An Exact Solution in Rotating Flow

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Summary. A study is made of the unsteady flow engendered in a viscous incompressible, rotating fluid by an infinite flat plate set suddenly in motion in its own plane. The exact solution of the problem is obtained by the method of Laplace Transforms. The effect of rotation on the drag is obtained and several limiting cases are deduced from the solution. It is shown that the rotation induces lateral motion and that vorticity is generated by rotation. It is found that it tends to oppose the penetration of longitudinal flow. Finally, the steady solution obtained from the unsteady problem provides the plausible argument of the fact that for small rotational effects, the lateral velocity $\sim t^{3/2}$ for large time, a result obtained earlier by us while considering the effects of small rotation (see [3]).

1. Introduction. Rotation plays a significant role in several important phenomenon in cosmical fluid dynamics. Similarly, a great deal of meteorology depends upon the dynamics of a revolving fluid. The large scale and moderate motions of the atmosphere are greatly affected by the vorticity of the earth's rotation. In the case of an infinite liquid, rotating as a rigid body about an axis, the amount of energy possessed by the liquid is infinite and it is of great interest to know how small disturbances propagate in such a liquid.

It is common practice in fluid dynamics to start with simple models to investigate various effects. The problem, although idealized, retains the essential features of the investigation. We have earlier discussed the effect of small rotation in the case of an oscillating sphere in a viscous fluid ([1] and [2]) and on the fluid flow induced by an infinite flat plate suddenly set into motion with constant velocity in its own plane [3]. (It was found in [3] that the lateral flow develops as $\tau^{3/2}$ with time which gave an erroneous impression of non-existence of steady solution. The present study which is a generalization of the work done in [3] explains this anomalous situation and shows that the steady solution in fact exists.) Here we will consider the effect of rotation on the Rayleigh Problem for any Ω . The investigation should yield valuable qualitative information regarding (i) boundary layer development and (ii) propagation of waves and their interaction with the boundary layer.

It turns out that an exact solution of the problem is possible in view of the well-known linearity of the Rayleigh Problem. The exact formulation of the problem can be stated as follows. The entire system consisting of the fluid and the plate is rotating with uniform angular velocity $\Omega \mathbf{k}$. The plate located at $z = 0$ is suddenly moved in its own plane with a constant velocity U_0 in the negative x -direction relative to the rotating coordinate system. It is shown that rotation tends to increase the drag. It tends to oppose the penetration of longitudinal flow fields while it induces a lateral flow whose depth of penetration increases with rotation. Another effect of the rotation here is to produce vorticity. These results are in broad agreement with the conclusions arrived at in earlier investigations (see [1], [2], [3] and [4]).

2. Equation of motion. An infinite flat plate (located at $z = 0$) and the fluid (which is in contact with the plate and occupies the whole of the region $z > 0$) are in uniform rotation Ω . For the sake of simplicity, the angular velocity Ω is taken parallel to the z -axis. The plate is suddenly moved in its own plane with a constant velocity U_0 in the negative x -direction relative to rotating coordinate system. The fluid is assumed to be incompressible and viscous. Referred to the rotating frame of reference, the momentum equation becomes

$$(1) \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + 2\Omega \times \mathbf{u} + \Omega \times (\Omega \times \mathbf{r}) = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{u}$$

where P is the pressure, $\mathbf{u}$ the velocity of the fluid, ρ its density and r is the radial coordinate:

$$r^2 = x^2 + y^2.$$

The equation of continuity is

$$(2) \quad \text{div } \mathbf{u} = 0$$

As in the non-rotating case, the physical variables are functions of z and t only. (See for example HIDE & ROBERTS, Hydromagnetic flow due to an oscillating plane, *Rev. Modern Phys.*, **32** (1960) 799–806, who treat the problem of the oscillating (linear) plate in the presence of rotation by similar approach.) From (2) and the continuity of the normal component of velocity at the plane, it follows that

$$(3) \quad u_z = 0$$

and (1) can be reduced to

$$(4) \quad \frac{\partial \mathbf{u}}{\partial t} - \nu \nabla^2 \mathbf{u} + 2\Omega \times \mathbf{u} = -\nabla \left(\frac{P}{\rho} - \frac{1}{2} r^2 \Omega^2 \right).$$

