit submanifold. For example, let $f : R^1 \rightarrow R^1$ be a square wave of bounded variation which is damped to 0 at 0, and which oscillates like $x^2 \sin^2 1/x$. Then $M^1 = \text{graph } f$ is rectifiable and embedding by arc length gives a regular Lipschitz homeomorphism of R onto M^1 . But M^1 satisfies no implicit relation at the origin.

We shall now show why the Lipschitz implicit submanifolds are of general interest: they are precisely the class of submanifolds to which the Cairns-Whitehead smoothing technique can be applied locally. Whitney [4] defined a k -plane P to be transverse to M^n at a point if a certain analytic condition [4; 3.1] was satisfied. Cairns [2] interpreted this condition geometrically as follows: P is transverse to M^n at x if there is a neighborhood U of x in M^n such that the set of all secant lines between points of U have their angles with P uniformly bounded away from zero. Cairns [2] and Whitehead [3] then succeeded in proving that the existence of a continuous field of transverse planes is sufficient for the existence of an arbitrarily small isotopic deformation of M^n onto a smooth submanifold of R^{n+k} . Roughly, the proof consists of using the transverse plane at a point x to obtain a local smoothing for M^n near x and then to use the global transverse field to paste these local smoothings together.

Theorem 2.2. *A necessary and sufficient condition for M^n to have a transverse k -plane at the point $p \in M^n$ is that M^n satisfy a Lipschitz implicit relation near p .*

Proof. This condition is sufficient by corollary 1.2. For the necessity, suppose that P is transverse to M^n at p , and let U denote a neighborhood of p whose secants have positive angle with P . Let Q be the normal n -plane to P at p , and let $\pi : R^{n+k} \rightarrow Q$ be the projection parallel to P . Then $\pi \mid M^n : M^n \rightarrow Q$ is a regular Lipschitz homeomorphism near p . Take coordinates $x \in Q$, $y \in P$ for R^{n+k} . Then $f(x, y) = y - (\pi \mid M^n)^{-1}(x)$ is the desired function.

3. Submanifolds with global implicit descriptions. We shall now consider the case where M^n has a global implicit description, i.e., there is a continuous function $f : R^{n+k} \rightarrow R^k$ which satisfies the hypotheses of the implicit function theorem I and which has $M^n = f^{-1}(0)$. If we do not assume that f satisfies a Lipschitz condition, then the k -planes which correspond to the range directions

of the implicit functions may not be transverse to M^n (in the sense of Whitney, Cairns, and Whitehead) since the implicit functions may not satisfy Lipschitz conditions (cf. 1.2 and the following example). However, these k -planes will have the property that their principal intersection with M^n is simple (the intersection consists of one point, and the intersection number is ± 1). We shall say that such planes are *topologically transverse* to M^n , and we shall call a continuous field of such planes a *topologically transverse field* for M^n . Following Whitehead, it is convenient to represent such a field by a mapping $P : M^n \rightarrow G_{k,n}$ where $G_{k,n}$ has the *angle metric* and $x + P(x)$ is the geometric realization of the topologically transverse plane at x . Since $G_{k,n}$ is a differentiable manifold and M^n is a closed subset of R^{n+k} , we can speak of P as being Lipschitzian or smooth as we wish.

Theorem 3.1. *If $M^n \subset R^{n+k}$ satisfies a global implicit relation, then M^n has a smooth field of topologically transverse k -planes. Moreover, this field has a natural product structure induced by the defining function.*

