Continuity of Envelopes of Plurisubharmonic Functions

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1. Introduction. Let D be a domain in $\mathbf{C}^n$ and let g be an extended real-valued function on $\bar{D}$. If u is any function with domain D , by u^* we mean the function defined by $u^*(z) = \limsup_{x \rightarrow z, x \in D} u(x)$, $z \in \bar{D}$. Let $U_g(D)$ be the set of all functions u plurisubharmonic in D which satisfy $u^* \leq g$. Let Φ_g^D be the upper envelope of all such functions, that is $\Phi_g^D(z) = \sup \{u(z) : u \in U_g\}$. We are primarily interested in two special cases.

The first is when $g = +\infty$ except on some subset K of ∂D . This case has been studied by Bremermann, who introduced such lower envelopes to solve an extended Dirichlet problem for plurisubharmonic functions. In one (complex) dimension such upper envelopes are harmonic and therefore continuous—even infinitely differentiable—in D . In higher dimensions they are plurisubharmonic or almost plurisubharmonic rather than harmonic, so questions of continuity are somewhat more delicate and have in fact been left open. Some results were obtained in [4] for a related Dirichlet problem involving multiply superharmonic rather than plurisuperharmonic functions; they were proved by probabilistic considerations. Here we show by analytic methods that analogous results hold for plurisubharmonic functions. In particular we give some (very restrictive) sufficient conditions for continuity and some examples to show that the behavior of these envelopes can be almost arbitrarily bad, even for continuous boundary functions.

The second case is when $g \equiv +\infty$ on ∂D , and g is either continuous or plurisuperharmonic on D . Again, in one dimension such envelopes are known to be continuous [1]. In higher dimensions, their continuity behavior depends on the boundary of the region, and they are not continuous in general.

2. A continuity lemma. The underlying observation in this paper is that the continuity behavior of an upper envelope is determined by its behavior at the boundary.

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Lemma 1. *Let D be a bounded domain in $\mathbb{C}^n$ and let $g : \bar{D} \rightarrow [-\infty, \infty]$ have the property that its restriction to D is either uniformly continuous or identically $+\infty$. If Φ_g^{D*} is dominated by g in $\bar{D}$ and is continuous at ∂D , then Φ_g^D is continuous in D .*

Proof. Note that the restriction of Φ_g^{D*} to D is in $U_o(D)$, so $\Phi_g^{D*} \leq \Phi_g^D$. The opposite inequality is clear. Thus Φ_g^D is plurisubharmonic hence upper semi-continuous in D . We will show it is also lower semi-continuous.

Φ_g^{D*} is continuous at ∂D , which is compact, and g is either uniformly continuous or infinite on D , so for $\epsilon > 0$ there exists $\delta > 0$ such that

$$(1) \quad x \in \bar{D}, x_0 \in \partial D \quad \text{and} \quad |x - x_0| < 3\delta \quad \text{imply} \quad |\Phi_g^{D*}(x) - \Phi_g^{D*}(x_0)| < \frac{\epsilon}{2};$$

$$(2) \quad y, z \in D \quad \text{and} \quad |y - z| < \delta \quad \text{imply} \quad g(y) \geq g(z) - \epsilon.$$

If $y \in \mathbb{C}^n$ is such that $|y| < \delta$, define H_y by

$$(3) \quad \begin{aligned} H_y(z) &= \max \{ \Phi_g^{D*}(z), \Phi_g^{D*}(z + y) - \epsilon \} & \text{if } z, z + y \in \bar{D} \\ &= \Phi_g^{D*}(z) & \text{if } z \in \bar{D}, y + z \notin \bar{D}. \end{aligned}$$

We claim the restriction of H_y to D is in $U_o(D)$.