The z -component of (4) shows that $(P/\rho) - \frac{1}{2} r^2 \Omega^2$ is independent of z and as it is independent of x and y , it can be taken to be constant in the region $z > 0$ occupied by the fluid, so that the motion of fluid is governed by the equation

$$(5) \quad (\partial/\partial t - \nu \nabla^2) \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} = 0.$$

The initial and the boundary conditions necessary to determine the unique solution are the following.

As the fluid is initially at rest with respect to the rotating frame of reference, we have

$$(6) \quad \text{at } t = 0: \quad \mathbf{u} = 0 \quad \text{for } z > 0.$$

Since the fluid is at rest at large distances from the plate, it follows that

$$(7) \quad t > 0: \quad \mathbf{u} = 0 \quad \text{as } z \rightarrow \infty.$$

No slip condition at the plate yields

$$(8) \quad t > 0: \mathbf{u} = -U_0 \mathbf{i} \quad \text{at } z = 0$$

remembering that rotating axes are used.

Before attempting the solution, we shall reduce equations (5), (6), (7) and (8) to non-dimensional form. Setting

$$(9) \quad z = \frac{\nu}{U_0} \xi, \quad \mathbf{u} = U_0 \mathbf{U}, \quad t = \frac{\nu}{U_0^2} \tau, \quad \Omega = \frac{U_0^2}{\nu} \omega$$

equation (5) reduces to

$$(10) \quad \left(\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial \xi^2} \right) \mathbf{U} + 2\boldsymbol{\omega} \times \mathbf{U} = 0$$

and the initial and boundary conditions (6), (7) and (8) reduce to

$$(11) \quad \tau = 0 : \mathbf{U} = 0 \quad \text{for all } \xi > 0$$

$$(12) \quad \left. \begin{array}{l} \tau > 0 : \mathbf{U} = 0 \quad \text{at } \xi = \infty \\ \mathbf{U} = -\mathbf{i} \quad \text{at } \xi = 0 \end{array} \right\}.$$

3. Transformed equations and the solution. From (10) and (11), the Laplace transform of $\mathbf{U}$ satisfies the equation

$$(13) \quad \left(p - \frac{d^2}{d\xi^2} \right) \bar{\mathbf{U}} + 2\boldsymbol{\omega} \times \bar{\mathbf{U}} = 0$$

x - and y -components of this equation satisfy

$$(14) \quad \left(p - \frac{d^2}{d\xi^2} \right) \bar{\mathbf{U}}_x - 2\omega \bar{\mathbf{U}}_y = 0$$

and

$$(15) \quad \left(p - \frac{d^2}{d\xi^2} \right) \bar{\mathbf{U}}_y + 2\omega \bar{\mathbf{U}}_x = 0$$

which form a coupled system of equations.

The boundary conditions are

$$(16) \quad \begin{aligned} \bar{U} &= -i/p \quad \text{at} \quad \xi = 0 \\ &\rightarrow 0 \quad \text{as} \quad \xi \rightarrow \infty. \end{aligned}$$

The solution of (14), (15) and (16) is

$$(17) \quad \bar{U}_x = -\frac{1}{2p} [\exp \{-(p + 2i\omega)^{1/2}\xi\} + \exp \{-(p - 2i\omega)^{1/2}\xi\}]$$

$$(18) \quad \bar{U}_y = \frac{i}{2p} [\exp \{-(p + 2i\omega)^{1/2}\xi\} - \exp \{-(p - 2i\omega)^{1/2}\xi\}]$$