Proof. Let $f : R^{n+k} \rightarrow R^k$ be a function which describes M^n implicitly and let $\{U_i\}$ be a locally finite covering of M^n by coordinate neighborhoods on which the coordinates are specified by implicit functions of f . Let P^i denote the k -plane through the origin which corresponds to the range direction for the implicit function corresponding to U_i . Then for each $x \in U_i$, $f|_{(x + P^i)}$ has continuous partial derivatives at x . Let $\{a_i\}$ be a *smooth partition of unity* subordinate to $\{U_i\}$ (the functions a_i form a partition of unity for M^n and are smooth in the differentiable structure of R^{n+k}). We want to paste the planes P^i together by $\{a_i\}$ to obtain a global topologically transverse field. We must be careful that the topological transversality is not lost during the pasting. We shall begin by choosing a preferred basis $[Y_1^i, \dots, Y_k^i]$ for each P^i . We want to form a continuous field of k -frames $[Y_{(x)}^i = \sum a_i(x) Y_j^i(x)]$ in such a way that the plane spanned by $[Y_{(x)}^i, \dots, Y_k^i(x)]$ is a topologically transverse k -plane to M^n at x . The preferred basis for $x + P^i$ is the set of gradient vectors $Y_j^i = \nabla(f_j|_{x + P^i})$. By the hypothesis of the implicit function theorem, $[Y_1^i, \dots, Y_k^i]$ is a linearly independent set. Now suppose that $x \in U_1 \cap \dots \cap U_r$. Let Q_x denote the subspace which is spanned by $P^1, \dots, P^r$. One easily shows

Lemma 3.2. $f|_{(x + Q_x)}$ is differentiable at x .

Thus we can form the k -frame $Y_j^*(x) = \nabla(f_j|_{Q_x})(x)$. Let P^* denote the plane spanned by $[Y_1^*, \dots, Y_k^*]$ and let Q^* denote its orthogonal in Q_x . Then each P^i is the graph of a linear function $A^i : P^* \rightarrow Q^*$. It is evident that

Lemma 3.3. *For each $i = 1, \dots, r$ and $j = 1, \dots, k$, the direction Y_j^i in Q_x coincides with the direction $(Y_j^*, A^1(Y_j^*))$ on the graph of A^i .*

It follows immediately from this lemma that all of the bases $[Y_1^i, \dots, Y_k^i]$ are similarly oriented and that when we form convex combinations of these bases, we always obtain a k -frame in Q_x . Therefore $[Y_1, \dots, Y_k]$ is a global k -frame

field on M^n as desired. Let $P(x) \subset Q_x$ denote the k -plane through the origin which is spanned by $[Y_1(x), \dots, Y_k(x)]$. Computing the derivatives of f at x in the directions $Y_1(x), \dots, Y_k(x)$, we see that $P(x)$ is an admissible range direction for an implicit function of f at x , and hence $x + P(x)$ is topologically transverse to M^n at x . Since the frames vary smoothly, it is clear that the planes also vary smoothly.

We need another property of this topologically transverse field, namely that it can be used to define a *topological tubular neighborhood* of M^n . Precisely, a *topological tubular neighborhood* of M^n is a neighborhood of M^n which is the image of a homeomorphism $\theta : B \rightarrow R^{n+k}$ where B is a normal disk bundle over M^n . In our case, $B = M^n \times D^k$ (k -disk) and θ is defined by $\theta(x, t) = x + t$ where $x \in M^n$ and $t \in P(x)$.

Theorem 3.4. *Let $P : M^n \rightarrow G_{k,n}$ be a Lipschitzian field of topologically transverse planes on M^n with the property that at each $x \in M^n$, $x + P(x)$ can be used as the range direction for an implicit chart of M^n . Then there is a positive function $\epsilon : M^n \rightarrow R$ such that the union of the $\epsilon(x)$ -disks $D_x \subset x + P(x)$ defines a topological tubular neighborhood of M^n .*

Proof. Let $P^0(x)$ denote the orthogonal complement of $P(x)$. Since $P(x)$ is the range direction for a local implicit description of M^n at x , $P^0(x)$ is the domain direction. By continuity of P , P defines a Lipschitzian transverse field on $x + P^0(x)$. By [3; §7], P defines a local topological tubular neighborhood of $x + P^0(x)$. But by the implicit relationship between M^n and $x + P^0(x)$ near x , P also defines a local topological tubular neighborhood of M^n near x . Therefore, by [3; §4] we obtain a global topological tubular neighborhood of M^n from P . The ϵ function is constructed by the standard argument (cf. [3; §5]).