Let $d(z, B)$ be the distance between the point z and the set B . If $z \in D$ and $d(z, \partial D) > \delta$, both $\Phi_g^{D*}(\cdot)$ and $\Phi_g^{D*}(\cdot + y)$ are plurisubharmonic in a neighborhood of z , hence so is $H_y(\cdot)$. On the other hand if $z \in \bar{D}$ and $d(z, \partial D) < 2\delta$, $H_y(z) = \Phi_g^{D*}(z)$: this is true by definition if $z + y \notin \bar{D}$; if $z + y \in \bar{D}$, choose $z_0 \in \partial D$ such that $|z_0 - z| < 2\delta$. Certainly $|z + y - z_0| < 3\delta$ so $\Phi_g^{D*}(z + y)$ and $\Phi_g^{D*}(z)$ each differ from $\Phi_g^{D*}(z_0)$ by $\epsilon/2$ or less, implying $\Phi_g^{D*}(z) \geq \Phi_g^{D*}(z + y) - \epsilon$, and $H_y(z) = \Phi_g^{D*}(z)$. Thus H_y is plurisubharmonic in D , and dominated by g on the boundary. By (2), if $z \in D$ and $d(z, \partial D) \geq 2\delta$:

$$\Phi_g^{D*}(z + y) - \epsilon \leq g(z + y) - \epsilon \leq g(z).$$

As $\Phi_g^{D*}(z) \leq g(z)$ by hypothesis, $H_y(z) \leq g(z)$ for all z . Then $H_y \in U_o$. Consequently, $H_y \leq \Phi_g^{D*}$. It follows that if $x, z \in \bar{D}$ are such that $|z - x| < \delta$, then

$$\Phi_g^{D*}(z) \geq H_{x-z}(z) \geq \Phi_g^{D*}(z + x - z) - \epsilon = \Phi_g^{D*}(x) - \epsilon,$$

which shows Φ_g^{D*} is lower semi-continuous at x , and completes the proof.

3. Bremermann's Dirichlet problem. If we take g to be identically $+\infty$ off some subset $K \subset \partial D$, the function Φ_g^D coincides with the Dirichlet solutions introduced by Bremermann. According to Lemma 1, such a function is continuous if it is continuous at the boundary and dominated by g on K .

We will say a point $\zeta \in \partial D$ is strongly regular if there is a plurisuperharmonic barrier at ζ , that is, if there is a neighborhood N of ζ and a function f defined and plurisuperharmonic in $N \cap D$ which has the limit 0 at ζ , and is bounded away from zero outside any neighborhood of ζ . If g is defined and continuous on

∂D , and if each point of ∂D is strongly regular, then for all $\zeta \in \partial D$, $\lim_{z \rightarrow \zeta} \Phi_g^D(z) = g(\zeta)$. ([3]; though apparently more general, the definition of a plurisuperharmonic barrier there is equivalent to the above.) Thus Φ_g^D is continuous on $\bar{D}$. In particular, since each boundary point of a bounded strictly pseudo-convex domain is strongly regular [2], we have:

Theorem 1. *Let D be a bounded strictly pseudo-convex domain. If g is continuous on ∂D , the function Φ_g^D is continuous in $\bar{D}$.*

If we relax the requirement that g be continuous, it is possible to find boundary functions g for which the behavior of Φ_g is almost arbitrarily bad. A large class of such examples can be constructed on the basis of the following observations.

a. If u is subharmonic in the complex plane then the function v defined by

$$(4) \quad v(z) = u(z_1), \quad \text{where } z = (z_1, z_2), z_1 \in \mathbf{C},$$

is plurisubharmonic in $\mathbf{C}^2$.

b. If D is any bounded domain in $\mathbf{C}^2$, v is defined by (4), and h is plurisubharmonic in D and satisfies $\limsup_{z \rightarrow \zeta} h(z) \leq v(\zeta)$ for every $\zeta \in \partial D$, then $h \leq v$ in D .

The first statement is clear and the second follows from the maximum principle for ordinary subharmonic functions applied to the restrictions of v and h to one (complex) dimensional varieties V of the form $V = D \cap \{z : z_1 = \alpha\}$, where $\alpha \in \mathbf{C}$.

