*Transitive Subsemigroups of the Moebius Group*¹

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Communicated by PAUL R. GARABEDIAN

This paper is concerned with those subsemigroups of the Moebius group that are generated by a pair of infinitesimal transformations. If at least one of these has a discriminant whose imaginary part vanishes, simple necessary and sufficient conditions are given for the generated subsemigroup to be transitive on the extended complex plane. The other interesting result that is found is that if one chooses the usual topology on the set of pairs of infinitesimal transformations, then the subset of pairs that generate a transitive subsemigroup is open; in a sense one can thus assert the continuous dependence of transitivity on the pair of infinitesimal transformations.

I. Introduction. The infinitesimal transformations of the Moebius group are quadratic functions with complex coefficients a , $2b$ and c . This implies that a one-parameter semigroup of this group is given by the solution of the ordinary differential equation:

$$(1) \quad \frac{dw}{dt} = aw^2 + 2bw + c$$

with the initial condition $w(0, z) = z$ for all non-negative (or non-positive) values of the parameter t . Thus for a fixed value of t the solution $w(t, z)$ considered as a function of the initial value z is a Moebius transformation and the family of these transformations for non-negative t defines the one-parameter semigroup generated by the infinitesimal transformation $\epsilon = aw^2 + 2bw + c$; this one-parameter semigroup will be denoted by $S(\epsilon)$.

One defines the subsemigroup $S(\epsilon, \eta)$ generated by the pair of infinitesimal transformations ϵ and η to be the smallest subsemigroup topologically closed with respect to the whole group that contains both $S(\epsilon)$ and $S(\eta)$. Thus it con-

¹ The main results are part of the author's doctoral dissertation. He wishes to acknowledge the guidance and inspiration of the late Professor Charles Loewner. Additional research and simplification of some proofs were supported by a summer Research Grant of the University of Oregon.

sists of all finite products of elements of $S(\epsilon)$ and $S(\eta)$ together with all non-singular limits. One denotes by $T(\epsilon, \eta)$ the subsemigroup that consists only of all finite products of elements of $S(\epsilon)$ and $S(\eta)$; clearly $T(\epsilon, \eta) \subset S(\epsilon, \eta)$.

An infinitesimal transformation $\epsilon = aw^2 + 2bw + c$ whose discriminant $b^2 - ac$ has non-zero imaginary part is called loxodromic; if $b^2 - ac$ is real, then ϵ is called elliptic, hyperbolic or parabolic depending upon whether the discriminant is negative, positive, or zero. If $w = T(z)$ induces an inner automorphism on the Moebius group, its differential at the identity denoted by Ad_τ is a non-singular linear transformation on the Lie algebra that preserves the bracket operation and further leaves the discriminant invariant; such transformations will be freely used to simplify the form of the pair of infinitesimal transformations. One may even allow improper Moebius transformations, i.e. $w = T(\bar{z}) = (a\bar{z} + b)/(c\bar{z} + d)$ for the purpose of achieving a normal form for ϵ and η . One observes that $S(\epsilon)$ is in fact a one-parameter subgroup if and only if ϵ is elliptic.

It is useful to adopt a hydrodynamical visualization of the infinitesimal transformation ϵ ; one can think of it as inducing a flow on the extended complex plane defined by the direction field of the differential equation. Unless ϵ is parabolic, there are two fixed points; if further ϵ is not elliptic, one of these is called the source of the flow, the other the sink in accordance with the standard terminology. In the parabolic case there is only one fixed point which is both the source and the sink of the flow. Through each point z_0 of the extended complex plane that is not a fixed point of ϵ there passes a unique orbit defined by the flow; this orbit may be divided into two portions—those points z_1 on it that are the successors of z_0 (i.e. there exists $w = T(z) \in S(\epsilon)$ such that $z_1 = T(z_0)$) and those points z_2 that are the predecessors of z_0 (i.e. there exists $w = T(z) \in S(\epsilon)$ such that $z_0 = T(z_2)$). If ϵ and η have a common fixed point, then $S(\epsilon, \eta)$ can never be transitive; hence in all subsequent theorems the assumption will always be made that they have no common fixed point.

A circle C is called a barrier if the flow determined by both ϵ and η is such that either all points in the interior of C must remain in the interior or all points in the exterior must remain in the exterior. A similar definition can be made for a line. Clearly $S(\epsilon, \eta)$ can never be transitive if a barrier exists.

II. In case neither ϵ nor η is loxodromic, the following result holds:

Theorem 1. *Let ϵ and η be infinitesimal transformations with no common fixed points and each having real discriminant. If both are elliptic and the cross-ratio of the roots is not a positive real number, $S(\epsilon, \eta)$ is transitive. In all other cases $S(\epsilon, \eta)$ is not transitive. If both ϵ and η are elliptic $S(\epsilon, \eta)$ is a subgroup. In all other cases $S(\epsilon, \eta)$ is a subgroup if and only if ϵ and η have a joint orbit and for some $s \geq 0$, $t \geq 0$, $s\epsilon + t\eta$ is elliptic.*

Proof. If both ϵ and η are elliptic, $S(\epsilon, \eta)$ is identical to the subgroup generated by ϵ and η . If the cross-ratio of the roots has non-zero imaginary part, then $S(\epsilon, \eta)$ equals the Moebius group; if the cross-ratio of the roots is a negative

real number, then $S(\epsilon, \eta)$ is to within an inner automorphism the isometries of the spherical geometry, a three dimensional transitive subgroup of the Moebius group; if the cross-ratio of the roots is a positive real number, then $S(\epsilon, \eta)$ is to within an inner automorphism the isometries of the hyperbolic geometry, a three dimensional subgroup that is not transitive. [1]

If ϵ is elliptic, η hyperbolic, one can assume that $\epsilon = iw$, $\eta = \alpha(w - 1)(w - \gamma)$ where $\alpha(1 - \gamma) > 0$ and $|\gamma| \geq 1$. If $|\gamma| = 1$, the unit circle is a joint orbit. If $|\gamma| > 1$, then for any b , $1 < b < |\gamma|$, the circle centered at the origin of radius b is a barrier, for it is an orbit of ϵ and as η has 1 as its source, γ as its sink (since $\alpha(1 - \gamma) > 0$) the flow determined by η transforms points in the exterior of these circles into points that again lie in the exterior.

If both ϵ and η are hyperbolic, one can assume $\epsilon = w$, $\eta = \alpha(w - 1)(w - \gamma)$ where $|\gamma| \geq 1$, $\alpha^2(1 - \gamma)^2 > 0$. If γ is real, so is α and hence the real axis is a joint orbit. If $\text{Im } \gamma \neq 0$, then the line L through 1 and γ does not pass through the origin, and it is clearly an "orbit" (union of two line segments each of which is an orbit) of η . But as ∞ is the sink of ϵ it is easily seen that ϵ transforms points in the halfplane determined by L that does not include the origin into points that are again in this halfplane; hence L is a barrier.

