The Continuous Part of a Markov Operator

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Summary. Two theorems are proved for a Markov operator. Theorem 1 states that, for a Markov operator P_1 which is strictly dominated by a conservative, ergodic Markov operator, $P_1^n 1 \downarrow 0$ a.e.. Theorem 2 is concerned with a Markov operator P which possesses a probability transition function. It is shown that if P is conservative and ergodic and if P_n is the continuous part of P^n then either $P_n = 0$ for all n or $P_n 1 \uparrow 1$ a.e..

Let X be a non-empty set, $\mathfrak{B}$, a σ -algebra of subsets of X and λ , a σ -finite measure on $\mathfrak{B}$. Let $L_\infty(\lambda)$ be the collection of all real valued, λ -essentially bounded, $\mathfrak{B}$ -measurable functions. For two functions f, g in $L_\infty(\lambda)$, $f = g$, $f < g$ are to mean that the equality and the inequality, respectively, are satisfied except on a λ -null set. Sometimes we still indicate $=$ a.e. (λ) or $<$ a.e. (λ) for emphasis. Let P be a linear operator on $L_\infty(\lambda)$ to $L_\infty(\lambda)$ satisfying the following conditions:

- p1. if $f \geq 0$ a.e. (λ) then $Pf \geq 0$ a.e. (λ),
- p2. if $f_n \downarrow 0$ a.e. (λ) then $Pf_n \downarrow 0$ a.e. (λ),
- p3. $P1 \leq 1$ a.e. (λ).

Such an operator is a λ -measurable Markov operator of E. Hopf or simply, a Markov operator. For any set A in $\mathfrak{B}$, let 1_A represent the function which takes the value 1 on A and 0 on the complement A' of A , and I_A represent the Markov operator defined by

$$I_A f(x) = 1_A(x)f(x).$$

For an arbitrary Markov operator P we let

$$P_A = \sum_{n=0}^{\infty} P(I_A \cdot P)^n.$$

P_A operating on nonnegative elements of $L_\infty(\lambda)$ has a well defined meaning. In particular $P_A 1_A$ is a nonnegative function which is ≤ 1 (cf Section VI of [6]). A set A in $\mathfrak{B}$ is said to be *conservative* if for every λ -non-null subset B of A , $P_B 1_B = 1$ on B . P is said to be *conservative* if X is conservative. A set C in $\mathfrak{B}$ is said to be *P -closed* if $P1_C = 1$ on C . A P -closed set C is *indecomposable* if

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if it does not contain two disjoint, λ -non-null P -closed sets. A conservative Markov operator P is *ergodic* if X is indecomposable. A conservative, ergodic Markov operator P is characterized by the fact: for every λ -non-null set $A \in \mathfrak{B}$, $P_A 1_A = 1$ a.e. (λ) .

Theorem 1. *If P is a conservative, ergodic, Markov operator and P_1 is another Markov operator such that $P_1 \leq P$ and $P - P_1 \neq 0$ then $P_1^n \downarrow 0$ a.e. (λ) .*

Proof. First we shall prove the theorem for the special case that P_1 is of the form $I_A \cdot P$ where A is a λ -non-null set and A' is its complement.

The following equality can be easily proved by mathematical induction.

$$(1) \quad P^n = I_A P^n + (I_A \cdot P) I_A P^{n-1} + (I_A \cdot P)^2 I_A P^{n-2} \\ + \cdots + (I_A \cdot P)^n I_A + (I_A \cdot P)^n I_{A'}.$$

It follows from (1) that

$$P^n 1 = I_A P^n 1 + (I_A \cdot P) I_A P^{n-1} 1 + \cdots + (I_A \cdot P)^n 1_A + (I_A \cdot P)^n 1_{A'}.$$