and inversion gives

$$(19) \quad \begin{aligned} U_x &= \frac{1}{4} \left[\exp(1 + i)\beta \operatorname{erfc} \left\{ \frac{\eta}{2} + (1 + i) \frac{\beta}{\eta} \right\} \right. \\ &\quad + \exp(-(1 + i)\beta) \operatorname{erfc} \left\{ \frac{\eta}{2} - (1 + i) \frac{\beta}{\eta} \right\} \\ &\quad + \exp(1 - i)\beta \operatorname{erfc} \left\{ \frac{\eta}{2} + (1 - i) \frac{\beta}{\eta} \right\} \\ &\quad \left. + \exp(-1 + i)\beta \operatorname{erfc} \left\{ \frac{\eta}{2} + (-1 + i) \frac{\beta}{\eta} \right\} \right] \end{aligned}$$

$$(20) \quad \begin{aligned} U_y &= \frac{i}{4} \left[\exp(1 + i)\beta \operatorname{erfc} \left\{ \frac{\eta}{2} + (1 + i) \frac{\beta}{\eta} \right\} \right. \\ &\quad + \exp(-(1 + i)\beta) \operatorname{erfc} \left\{ \frac{\eta}{2} - (1 + i) \frac{\beta}{\eta} \right\} \\ &\quad - \exp(1 - i)\beta \operatorname{erfc} \left\{ \frac{\eta}{2} + (1 - i) \frac{\beta}{\eta} \right\} \\ &\quad \left. - \exp(-1 - i)\beta \operatorname{erfc} \left\{ \frac{\eta}{2} - (1 - i) \frac{\beta}{\eta} \right\} \right] \end{aligned}$$

where

$$\beta = \xi \sqrt{\omega}, \quad \eta = \frac{\xi}{\sqrt{\tau}}$$

(19) and (20) can be written in the form (see [5]):

$$(21) \quad \begin{aligned} U_x &= -\frac{1}{2}[e^\beta \{H_1 \cos(2XY - \beta) - H_2 Y \sin(2XY - \beta)\} \\ &\quad + e^{-\beta} \{H'_1 \cos(2X_0 Y - \beta) - H'_2 Y \sin(2X_0 Y - \beta)\}], \quad \eta^2 > 2\beta \\ &\quad - \frac{1}{2}[e^\beta \{H_1 \cos(2XY - \beta) - H_2 Y \sin(2XY - \beta)\} \\ &\quad + e^{-\beta} \{2 \cos \beta - H''_1 \cos(2X_0 Y - \beta) - H''_2 Y \sin(2X_0 Y - \beta)\}], \quad \eta^2 < 2\beta \end{aligned}$$

$$\begin{aligned}
 (22) \quad U_v &= \frac{1}{2}[e^\beta \{H_1 \sin(2XY - \beta) + H_2 Y \cos(2XY - \beta)\} \\
 &\quad - e^{-\beta} \{H'_1 \sin(2X_0 Y - \beta) + H'_2 Y \cos(2X_0 Y - \beta)\}], \quad \eta^2 > 2\beta \\
 &\quad \frac{1}{2}[e^\beta \{H_1 \sin(2XY - \beta) + H_2 Y \cos(2XY - \beta)\} \\
 &\quad + e^{-\beta} \{2 \sin \beta + H''_1 \sin(2X_0 Y - \beta) - H''_2 Y \cos(2X_0 Y - \beta)\}], \\
 &\quad \eta^2 < 2\beta
 \end{aligned}$$