If ϵ is elliptic, η parabolic one can assume $\epsilon = i(w - 1)(w + 1)$, $\eta = \delta \neq 0$ where $\text{Re } \delta \geq 0$. The imaginary axis is an orbit of ϵ ; if $\text{Re } \delta = 0$ it is also an orbit of η . If $\text{Re } \delta > 0$, then points in the right halfplane are transformed by the flow due to η into points that are again in this halfplane so the imaginary axis is a barrier.

If ϵ is hyperbolic, η parabolic, one can assume that $\epsilon = (w - 1)(w + 1)$, $\eta = \delta \neq 0$. If $\text{Im } \delta = 0$, the real axis is a joint orbit; if $\text{Im } \delta \neq 0$ it still is easily seen to be a barrier.

If both generators are parabolic, one can assume that $\epsilon = 1$, $\eta = \alpha w^2$ where $\alpha \neq 0$. Clearly if $\text{Im } \alpha = 0$, the real axis is a joint orbit but in any case it always is a barrier.

One easily sees that except when both generators are elliptic, $S(\epsilon, \eta)$ cannot be a group unless ϵ and η have a joint orbit for otherwise one is able to find barriers which are barriers in one direction only. Without loss of generality one can assume that the real axis is a joint orbit when a joint orbit exists, and then a necessary and sufficient condition for $S(\epsilon, \eta)$ to be the isometries of the hyperbolic geometry for the halfplane is that for some $s \geq 0$, $t \geq 0$, $s\epsilon + t\eta$ be elliptic; otherwise $S(\epsilon, \eta)$ is not even a group. [2]

III. Since $T(\epsilon, \eta)$ is contained in $S(\epsilon, \eta)$, it is clear that if the former is transitive, so is the latter; the converse is also true as stated in:

Theorem 2. *If $S(\epsilon, \eta)$ is transitive, so is $T(\epsilon, \eta)$.*

Proof. If both ϵ and η are not loxodromic, then by Theorem 1, $S(\epsilon, \eta)$ is transitive only if both ϵ and η are elliptic. But since $S(\epsilon)$ and $S(\eta)$ are then in fact one-parameter subgroups, it follows that $T(\epsilon, \eta) = S(\epsilon, \eta)$ since all one-parameter subgroups of the Moebius group are closed and Malcev's theorem [3]

asserts that in this case $T(\epsilon, \eta)$ is closed. Hence one need only consider the case where at least one generator is loxodromic and since $S(\epsilon, \eta)$ is transitive, it follows that ϵ and η have no common fixed point. One may assume therefore that $\epsilon = \lambda w$, $\operatorname{Re} \lambda > 0$, $\operatorname{Im} \lambda \neq 0$ and η leaves neither 0 nor ∞ fixed. The proof that $T(\epsilon, \eta)$ is transitive will consist of proving two lemmas:

Lemma 1. *Let $\epsilon = \lambda w$, $\operatorname{Re} \lambda > 0$, $\operatorname{Im} \lambda \neq 0$, and suppose η leaves neither 0 nor ∞ fixed. Then given any point z of the extended complex plane, there exists a Moebius transformation $w = T(z) \in T(\epsilon, \eta)$ such that $T(0) = z$.*

Proof. The integral curves or orbits of $\epsilon = \lambda w$ are logarithmic spirals, and the flow is outward on these spirals as $\operatorname{Re} \lambda > 0$ assures that 0 is the source, ∞ the sink of ϵ . Since 0 is not a fixed point of η , there exists an element of $S(\eta)$ that takes 0 into some point z^* of the spiral through z ($z \neq 0$, $z \neq \infty$) such that $|z^*| < |z|$, and then there is an element of $S(\epsilon)$ that takes z^* into z . As for ∞ , since it is not a fixed point of η , one can find a finite complex number c and an element of $S(\eta)$ that takes c into ∞ , and since there is an element of $T(\epsilon, \eta)$ that takes 0 into c , there is an element of $T(\epsilon, \eta)$ that takes 0 into ∞ .

Lemma 2. *Suppose $\epsilon = \lambda w$, $\operatorname{Im} \lambda \neq 0$ and suppose η does not leave 0 fixed. Then there is a neighborhood V of 0 such that if $z \in V$, there is an element of $T(\epsilon, \eta)$ that takes z into 0.*

Proof. Let J be that portion of the orbit of η through 0 that consists of predecessors of 0. One can find a circle centered at the origin of sufficiently small radius such that if two complete loops of any spiral orbit of ϵ lie entirely inside this circle, then there is a point of intersection of J and these two loops; let t be the radius of this circle. If $\mu = |\operatorname{Re} \lambda|/|\operatorname{Im} \lambda|$, then let $V = \{z : |z| < te^{-4\pi\mu}\}$. If $z \in V$, two complete loops of the spiral through z consisting of successors of z must lie inside the circle centered at the origin of radius t and hence there is an element of $S(\epsilon)$ that takes z into a point on J and thus an element of $T(\epsilon, \eta)$ that takes z into 0.

An immediate corollary of this lemma is that if $A = \{z : \text{there exists an element of } T(\epsilon, \eta) \text{ taking } z \text{ into } 0\}$, $B = \{z : \text{there exists an element of } S(\epsilon, \eta) \text{ taking } z \text{ into } 0\}$, then $A = B = \text{open set for } \epsilon \text{ and } \eta \text{ as in Lemma 2}$.

But then if $S(\epsilon, \eta)$ is transitive so that B is just all points of the complex plane, in view of Lemma 1 and the fact that $A = B$, $T(\epsilon, \eta)$ is also transitive.

IV. If one generator is loxodromic and the other parabolic, and they have no common fixed point, one can assume the normal form: $\epsilon = \lambda w$, $\eta = \alpha(w - 1)^2$ where $\operatorname{Re} \lambda > 0$, $\operatorname{Im} \lambda > 0$, $\alpha \neq 0$. One selects the unique value of the argument of a complex number to be the value greater than $-\pi$ but less than or equal to π ; clearly $0 < \arg \lambda < \pi/2$. For ϵ and η as above, there is a simple inequality that determines whether or not $S(\epsilon, \eta)$ is transitive:

Theorem 3. *Suppose $\epsilon = \lambda w$, $\eta = \alpha(w - 1)^2$, $0 < \arg \lambda < \pi/2$, $\alpha \neq 0$, $-\pi < \arg \alpha \leq \pi$. Then $S(\epsilon, \eta)$ is transitive if and only if:*

$$(2) \quad -\arg \lambda < \arg \alpha < \arg \lambda.$$

In particular if $\alpha > 0$, $S(\epsilon, \eta)$ is always transitive.