Hence

$$1 = \{1_A + (I_A \cdot P) 1_A + \cdots + (I_A \cdot P)^n 1_A\} + (I_A \cdot P)^n 1_{A'},$$

and

$$(2) \quad 1 - \{1_A + (I_A \cdot P) 1_A + \cdots + (I_A \cdot P)^n 1_A\} = (I_A \cdot P)^n 1_{A'}.$$

Since P is conservative and ergodic, $P_A 1_A = 1$ a.e. (λ) , the left hand side of (2) approaches 0 a.e. (λ) as $n \rightarrow \infty$ so that $(I_A \cdot P)^n 1_{A'} \downarrow 0$ a.e. (λ) . Hence $(I_A \cdot P)^n 1 = (I_A \cdot P)^{n-1} I_A \cdot P 1 = (I_A \cdot P)^{n-1} 1_A$, and the conclusion of the theorem follows.

For the general case, there is a positive number ϵ and a λ -non-null set A such that $P_1 1 < 1 - \epsilon$ on A . Define another Markov operator $\bar{P}$ as follows.

$$\bar{P} = I_A P_1 + I_{A'} P.$$

Then $P_1 \leq \bar{P} \leq P$. It is sufficient to prove $\bar{P}^n 1 \downarrow 0$ a.e. (λ) .

We shall now establish the following assertion: (I) For every positive integer k , there exist a positive number c_k and a sequence $\{g_n^{(k)}\}$ of functions such that $0 \leq g_n^{(k)} \leq c_k$, $\lim_{n \rightarrow \infty} g_n^{(k)} = 0$ a.e. (λ) and

$$(3) \quad \bar{P}^n 1 \leq (1 - \epsilon)^k + g_n^{(k)}.$$

Applying formula (1) to $\bar{P}$ and noting that $I_A \cdot \bar{P} = I_A \cdot P$, we obtain:

$$\bar{P}^n 1 = \{I_A \bar{P}^n + (I_A \cdot P) I_A \bar{P}^{n-1} + \cdots + (I_A \cdot P)^{n-1} I_A \bar{P}\} 1 + (I_A \cdot P)^n 1 \\ \leq \{I_A + (I_A \cdot P) I_A + \cdots + (I_A \cdot P)^{n-1} I_A\} \bar{P} 1 + (I_A \cdot P)^n 1.$$

Since $\bar{P} 1 = P_1 1 < 1 - \epsilon$ on A , it follows

$$\bar{P}^n 1 \leq \{1_A + (I_A \cdot P) 1_A + \cdots + (I_A \cdot P)^{n-1} 1_A\} (1 - \epsilon) + (I_A \cdot P)^n 1 \\ \leq (1 - \epsilon) + (I_A \cdot P)^n 1.$$

Hence assertion (I) holds true for $k = 1$ with $c_1 = 1$, $g_n^{(1)} = (I_A \cdot P)^n 1$. Now we assume that (I) holds true for k and proceed to show that (I) also holds for $k + 1$:

$$\begin{aligned} \bar{P}^n 1 &= \{I_A \bar{P}^n 1 + (I_A \cdot P) I_A \bar{P}^{n-1} 1 + \cdots + (I_A \cdot P)^{n-2} I_A \bar{P}^2 1\} + (I_A \cdot P)^{n-1} \bar{P} 1 \\ &\leq I_A \bar{P} \{(1 - \epsilon)^k + g_{n-1}^{(k)}\} + (I_A \cdot P) I_A \bar{P} \{(1 - \epsilon)^k + g_{n-2}^{(k)}\} \\ &\quad + \cdots + (I_A \cdot P)^{n-2} I_A \bar{P} \{(1 - \epsilon)^k + g_1^{(k)}\} + (I_A \cdot P)^{n-1} \bar{P} 1. \\ &= \{I_A \bar{P} + (I_A \cdot P) I_A \bar{P} + \cdots + (I_A \cdot P)^{n-2} I_A \bar{P}\} (1 - \epsilon)^k \\ &\quad + \{I_A \bar{P} g_{n-1}^{(k)} + (I_A \cdot P) I_A \bar{P} g_{n-2}^{(k)} + \cdots + (I_A \cdot P)^{n-2} I_A \bar{P} g_1^{(k)}\} + (I_A \cdot P)^{n-1} \bar{P} 1. \end{aligned}$$