Now let u be subharmonic in the complex plane, and define v by (4). Let D be any bounded domain in $\mathbf{C}^2$, and take g to be infinite in D , and equal to the restriction of v to ∂D on the boundary. Noting that $v \in U_g$, we see from (b) that

$$(5) \quad \Phi_g^D = v \quad \text{in } D.$$

In the following three examples, D is any bounded domain in $\mathbf{C}^2$, $z = (z_1, z_2)$ is a point in $\mathbf{C}^2$, and v and g are as defined above.

Example 1. Choose $u(z_1) = \log |z_1|$. Then $\Phi_g^D(z) = \log |z_1|$. If D contains the origin, Φ_g^D is infinite at interior points of D but not identically infinite; this is a counter example to a conjecture of Bremermann [2].

Example 2. Let u be any subharmonic function with infinities dense in $\mathbf{C}$. Then Φ_g^D has infinities dense in D .

Example 3. Let u be as in Example 2, and set $u' = \max(u, -b)$ where $b > 0$. u' will be bounded in D . For b sufficiently large, u' will have discontinuities on a set of positive two-dimensional Lebesgue measure, hence $\Phi_g^D(z) = u'(z_1)$, $z \in D$, will be discontinuous on a set of positive four-dimensional Lebesgue measure.

4. Greatest plurisubharmonic minorants. In this section we consider the special case where the dominant function g is continuous or plurisuperharmonic in D , and either undefined or $+\infty$ on ∂D . It is easy to see that if g has a plurisubharmonic minorant, it has a greatest plurisubharmonic minorant, namely Φ_g^D . The case where g is plurisuperharmonic can be derived from the case where g is continuous by noting that if g is plurisuperharmonic, then $\Phi_g^D = \Phi_f^D$, where f is the greatest harmonic minorant of g , which is continuous.

The following lemma is standard:

Lemma 2. *Let D and D' be regions in $\mathbb{C}^n$ with $D \cap D' \neq \emptyset$ and let u and v be plurisubharmonic in D and D' respectively. Then if $u \leq v$, in $D \cap D'$, the function*

$$w = \begin{cases} v & \text{in } D' \\ u & \text{in } D - D' \end{cases}$$

is plurisubharmonic if it is upper semi-continuous.

Lemma 3. *Let D' be relatively compact in D , and let g be an extended real-valued function on $\bar{D}$ which has a plurisubharmonic minorant. If $\Phi_g^{D*} \leq g$ and if h is defined on $\bar{D}'$ by*

$$h(x) = \begin{cases} g(x) & x \in D' \\ \Phi_g^D(x) & x \in \partial D', \end{cases}$$

then $\Phi_h^{D'} = \Phi_g^D$ in D' .

Proof. Under the hypotheses, $\Phi_g^{D*} = \Phi_g^D$ in D , hence the latter is plurisubharmonic; its restriction to D' is in $U_h(D')$, so $\Phi_h^{D'} \geq \Phi_g^D$ in D' . On the other hand, if $v \in U_h(D')$, let

$$H(x) = \begin{cases} \Phi_g^D & \text{in } D - D' \\ \max \{ \Phi_g^D, v \} & \text{in } D'. \end{cases}$$

Then by Lemma 2, $H(x)$ is plurisubharmonic in D , and $H \in U_g(D)$. Thus $H \leq \Phi_g^D$, so in particular $v \leq \Phi_g^D$ in D' ; thus, taking the supremum over all such v , we see $\Phi_h^{D'} \leq \Phi_g^D$. Q.E.D.