Proof. If $\operatorname{Re} \alpha \leq 0$, then one can show that the flow determined by η takes points in the exterior of the unit circle into the exterior; as $\operatorname{Re} \lambda > 0$, ∞ is the sink of ϵ so that one sees the same is always true for ϵ and the unit circle is a barrier.

If $-\pi/2 < \arg \alpha \leq -\arg \lambda$, let L be the line through $z = 1$ whose angle with the positive real axis is $-\arg \alpha$; let K be the circle through $z = -1$ that is tangent to L at $z = 1$. Both L and K are orbits of η . To show K is a barrier it suffices to prove that the direction field determined by ϵ never points towards the interior of K at any point of K . If ψ is the angle between the radius vector and the tangent to K , one verifies that $\tan^2 \psi$ assumes its minimum values at $z = \pm 1$. But at $z = 1$, $\psi = \arg \alpha$, and at $z = -1$, $\psi = \pi + \arg \alpha$ and hence as $\arg \lambda \leq -\arg \alpha$, one sees that $\arg \lambda \leq \psi_{\text{minimum}}$ and K must be a barrier. A similar argument applies if $\arg \lambda \leq \arg \alpha < \pi/2$.

To complete the proof one must show that in all other cases $S(\epsilon, \eta)$ is transitive. It suffices to prove that given any neighborhood of the origin, there is a transformation of $S(\epsilon, \eta)$ that takes 1 into this neighborhood. For both Lemma 1 and Lemma 2 apply and further as $z = 1$ is both the source and sink of η , given any neighborhood of $z = 1$ and any point of the extended complex plane, there is an element of $S(\eta)$ that takes the point into the given neighborhood.

If $\alpha > 0$, the real axis is an orbit of η and the flow is toward $z = 1$ for all x on the real axis, $x < 1$. Clearly there is an element of $S(\epsilon)$ that takes $z = 1$ into a point on the negative real axis and then an element of $S(\eta)$ that takes this point into the origin.

If $-\arg \lambda < \arg \alpha < 0$, let L be as above and let C be the circle through the origin, tangent to L at $z = 1$. Both C and L are orbits of η and the flow is as indicated in Figure 1:

FIGURE 1.

Since $-\arg \alpha < \arg \lambda$, the direction field determined by ϵ points into the interior of C at $z = 1$; hence there is an element of $S(\epsilon)$ that takes $z = 1$ into a point of C lying in the upper halfplane (realizing that the orbits of ϵ are spirals with distance from the origin increasing). But now one can find an element of $S(\eta)$ that takes this point into the origin.

Finally, if $0 < \arg \alpha < \arg \lambda$, let L be the line through $z = 1$ whose angle with the positive real axis is $\pi - \arg \alpha$; for each $s \geq 0$ let C_s be the circle through the point $z = -s$ that is tangent to L at $z = 1$. The direction of the flow due to η is as indicated in Figure 2:

FIGURE 2.

Clearly the angle ψ at the point $z = -s$ on the circle C_s is equal to $\arg \alpha$, independently of s . Since $0 < \arg \alpha < \arg \lambda$, it follows that the direction field determined by ϵ points at each point $z = -s$ into the interior of the circle C_s . Hence given any point $z = -s$, it is possible by applying first an element of $S(\epsilon)$ and then an element of $S(\eta)$ to take $z = -s$ into some point $-s'$ where $0 < s' < s$ since from Figure 2 one sees that the flow along each circle C_s is clockwise. By a compactness argument one can show that given any $c > 0$, there exists a $\delta > 0$ such that for all w , $-s \leq w \leq -c$, one can as above transform w into a point w^* also on the negative real axis with $w^* - w > \delta$. Hence for each $z = -s$, given $c > 0$, there is an element of $T(\epsilon, \eta)$ such that $-s$ is taken into a point whose distance from the origin is less than c ; since $z = 1$ can be taken by some element of $S(\epsilon)$ onto the negative real axis, it follows that given any neighborhood of the origin, there is an element of $S(\epsilon, \eta)$ that takes $z = 1$ into this neighborhood and hence $S(\epsilon, \eta)$ is transitive.

V. If one generator is loxodromic and the other elliptic and they have no common fixed point, one can assume the normal form: $\epsilon = iw$, $\eta = \alpha(w-1)(w-\gamma)$ where $|\gamma| \geq 1$, $\operatorname{Re} \alpha(1-\gamma) > 0$ and $\operatorname{Im} \alpha(1-\gamma) \neq 0$. Thus 1 is the source and γ is the sink of η . Let $\mu = \operatorname{Im} \alpha(1-\gamma)/\operatorname{Re} \alpha(1-\gamma)$ and let $\phi = \arg \gamma$.

Theorem 4. Suppose $\epsilon = iw$, $\eta = \alpha(w - 1)(w - \gamma)$ where $|\gamma| \geq 1$, $\operatorname{Re} \alpha(1 - \gamma) > 0$ and $\mu \neq 0$. Then $S(\epsilon, \eta)$ is transitive if and only if:

$$(3) \quad \mu^2 > \frac{(|\gamma| - 1)^2}{2|\gamma|(1 - \cos \phi)}.$$

In particular if $|\gamma| = 1$, $\gamma \neq 1$, $S(\epsilon, \eta)$ is transitive but if $\gamma > 1$, $S(\epsilon, \eta)$ is never transitive.

Proof. To show that $S(\epsilon, \eta)$ is transitive it suffices to prove that given any neighborhood of $z = 1$, there is an element of $S(\epsilon, \eta)$ that takes $z = \gamma$ into this neighborhood (observe that $z = 1$ plays the role of $z = 0$ in Lemmas 1 and 2). If $|\gamma| = 1$, $\gamma \neq 1$, since $S(\epsilon)$ consists of rotations about the origin, there is clearly an element of $S(\epsilon, \eta)$ that takes γ into 1 and hence $S(\epsilon, \eta)$ is transitive.