It would be interesting to obtain an analytic characterization of when a topologically transverse field determines a topological tubular neighborhood of M^n . The problem, of course, is the possibility of the intersection of close fibers arbitrarily close to a compact part of M^n . The following is true: If $M^n \subset R^{n+k}$ with $k \geq n$ is an implicit submanifold and if P is a topologically transverse field of range directions of implicit functions on M^n , then a necessary condition for P to define a topological tubular neighborhood of M^n is that the angle derivatives of P be finite, i.e. for each $x \in M^n$,

$$\limsup_{y \rightarrow x} \frac{|\alpha(P(x), P(y))|}{\|x - y\|} < \infty.$$

If $k < n$, then it is easy to give an example to show that we can have finite angle derivatives, and still not have a tubular neighborhood defined.

Theorem 3.5. *If $M^n \subset R^{n+k}$ satisfies a global implicit relation, then M^n can be smoothed by small isotopic deformations.*

Proof. By 3.1, M^n has a smooth topologically transverse field which is a product. By 3.4, this field defines a topological tubular neighborhood of M^n

whose fibers are linear spaces. Therefore, the results of [5; §4] apply and we can smooth M^n as desired.

Remark. If M^n is a Lipschitz implicit submanifold, then the topologically transverse k -planes are actually transverse, and so M^n can be smoothed by the Cairns–Whitehead approach. Whitney proved this theorem in this case [4]. His proof amounts to the development of the Cairns–Whitehead theory for the special case when the global transverse field is a product over M^n .

4. Implicit submanifolds of codimension one. Suppose that $M^n = f^{-1}(0)$ where $f : R^{n+k} \rightarrow R^1$ is a function which satisfies the hypothesis of the implicit function theorem II along M^n . Since M^n obtains its charts by continuous implicit functions, an argument similar to the one used in the proof of 2.2 shows that locally, M^n can also be described implicitly by functions which satisfy the hypothesis of the implicit function theorem I, *i.e.* M^n is an implicit submanifold. However, we have not been able to show that global description by transversely monotone functions implies that there is a global description by transversely differentiable functions. Therefore, we cannot apply the results of §3 to smooth M^n . The following result does provide a different kind of generalization of Whitney's theorem.

Theorem 4.1. *Suppose that $M^n = f^{-1}(0)$ where f is a continuous function which satisfies the hypothesis of the implicit function theorem II at each point of M^n . Then M^n has a topologically transverse field, and consequently, M^n can be smoothed by small isotopic deformations.*

The proof of this theorem is very similar to the proof of 3.1. We cover M^n by implicit coordinate neighborhoods, choose a locally finite refinement, and then paste the range direction of the implicit functions together with a partition of unity. These range directions are always topologically transverse lines. The only difficulty comes in assuring that we do not lose the transverse condition during the pasting.

Lemma 4.2. *The set of range directions for implicit functions which define M^n near a point p is convex.*

Proof. Let D, E denote positively directed range directions for implicit functions which define M^n on a neighborhood U of p in M^n , let P denote the plane through p which is spanned by D and E , and let F denote any direction in P between D and E . Since M^n is a graph with respect to the direction $D(E)$, it follows that $P \cap U$ is a curve and that this curve intersects in at most one point with each line in P parallel to F . Therefore, F is topologically transverse to M^n at p . Since D and E are range directions on all of U , the above argument shows that F is topologically transverse to M^n at each point of U . Let F^0 denote the n -plane orthogonal to F , and define $g : p + F^0 \rightarrow p + F$ by $(x, g(x)) = (x + F) \cap U$ for each x in $p + F^0$. g is a continuous function which describes

M^n on the neighborhood U and which has range direction F . This completes the proof.

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