Theorem 2. *Let g be either continuous or plurisuperharmonic on D , $+\infty$ on ∂D . Suppose g has a plurisubharmonic minorant. Then the set of discontinuities of Φ_g^D cannot have compact closure in D .*

Proof. By previous remarks, it is enough to prove this when g is continuous. Clearly $\Phi_g^{D*} \leq g$. If the set of discontinuities of Φ_g^D has compact support, choose a relatively compact subdomain D' of D which contains the closure of the set of discontinuities of Φ_g^D . Let

$$h(x) = \begin{cases} g(x) & x \in D' \\ \Phi_g^D(x) & x \in \partial D'. \end{cases}$$

By Lemma 3, $\Phi_h^{D'} = \Phi_g^D$ in D ; as the latter is continuous on $\partial D'$, it must be continuous in D' by Lemma 1, and the set of discontinuities of Φ_g^D must be empty. Q.E.D.

It is not *a priori* obvious that the continuity behavior of the greatest plurisubharmonic minorant of a continuous function should depend on the boundary of the region. In fact it does, as the next theorem and following example show.

Theorem 3. *If D is a bounded domain in $\mathbf{C}^n$, all of whose boundary points are strongly regular, and if g is continuous on $\bar{D}$, then Φ_g^D is continuous, and $\Phi_g^{D*} = g$ on ∂D .*

Proof. In view of Lemma 1, it is enough to show that $\lim_{\epsilon \rightarrow x} \Phi_g^D(z) = g(x)$, $x \in \partial D$. Let $x \in \partial D$ and let N be a ball centered at x of small enough radius so that the oscillation of g on N is less than $\epsilon/3$. Define h on $\partial(N \cap D)$ by

$$h(x) = \begin{cases} \Phi_g^D(x) - \epsilon & x \in D \cap \partial N \\ g(x) - \frac{1}{2}\epsilon & \text{otherwise,} \end{cases}$$

and let

$$H(z) = \begin{cases} \Phi_g^D(z) - \epsilon & \text{if } z \in D - N \cap D \\ \Phi_h^{N \cap D^*}(z) & \text{if } z \in N \cap D. \end{cases}$$

Using the fact that the points of $D \cap \partial N$ are strongly regular in $N \cap D$, it is readily verified that H is plurisubharmonic and lies between $\Phi_g^D - \epsilon$ and g in D . Furthermore, h is continuous at the strongly regular point x , so $\lim_{y \rightarrow x} H(y) = h(x)$. Thus

$$\begin{aligned} g(x) - \frac{1}{2}\epsilon &= \lim_{y \rightarrow x} H(y) \leq \liminf_{y \rightarrow x} \Phi_g^D(y) \\ &\leq \limsup_{y \rightarrow x} \Phi_g^D(y) \leq g(x), \end{aligned}$$

hence Φ_g^D has the limit g at ∂D .

Q.E.D.

The next example shows that without regularity conditions on the boundary, the conclusions of both Theorem 1 and Theorem 3 are false in general and in fact are even false for some domains which are pseudo-convex but not strictly so.

Example 4. Let u be subharmonic and bounded in $\mathbf{C}$ and suppose the region of negativity of u is bounded and non-empty. Define plurisubharmonic functions v and w in $\mathbf{C}^2$ by:

$$\begin{aligned} v(z) &= u(z_1) + |z_2| \\ w(z) &= u(z_1). \end{aligned}$$

Let D be the bounded pseudo-convex domain: $D = \{z : v(z) < 0\}$. Let $g = w - v$; g is a continuous plurisuperharmonic function. We claim $\Phi_g^D = w$ in

D . As v is negative in D , $w \leq g$ in D , so $\Phi_g^D \geq w$. But outside D , v is positive; in particular $w \geq g$ on ∂D . Thus if we define

$$f(x) = \begin{cases} w(x), & x \in \partial D \\ +\infty & \text{in } D, \end{cases}$$

$\Phi_g^D \leq \Phi_f = w$ in D , where the last equality follows from (5).

To get the desired counter example we need only choose u to be discontinuous—we can even choose it as in Example 3—in which case Φ_g^D will be discontinuous on a set of positive four-dimensional Lebesgue measure.

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