A necessary condition that $S(\epsilon, \eta)$ be transitive is that on each circle centered at the origin there be at least one point where the flow due to η is directed into the interior of the circle; on each circle there must be a point z such that:

$$(4) \quad \operatorname{Re} \left[\frac{\alpha(z - 1)(z - \gamma)}{z} \right] < 0 \quad \text{or} \quad \operatorname{Re} \left[\alpha \left(z + \frac{\gamma}{z} - (1 + \gamma) \right) \right] < 0.$$

But the map $w = \alpha(z + \gamma/z - (1 + \gamma))$ takes the family of concentric circles centered at the origin into a family of confocal ellipses, the foci being the images under the map of the two square roots of γ , except that the circle in the family of radius $|\gamma|^{1/2}$ is transformed into the line segment joining the foci. Thus (4) is equivalent to the requirement that under the above map at least one of the two square roots of γ is transformed into a point that lies in the left halfplane. For if at least one focus is in the left halfplane, then every ellipse of the image family has points on it that lie in the left halfplane, and if neither focus lies in the left halfplane, neither does any point on the line segment joining these foci. Thus (4) is satisfied for at least one point on each circle centered at the origin if and only if it is satisfied by at least one of the two square roots of γ . A direct calculation then shows that this last condition is equivalent to (3) and hence the necessity of (3) is established.

To prove the sufficiency of (3) assuming $|\gamma| > 1$, one chooses an element of $S(\epsilon)$ that takes γ into some point on $|z| = |\gamma|$ where the flow due to η is directed into the interior; the remaining details are completely analogous to the proof in the last case of Theorem 3 and involve a similar compactness argument.

VI. If one generator is loxodromic and the other hyperbolic and they have no common fixed point, then one can assume two possible normal forms, each of which will be considered separately:

$$\begin{array}{ll} \text{a)} & \epsilon = w, \quad \eta = \alpha(w - 1)(w - \gamma) \\ \text{b)} & \epsilon = -w, \quad \eta = \alpha(w - 1)(w - \gamma) \end{array}$$

where in both cases $|\gamma| \geq 1$, $\operatorname{Im} \gamma \geq 0$, $\operatorname{Re} \alpha(1 - \gamma) > 0$, and $\operatorname{Im} \alpha(1 - \gamma) \neq 0$; μ and ϕ are defined as in V.

Theorem 5. Suppose $\epsilon = w$, $\eta = \alpha(w - 1)(w - \gamma)$ where $|\gamma| \geq 1$, $\operatorname{Re} \alpha(1 - \gamma) > 0$ and $\mu \neq 0$. Then $S(\epsilon, \eta)$ is never transitive.

Proof. Let L be the line that is the perpendicular bisector of the line segment joining 1 and γ ; if $|\gamma| = 1$, then the origin lies on L ; if $|\gamma| > 1$, then the origin and γ are in opposite halfplanes determined by L . In either case the elements of $S(\epsilon)$ transform the halfplane determined by L that contains γ into itself.

The transformation

$$w = \frac{z - 1}{z - \gamma}$$

takes L into the unit circle, with the halfplane that contains γ going into the exterior. But since γ is the sink of η , it is clear that the flow induced by η in the w plane is directed at each point of the unit circle into the exterior, and thus the flow due to η in the z plane is directed at all points of L into the half plane in which γ lies. Hence L is a barrier and $S(\epsilon, \eta)$ is not transitive.

Theorem 6. Suppose $\epsilon = -w$, $\eta = \alpha(w - 1)(w - \gamma)$ where $|\gamma| \geq 1$, $\operatorname{Re} \alpha(1 - \gamma) > 0$, $\mu \neq 0$. Then $S(\epsilon, \eta)$ is transitive if and only if:

$$(5) \quad \mu^2 > \frac{2|\gamma|(1 - \cos \phi)}{(|\gamma| - 1)^2}.$$

In particular if $\gamma > 1$, $S(\epsilon, \eta)$ is always transitive; if $|\gamma| = 1$, $\gamma \neq 1$ it never is transitive.

Proof. One may assume $\operatorname{Im} \gamma \geq 0$. If $\gamma > 1$, there is an element of $S(\epsilon)$ that takes γ into 1 and hence $S(\epsilon, \eta)$ is transitive.

Since the elements of $S(\epsilon)$ transform lines through the origin into themselves, a necessary condition for $S(\epsilon, \eta)$ to be transitive is that on each line through the origin there be at least one point where the flow due to η is directed into the halfplane determined by that line which contains the point $z = 1$. A derivation analogous to that used in the proof of Theorem 4 shows that this geometric condition is equivalent to (5) and hence the necessity of (5) is established.

The proof of the sufficiency of (5) is more complicated than the corresponding proof in Theorem 4 because now $S(\epsilon)$ is not a one-parameter subgroup. Assume $0 < \phi < \pi$ and let L_s be the line through the origin with the slope s for $0 < \tan^{-1} s < \phi$ and let C_s be the image in the w plane of L_s under the map

$$w = T(z) = \frac{z - 1}{z - \gamma};$$

if $\epsilon' = \operatorname{Ad}_T \epsilon$, $\eta' = \operatorname{Ad}_T \eta$ then on each C_s there is a point where the flow due to η' is into the interior of C_s , provided (5) is satisfied. One observes that the origin is in the interior of all circles C_s and that the fixed points of ϵ' are 1 and

$1/\gamma$ with $1/\gamma$ the sink and 1 the source and the fixed points of η' are 0 and ∞ , with 0 the source and ∞ the sink. The flow on the spirals that are the orbits of η' is outward, counterclockwise if $\mu > 0$, clockwise if $\mu < 0$. It suffices to prove that if (5) is satisfied $S(\epsilon', \eta')$ is transitive.

If $\psi(s, z)$ is the angle between 0 and π that is made by the radius vector to the point z on the circle C_s and the tangent to the circle at that point, then there are two points, one the negative of the other, on each circle where $\tan^2 \psi$ is a minimum. At one point the angle ψ is acute, at the other it is obtuse. The map $w^*(t)$ (which can be explicitly calculated) from the interval $(0, \phi)$ on the t axis into the w plane defined by letting correspond to each t the unique point on $C_{\tan t}$ where $\tan^2 \psi$ is a minimum and ψ is acute is a C^1 map without singular points, i.e. $(d/dt)w^*(t) \neq 0$ all t above. The range of this map in the w plane will be called the acute curve and denoted by Γ ; similarly one can define the obtuse curve, denoted by σ , which is the range of the map $-w^*(t)$. The significance of these curves is that if (5) is satisfied, then if $\mu > 0$, the flow due to η' is directed at each point of Γ into the interior of the appropriate circle C_s passing through that point and if $\mu < 0$ a similar assertion is true at each point of σ .

Case $\phi = \pi$: As γ is real, one may in this case actually always achieve a normal form where $\mu > 0$. Further, $\gamma \neq -1$ since (5) is satisfied. Letting $\gamma = -r$ where $r > 1$, one finds that Γ is just the portion of the circle $|w| = r^{-1/2}$ lying in the lower halfplane. If one defines $w^*(0) = r^{-1/2}$, $w^*(\pi) = -r^{-1/2}$, Γ is then a semicircle when these two points are added. To prove $S(\epsilon', \eta')$ transitive, it suffices to show that given any neighborhood of 0, the source of η' , there is an element of $S(\epsilon', \eta')$ that takes $-r^{-1}$, the sink of ϵ' , into this neighborhood.