where

$$\begin{aligned}
 X &= \frac{\eta}{2} + \frac{\beta}{\eta}, \quad Y = \frac{\beta}{\eta}, \quad X_0 = \frac{\eta}{2} - \frac{\beta}{\eta} > 0 \quad \text{if} \quad \eta^2 > 2\beta, \\
 H_1 &= \sum_{n=0}^{\infty} \gamma_n (XY)^{2n}, \quad H_2 = X \sum_{n=0}^{\infty} (n+1) \gamma_{n+1} (XY)^{2n}, \\
 \gamma_{n+1} &= \frac{2}{(2n+1)\sqrt{\pi}} \left[\frac{e^{-X^2}}{(n+1)! X^{2n+1}} - \frac{\sqrt{\pi}}{n+1} \gamma_n \right], \quad \gamma_0 = \operatorname{erfc} X, \\
 H'_1 &= \sum_{n=0}^{\infty} \gamma'_n (X_0 Y)^{2n}, \quad H'_2 = X_0 \sum_{n=0}^{\infty} (n+1) \gamma'_{n+1} (X_0 Y)^{2n}, \\
 \gamma'_{n+1} &= \frac{2}{(2n+1)\sqrt{\pi}} \left[\frac{e^{-X_0^2}}{(n+1)! X_0^{2n+1}} - \frac{\sqrt{\pi}}{n+1} \gamma'_n \right], \quad \gamma'_0 = \operatorname{erfc} X_0, \\
 H''_1 &= \sum_{n=0}^{\infty} \gamma''_n (X_0 Y)^{2n}, \quad H''_2 = -X_0 \sum_{n=0}^{\infty} (n+1) \gamma''_{n+1} (X_0 Y)^{2n}, \\
 \gamma''_{n+1} &= \frac{-2}{(2n+1)\sqrt{\pi}} \left[\frac{e^{-X_0^2}}{(n+1)! X_0^{2n+1}} - \frac{\sqrt{\pi}}{n+1} \gamma''_n \right], \quad \gamma''_0 = 2 - \operatorname{erfc} X_0.
 \end{aligned}$$

4. Conclusions. 1. Figure 1 leads to the conclusion that the rotation tends to oppose the penetration of longitudinal flow fields while it induces a lateral flow whose depth of penetration increases with rotation (see Fig. 2).

2. The drag on the plate is given by

$$\left(\frac{\partial U_x}{\partial \xi} \right)_{\xi=0} = \frac{\cos(2\omega\tau)}{\sqrt{\pi\tau}} + 4\sqrt{\frac{\omega}{\pi}} \int_0^{\sqrt{\omega\tau}} \sin(2\alpha^2) d\alpha$$

and the lateral stress on the plate is given by

$$\left(\frac{\partial U_v}{\partial \xi} \right)_{\xi=0} = -\frac{\sin(2\omega\tau)}{\sqrt{\pi\tau}} + 4\sqrt{\frac{\omega}{\pi}} \int_0^{\sqrt{\omega\tau}} \cos(2\alpha^2) d\alpha$$

which shows that the stress on the plate increases with rotation (see Fig. 3 and 4).

3. Inviscid flow can be obtained from this solution and the result is that rotation produces no disturbance just as in irrotational case.

4. The steady solution can be obtained by letting $\tau \rightarrow \infty$ which gives

FIGURE 1. U_x vs. η FIGURE 2. U_y vs. η .

 FIGURE 3. $(\partial U / \partial \xi)_{\xi=0}$ vs. ω .

$$U_x = -e^{-\xi\sqrt{\omega}} \cos(\xi\sqrt{\omega})$$

$$U_y = e^{-\xi\sqrt{\omega}} \sin(\xi\sqrt{\omega}).$$

(In an earlier problem solved for small ω (see [3]), it was found that U_y develops as $\tau^{3/2}$ with time which is misleading. As shown above, the flow in fact attains steady state and the solution obtained in [3] for U_y (as $\tau \rightarrow \infty$) can be obtained from equation (20) by letting $\omega \rightarrow 0$ and then letting $\tau \rightarrow \infty$. Thus we notice that $Lt_{\tau \rightarrow \infty} Lt_{\omega \rightarrow 0} U_y \neq Lt_{\omega \rightarrow 0} Lt_{\tau \rightarrow \infty} U_y$.)

5. Finally, the consideration of the vorticity vector

$$\text{curl } \mathbf{u} = \left(-\frac{\partial u_y}{\partial z}, \frac{\partial u_x}{\partial z}, 0 \right)$$

leads us to the conclusion that the rotation generates vorticity.

FIGURE 4. $(\partial U/\partial \xi)_{\xi=0}$ vs. ω .

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