It is clear that there is an element of $S(\epsilon')$ that takes $r^{-1/2}$ into 0 since $r > 1$; by the corollary to Lemma 2 there must be an arc of Γ with $r^{-1/2}$ as one end-point such that for any point of the arc there is an element of $S(\epsilon', \eta')$ taking that point into 0. Now as $\mu > 0$ (counterclockwise case) the orbit of η' through

FIGURE 3.

$-r^{-1}$ consisting of successors of $-r^{-1}$ must intersect either the interval $(0, r^{-1/2})$ or Γ , observing that $r^{-1} < r^{-1/2}$. But since the flow due to ϵ' is as indicated in Figure 3, the first case is trivial and hence the problem is reduced to considering points on Γ . In view of the above remarks, and the fact that the usual compactness argument can always be applied, it suffices to show that for any point $w_0 = t^{-1/2}e^{-i\theta_0}$, $0 < \theta_0 < \pi$, there is an element of $S(\epsilon', \eta')$ that takes w_0 into $w_1 = t^{-1/2}e^{-i\theta_1}$, where $\theta_0 > \theta_1 > 0$, i.e. an element of $S(\epsilon', \eta')$ that moves w_0 counter-clockwise along Γ .

Let $w_0 = w^*(t_0)$, $0 < t_0 < \pi$, $s_0 = \tan t_0$; then the orbit of η' that consists of successors of w_0 enters at w_0 the interior of C_{s_0} , the unique circle through 1, $-r^{-1}$ and w_0 . But any circle through 1 and $-r^{-1}$ that also passes through a point in the lower halfplane interior to C_{s_0} must in the lower halfplane lie entirely inside C_{s_0} and hence must intersect Γ at a point that lies counter-clockwise relative to w_0 . Thus there is an element of $S(\eta')$ that takes w_0 onto a circle C_{s_1} that intersects Γ at w_1 lying counter-clockwise relative to w_0 and further the flow due to ϵ' is clockwise on the portion of C_{s_1} that lies in the lower halfplane, recalling that $-r^{-1}$ is the sink of ϵ' . But as the flow on the spirals that are orbits of η' is outward, it is possible to find an element of $S(\eta')$ that takes w_0 into a point w' on C_{s_1} , where w' lies in the lower halfplane and $|w'| > r^{-1/2}$; since w_1 is on C_{s_1} and on Γ so that $|w_1| = r^{-1/2}$, it is geometrically clear that if one moves on C_{s_1} from w' to w_1 , one must move clockwise so that there is an element of $S(\epsilon')$ taking w' into w_1 . This completes the proof of the sufficiency of (5) in case $\phi = \pi$.

If $0 < \phi < \pi$, one can easily verify that as $t \rightarrow 0$, $w^*(t) \rightarrow 0$ and as $t \rightarrow \phi$ $w^*(t) \rightarrow \infty$; one adds these limit points to the curves Γ and σ so that the map $w^*(t)$ is now defined on the compact set $[0, \phi]$. The points 0 and ∞ are then the only points on both the acute and obtuse curves. One finds by direct calculation that as t increases, the distance of $w^*(t)$ from the origin increases.

A detailed analysis shows that the points 1 and γ^{-1} both lie on the acute curve; these points are significant as they are the fixed points of ϵ' . In fact if t_1 , $0 < t_1 < \phi$ is such that $\tan t_1 = \text{Im } \gamma / (|\gamma|^2 + \text{Re } \gamma)$, then $w^*(t_1) = \gamma^{-1}$ and if t_2 , $0 < t_2 < \phi$ is such that $\tan t_2 = \text{Im } \gamma / (1 + \text{Re } \gamma)$ then $w^*(t_2) = 1$. It is useful to add to the family of circles C_s the circle C_0 that passes through 0, 1 and γ^{-1} and the line $C_{\tan \phi}$ that passes through γ^{-1} and 1. If one denotes by R the region in the lower halfplane bounded by C_0 and $C_{\tan \phi}$ (see Figure 4), then for all t , $t_1 < t < t_2$, $w^*(t)$ lies in R , i.e. the arc of the acute curve joining γ^{-1} and 1 lies entirely in R . This follows from the fact that the arc of every circle C_s , $0 < \tan^{-1} s < \phi$, that joins 1 and γ^{-1} and lies in the lower halfplane must lie entirely in R and the fact that a direct calculation shows that the argument of $w^*(t)$ increases continuously with increasing t so that if one moves along either Γ or σ in direction determined by increasing t , one moves continuously in the counter-clockwise direction in the w plane. Using all the information obtained, one can sketch the curves Γ and σ as shown in Figure 5. Observe that $0 < \phi < \pi$ implies that γ^{-1} lies in the lower halfplane. One remarks that

FIGURE 4.

through each point of σ there passes a unique circle C_s ; the same is true for all points of Γ except for 1 and γ^{-1} through which all the circles pass.

Assume (5) is satisfied; then $|\gamma| > 1$ and one divides the proof that $S(\epsilon', \eta')$ is transitive into two cases:

Case $\mu < 0$ —clockwise case: In this case the obtuse curve σ will be used. By arguments completely analagous to those previously described, it suffices to show that given any point other than 0 and ∞ on σ , there is an element of $S(\epsilon', \eta')$ that takes that point into another point also on σ but closer to the origin than the initial point.

FIGURE 5. Case $\operatorname{Re} \gamma < 0$

Define the right and left side of σ as the sides that are on one's right and left respectively as one moves along σ in the direction determined by increasing t . Since this direction is such that the distance from the origin increases and one moves counter-clockwise, the angle ψ^* between the radius vector to a point on σ and the tangent to σ at that point is an acute angle. Thus as $\mu < 0$, the flow at points of σ determined by η' is such that one moves initially to the right side of σ . Further this flow is directed into the interior of the unique circle C_s through that point. If t is such that $\tan t = s$, and $t' < t$, then the arc of the circle $C_{s'}$, $s' = \tan t'$, joining 1 and γ^{-1} that lies outside R is in the interior of C_s . Combining these remarks one sees that there is an element of $S(\eta')$ that takes the given point on σ into a point lying on the right side of σ on a circle $C_{s'}$ that intersects σ at a point closer to the origin than the initial point. But the flow due to ϵ' along $C_{s'}$ is counterclockwise on that portion of $C_{s'}$ that lies outside R . Thus there is an element of $S(\epsilon')$ that takes the image of the original point under the element of $S(\eta')$ into the point of intersection of σ and $C_{s'}$.

Case $\mu > 0$ —counter-clockwise case: Denote by K the portion of Γ that joins γ^{-1} and 1; thus K is the range of the map $w^*(t)$ restricted to the interval $t_1 \leq t \leq t_2$. With each point of K except 1 and γ^{-1} one associates the unique circle $C_{\tan t}$, $t_1 < t < t_2$ through that point; with γ^{-1} one associates the circle $C_{\tan t_1}$ and with 1 the circle $C_{\tan t_2}$ (all circles pass through 1 and $1/\gamma$). Then at each point of K the flow due to η' is into the interior of the associated circle through that point. Define in a manner determined by increasing t the right and left side of K . Let $\psi^*(t)$ be the angle between the radius vector and the tangent to K , defined for $t_1 \leq t \leq t_2$. As before one sees that ψ^* is always an acute angle. Let $\psi(t)$ be the angle between the radius vector to the point $w^*(t)$ on K and the tangent to the unique circle of the family C_s associated with that point, direction of tangent being determined by counter-clockwise motion along the circle. One can show that:

$$(6) \quad \tan \psi^*(t) = \frac{|\gamma|^2 - 1}{2|\gamma| \sin \phi} \tan^2 \psi(t), \quad t_1 \leq t \leq t_2.$$

Also since K is part of the acute curve $\psi(t)$ is always an acute angle.

Using (6) and calculating $\tan^2 \psi(t_1)$ and $\tan^2 \psi(t_2)$ one finds that $\psi(t_1) = \psi^*(t_1)$ and $\psi(t_2) = \psi^*(t_2)$. If $s_1 = \tan t_1$, $s_2 = \tan t_2$, then at γ^{-1} the curve K and the circle C_{s_1} are tangent and at 1 the curve K and the circle C_{s_2} are tangent. Also from (6) and using an explicit formula for $\tan^2 \psi(t)$ one finds that the function $\tan \psi^*(t)$ is increasing for $t_1 \leq t \leq \phi/2$ and decreasing for $\phi/2 \leq t \leq t_2$ (one can verify that $t_1 < \phi/2 < t_2$). Now at γ^{-1} the fact that the flow due to η' is directed into the interior of C_{s_1} means that $\mu > \tan \psi(t_1) = \tan \psi^*(t_1)$; similarly $\mu > \tan \psi^*(t_2)$. First consider the case when $\mu \geq \tan \psi^*(\phi/2)$.

In this case at all points of K the flow due to η' is directed to the left side of K except that possibly at $w^*(\phi/2)$ the flow is directed tangent to K . Referring to Figure 6 it is clear that this implies that if one moves along the orbit of η' through

FIGURE 6.

γ^{-1} in the direction of the flow (outward and counter-clockwise) one must intersect the portion of the circle C_0 that lies in the upper halfplane. But the flow due to ϵ' along the part of C_0 in the upper halfplane is counter-clockwise as γ^{-1} is the sink of ϵ' . Hence it is possible to find an element of $S(\epsilon', \eta')$ that takes γ^{-1} into the origin and hence $S(\epsilon', \eta')$ is transitive.

If $\mu < \tan \psi^*(\phi/2)$, one chooses a and b , $t_1 < a < \phi/2 < b < t_2$ so that $\mu = \tan \psi^*(a) = \tan \psi^*(b)$; then for $a < t < b$, $\mu < \tan \psi^*(t)$ while for both $t_1 \leq t < a$ and $b < t \leq t_2$, $\mu > \tan \psi^*(t)$. Thus for points $w^*(t)$ on K with $t_1 \leq t < a$ and $b < t \leq t_2$, the flow due to η' is directed to the left side; if $a < t < b$, it is directed to the right side and at $w^*(a)$ and $w^*(b)$ the flow is tangent to K .

If as above, one moves along the orbit of η' through γ^{-1} there are two possibilities: either one intersects the portion of C_0 that lies in the upper halfplane or one intersects K at a point $w^*(t)$ for some t , $a < t < b$. In the first case one then sees as above, that $S(\epsilon', \eta')$ is transitive (if both cases occur and this is not excluded, there is no problem for one only needs the fact that one can then transform γ^{-1} into a point on C_0 in the upper halfplane by an element of $S(\eta')$).

FIGURE 7.

In the second case one can assert that there is an element of $S(\eta')$ that takes γ^{-1} into a point that lies in the interior of the arc of K joining $w^*(a)$ and $w^*(b)$.

Clearly there is an element of $S(\epsilon', \eta')$ that takes $w^*(b)$ into the origin—observe that even though the flow due to η' is at $w^*(b)$ tangent to K , nevertheless the orbit of η' through $w^*(b)$ must always be (locally) on the left side of K to be consistent with the direction field induced by η' . To complete the proof it thus suffices to prove that given any point $w^*(t)$ where $a < t < b$ there is an element of $S(\epsilon', \eta')$ that takes $w^*(t)$ into $w^*(t')$ where $t < t' < b$, for once this is established, the remainder of the proof is along lines analagous to those already described. But at $w^*(t)$ the flow due to η' is directed to the right side of K , and into the interior of the circle C_s where $s = \tan t$. If $t' > t$ and $s' = \tan t'$, then the arc of the circle $C_{s'}$ joining 1 and γ^{-1} that lies inside R is in the interior of C_s . Thus there is an element of $S(\eta')$ that takes $w^*(t)$ into a point w' that lies on some circle C_{s_1} , $s_1 = \tan t_1$ where $t < t_1 < b$, and that also lies on the right side of K . The flow due to ϵ' on the arc of C_{s_1} that joins 1 and γ^{-1} and lies inside R is clockwise. Further since if one moves along K in the direction of increasing t one enters at $w^*(t)$ the interior of C_s , and, in fact, remains there, it follows that if one moves along the circle C_s clockwise, at $w^*(t)$ one passes from the right side of K to the left side of K . Applying this fact (with $t = t_1$) it is clear that there is an element of $S(\epsilon')$ that takes the point w' on the circle C_{s_1} into $w^*(t_1)$ as w' being on the right side of K , the direction of motion on C_{s_1} from w' to $w^*(t_1)$ is clockwise. This completes the proof in case $\mu < \tan \psi^*(\phi/2)$ and hence of Theorem 6.

Observe the similarity between condition (3) of Theorem 4 and (5) of Theorem 6: if $F(\gamma) = (|\gamma| - 1)^2/2 |\gamma| (1 - \cos \phi)$ then the first requires $\mu^2 > F(\gamma)$ and the second requires $\mu^2 > 1/F(\gamma)$.

VII. Since for $k > 0$, $S(\epsilon) = S(k\epsilon)$ and for $k < 0$, $S(\epsilon) = S(k\epsilon)$ if and only if ϵ is elliptic, to each infinitesimal transformation $\epsilon = aw^2 + 2bw + c$, a, b, c complex, that is not elliptic there corresponds a unique point on the unit sphere in E^6 and to each elliptic generator there corresponds a unique pair of antipodal points on the unit sphere in E^6 , and this correspondence is one-one, onto. After identification of the appropriate antipodal points for elliptic generators, one can make the unit sphere in E^6 into a topological space in the usual manner. Denote this space by h and put the product topology on $h \times h$. There is a natural identification between a pair of infinitesimal transformations ϵ and η and an element of $h \times h$ —one denotes this element by (ϵ, η) .

Theorem 7. *Let $D = \{(\epsilon, \eta) \in h \times h : S(\epsilon, \eta) \text{ is transitive}\}$. Then D is open.*

Proof. Given any element of D , one must find a neighborhood of the element contained in D , i.e., for all (ϵ', η') in the neighborhood, $S(\epsilon', \eta')$ is transitive. The proof of the existence of such a neighborhood is facilitated if one establishes the following two lemmas:

Lemma 3. *Suppose $\epsilon = \lambda w$, $\text{Im } \lambda \neq 0$ and η does not leave 0 fixed. Then there is a neighborhood U of 0 in the z plane, a neighborhood V of η in h and a $\delta > 0$ such that if $z \in U$, $\eta' \in V$ and $\epsilon' = \lambda' w$ where $|\lambda' - \lambda| < \delta$, there is an element of $T(\epsilon', \eta')$ that takes z into 0.*

Proof. This is a generalization of Lemma 2; the proof, although more detailed than that of Lemma 2, involves the same basic idea and hence is left to the reader.

Lemma 4. *Let U be a neighborhood of the origin in the z plane. Suppose $S(\epsilon, \eta)$ is transitive. Then there is a neighborhood R of (ϵ, η) in $h \times h$ such that if $(\epsilon', \eta') \in R$, there exists a transformation of $T(\epsilon', \eta')$ that takes ∞ into a point of U .*

Proof. By Theorem 2, $T(\epsilon, \eta)$ is transitive and hence there is an element of this semigroup taking ∞ into 0. The solution of the differential equation $dw/dt = a(t)w^2 + 2b(t)w + c(t)$, $0 \leq t \leq t_0$, t_0 finite, depends continuously upon the coefficients which here are piecewise constant and the lemma follows.

Since the non-singular linear transformation Ad_T depends continuously on T , it follows that $Ad_T(\eta)$ is jointly continuous in T and η ; thus if one denotes by I the identity of the Moebius group, given any neighborhood V_1 on η in h , there is a neighborhood V_2 of η in h and a neighborhood N of I in the Moebius group such that for all $\eta' \in V_2$ and all $T \in N$, $Ad_T(\eta') \in V_1$. Further given any neighborhood N of I in the Moebius group, one can find positive real numbers s and t such that for all a , $|a| < s$ and for all b , $|b| > t$, the unique Moebius transformation $w = T(z)$ such that $T(a) = 0$, $T(b) = \infty$ and $T(1) = 1$ belongs to N .

To prove the theorem, consider first those elements of D for which either ϵ or η is not elliptic. From Theorem 1 it follows that either ϵ or η must be loxodromic; also ϵ and η have no common fixed point. Hence one may assume that $\epsilon = \lambda w$, $\text{Re } \lambda > 0$, $\text{Im } \lambda \neq 0$ and η leaves neither 0 nor ∞ fixed.

Lemma 3 is applicable to this pair ϵ and η ; choose a neighborhood U of the origin, a neighborhood V_1 of η in h and a $\delta_1 > 0$ so that this lemma holds. Since $S(\epsilon, \eta)$ is transitive, Lemma 4 can be applied choosing as the neighborhood U of the origin the one above; one can in fact choose the neighborhood R of (ϵ, η) in $h \times h$ to be of the form $W_2 \times V_2$ where W_2 is a neighborhood of ϵ in h , V_2 a neighborhood of η in h . Choose a neighborhood N of I in the Moebius group and a neighborhood V of η in h such that if $\eta' \in V$, $T \in N$, then $Ad_T(\eta') \in V_1 \cap V_2$. For this N choose the positive real numbers s and t as described above. Next choose a neighborhood W_1 of ϵ in h such that if $\epsilon' \in W_1$, then ϵ' is loxodromic and if z_0 is the source of ϵ' , z_1 the sink, then $|z_0| < s$ and $|z_1| > t$. Choose $\delta_2 > 0$ so that if $\epsilon' = \lambda' w$ with $|\lambda' - \lambda| < \delta_2$, then $\epsilon' \in W_2$. Choose $\delta_3 > 0$ so that if $|a - \lambda| < \delta_3$, then $\text{Re } a > 0$. Let $\delta = \min \{\delta_1, \delta_2, \delta_3\}$. Choose a neighborhood W_3 of ϵ in h such that for all $\epsilon' \in W_3$, ϵ' is loxodromic and if one denotes by the (discriminant ϵ')^{1/2} that branch of the square root whose imaginary part has the same sign as the imaginary part of λ , then $|(\text{discriminant } \epsilon')^{1/2} - \lambda| < \delta$.

Let $W = W_1 \cap W_3$ and choose as the neighborhood of (ϵ, η) $W \times V$; it will now be shown that $W \times V \subset D$.

Suppose $(\epsilon', \eta') \in W \times V$, i.e. $\epsilon' \in W$, $\eta' \in V$. Then $\epsilon' \in W_1$ and hence ϵ' is loxodromic and if z_0 is its source, z_1 its sink, then the unique Moebius transformation $w = T(z)$ such that $T(z_0) = 0$, $T(z_1) = \infty$ and $T(1) = 1$ is an element of N . Let $\epsilon^* = Ad_T(\epsilon')$, $\eta^* = Ad_T(\eta')$. $S(\epsilon', \eta')$ is transitive if and only if $S(\epsilon^*, \eta^*)$ is transitive.

Observe that $\eta^* \in V_1 \cap V_2$ and $\epsilon^* = \lambda'w$ for some λ' where $\text{Re } \lambda' > 0$ as 0 must be the source, ∞ the sink of ϵ^* . ϵ^* is loxodromic and $\lambda'^2 = \text{discriminant of } \epsilon^* = \text{discriminant of } \epsilon'$ and as $\epsilon' \in W_3$, either $|\lambda' - \lambda| < \delta$ or $|\lambda' - \lambda| < \delta$. But as $\delta \leq \delta_3$, if $|\lambda' - \lambda| < \delta$, then $\text{Re } (-\lambda') > 0$ which is a contradiction; hence $|\lambda' - \lambda| < \delta$. As $\eta^* \in V_1$ and $|\lambda' - \lambda| < \delta_1$, given any $z \in U_1$ there is an element of $T(\epsilon^*, \eta^*)$ that takes z into 0 . Since $|\lambda' - \lambda| < \delta_2$, $\epsilon^* \in W_2$ and as $\eta^* \in V_2$, there is an element of $T(\epsilon^*, \eta^*)$ that takes ∞ into some point of U and hence there is an element of $T(\epsilon^*, \eta^*)$ that takes ∞ into 0 . In particular η^* leaves neither 0 nor ∞ fixed. As ∞ is the sink of ϵ^* , given any neighborhood of ∞ and any $z \neq 0$, there is an element of $S(\epsilon^*)$ taking z into that neighborhood. By the corollary to Lemma 2, $A = \{z : \text{there exists an element of } T(\epsilon^*, \eta^*) \text{ taking } z \text{ into } 0\}$ is an open set and as $\infty \in A$, it follows that $A = \text{extended complex plane}$. In view of Lemma 1 which applies to ϵ^* and η^* it follows that $S(\epsilon^*, \eta^*)$ is transitive.

If both ϵ and η are elliptic, then by Theorem 1 the fact that $S(\epsilon, \eta)$ is transitive implies that the cross-ratio of the roots of ϵ' and η' is not positive. Clearly there is a neighborhood R_1 of (ϵ, η) in $h \times h$ such that for all $(\epsilon', \eta') \in R_1$ the cross-ratio of the roots of ϵ' and η' is not positive. By Theorem 1 it follows that if $(\epsilon', \eta') \in R_1$ and both ϵ' and η' are elliptic, then $S(\epsilon', \eta')$ is transitive.

There exists a neighborhood R_2 of (ϵ, η) in $h \times h$ such that for all elements $(\epsilon', \eta') \in R_2$ the real part of the discriminant of both ϵ' and η' is negative.

Now one may assume $\epsilon = iw$, and η leaves neither 0 nor ∞ fixed. Lemma 3 is applicable; in fact the requirement that $|\lambda' - i| < \delta$ can be relaxed to $|\lambda' - i| < \delta$ or $|\lambda' + i| < \delta$. Lemma 4 is also applicable as $S(\epsilon, \eta)$ is transitive and one continues exactly as in the previous case where ϵ was equal to λw up to the point where W_1 is defined. One chooses a neighborhood W_1 of ϵ in h such that if $\epsilon' \in W_1$, then ϵ' is either elliptic or loxodromic, and one fixed point of ϵ' has absolute value less than s and the other absolute value greater than t ; denote these by z_0 and z_1 respectively. Choose $\delta_2 > 0$ so that if $\epsilon' = \lambda'w$ with either $|\lambda' - i| < \delta_2$ or $|\lambda' + i| < \delta_2$, then $\epsilon' \in W_2$ (this is possible since iw and $-iw$ are identified). Let $\delta = \min(\delta_1, \delta_2)$. Choose a neighborhood W_3 of ϵ in h such that for all $\epsilon' \in W_3$, if one denotes by $(\text{discriminant } \epsilon')^{1/2}$ that branch of the square root whose imaginary part is positive, then $|(\text{discriminant } \epsilon')^{1/2} - i| < \delta$. Let $W = W_1 \cap W_3$ and choose as the neighborhood of (ϵ, η) $W \times V$; one may assume the neighborhood is of the form that if (ϵ', η') belongs to the neighborhood, then so does $(-\epsilon', -\eta')$ since (ϵ, η) and $(-\epsilon, -\eta)$ are identified.

If $(\epsilon', \eta') \in W \times V$ and ϵ' is loxodromic, then $S(\epsilon', \eta')$ is transitive; one pro-

ceeds as before and finds $\epsilon^* = \lambda'w$ where ϵ^* is loxodromic and one may assume $\operatorname{Re} \lambda' > 0$ because if $\operatorname{Re} \lambda' < 0$, i.e. z_0 were the sink, z_1 the source of ϵ' , then one considers instead of (ϵ', η') the element $(-\epsilon', -\eta')$ and observes that z_0 is the source of $-\epsilon'$, z_1 the sink of $-\epsilon'$, and that $S(\epsilon', \eta')$ is transitive if and only if $S(-\epsilon', -\eta')$ is transitive. The rest of the proof is identical except that now one can conclude only that either $|\lambda' - i| < \delta$ or $|\lambda' + i| < \delta$ but as pointed out above this is in this case sufficient to insure both that $\epsilon^* \in W_2$ and that Lemma 3 holds.

Thus one concludes that there exists a neighborhood R_3 of (ϵ, η) in $h \times h$ such that for all $(\epsilon', \eta') \in R_3$ with ϵ' loxodromic, then $S(\epsilon', \eta')$ is transitive. By symmetry one can find a neighborhood R_4 of (ϵ, η) in $h \times h$ such that for all $(\epsilon', \eta') \in R_4$ with η' loxodromic, then $S(\epsilon', \eta')$ is transitive. Let $R = R_1 \cap R_2 \cap R_3 \cap R_4$ —this is then the desired neighborhood of (ϵ, η) and the proof of Theorem 7 is complete.

Corollary 1. Suppose $\epsilon = (1 + ti)w$, $\eta = \alpha(w - 1)(w - \gamma)$ where $|\gamma| \geq 1$, $\operatorname{Re} \alpha(1 - \gamma) > 0$, $\mu = \operatorname{Im} \alpha(1 - \gamma)/\operatorname{Re} \alpha(1 - \gamma) \neq 0$. As before $\phi = \arg \gamma$, $-\pi < \phi \leq \pi$; suppose

$$(7) \quad \mu^2 > \frac{(|\gamma| - 1)^2}{2|\gamma|(1 - \cos \phi)}.$$

Then there is an $M > 0$ such that for all t , $|t| > M$, $S(\epsilon, \eta)$ is transitive. The same result holds if $\epsilon = (-1 + ti)w$.

Proof. Combine Theorem 7 and Theorem 4.

Corollary 2. Suppose $\epsilon = (-1 + ti)w$, $\eta = \alpha(w - 1)(w - \gamma)$ where $|\gamma| \geq 1$, $\operatorname{Re} \alpha(1 - \gamma) > 0$, $\mu \neq 0$; suppose

$$(8) \quad \mu^2 > \frac{2|\gamma|(1 - \cos \phi)}{(|\gamma| - 1)^2}.$$

Then there is a $\delta > 0$ such that for all t , $|t| < \delta$, $S(\epsilon, \eta)$ is transitive.

Proof. Combine Theorem 6 and Theorem 7.

One observes that Corollaries 1 and 2 give examples where $S(\epsilon, \eta)$ is transitive when both ϵ and η are loxodromic.

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Date communicated: DECEMBER 27, 1967