Let

$$f_n = I_A \bar{P} g_{n-1}^{(k)} + (I_A \cdot P) I_A \bar{P} g_{n-2}^{(k)} + \cdots + (I_A \cdot P)^{n-2} I_A \bar{P} g_1^{(k)},$$

and

$$g_n^{(k+1)} = f_n + (I_A \cdot P)^{n-1} \bar{P} 1,$$

then $\bar{P}^n 1 \leq (1 - \epsilon)^{(k+1)} + g_n^{(k+1)}$. Since $\lim_{n \rightarrow \infty} g_n^{(k)} = \text{a.e.}(\lambda)$ and $g_n^{(k)} \leq c_k$ for $n = 1, 2, \dots$, we have $\bar{P} g_n^{(k)} \leq c_k$ for $n = 1, 2, \dots$ and $\lim_{n \rightarrow \infty} \bar{P} g_n^{(k)} = 0$ a.e. (λ) . Since $1 = 1_A + (I_A \cdot P) 1_A + (I_A \cdot P)^2 1_A + \cdots$, we have $f_n \leq c_k$ for $n = 1, 2, \dots$ and $\lim_{n \rightarrow \infty} f_n = 0$ a.e. (λ) . We have already proved that $\lim_{n \rightarrow \infty} (I_A \cdot P)^n 1 = 0$ a.e. (λ) . Hence $\lim_{n \rightarrow \infty} g_n^{(k+1)} = 0$ a.e. (λ) and $g_n^{(k+1)} \leq c_{k+1}$ where $c_{k+1} = c_k + 1$. Thus (I) is shown to hold true for $k + 1$. Clearly $\lim_{n \rightarrow \infty} \bar{P}^n 1 \leq (1 - \epsilon)^k$ a.e. (λ) may be obtained by letting $n \rightarrow \infty$ in (3). Since k is an arbitrary positive integer, $\lim_{n \rightarrow \infty} \bar{P}^n 1 = 0$ a.e. (λ) and the proof of Theorem 1 is complete.

A real valued function $P(x, A)$ of two variables, $x \in X$, $A \in \mathfrak{B}$, is called a *probability transition function* if the following conditions are satisfied.

- T1. For every fixed $x \in X$, $P(x, \cdot)$ is a measure.
- T2. For every fixed $A \in \mathfrak{B}$, $P(\cdot, A)$ is an $\mathfrak{B}$ -measurable function.
- T3. For all x , $P(x, X) \leq 1$.

Define $P^{(n)}(x, A)$, $n = 1, 2, \dots$, inductively by

$$\begin{aligned} P^{(1)}(x, A) &= P(x, A), \\ P^{(n+1)}(x, A) &= \int P^{(n)}(x, dy) P(y, A). \end{aligned}$$

Then $P^{(n)}(x, A)$, $n = 1, 2, \dots$, are also probability transition functions. A Markov operator is said to have a probability transition function $P(x, A)$ if, for all $f \in L_\infty(\lambda)$,

$$Pf(x) = \int P(x, dy) f(y)$$

for (λ) almost all x . If P has a probability transition function $P(x, A)$, then P^n has probability transition function $P^{(n)}(x, A)$. A λ -continuous Markov

operator P is a Markov operator for which there is an $\mathfrak{B} \times \mathfrak{B}$ measurable function $p(x, y)$ such that $Pf(x) = \int p(x, y)f(y)\lambda(dy)$, [6], [7].

Let P be a Markov operator with a probability transition function $P(x, A)$. Define a measure η on $\mathfrak{B} \times \mathfrak{B}$ as follows. If E is an $\mathfrak{B} \times \mathfrak{B}$ -measurable subset of $X \times X$,

$$(4) \quad \eta(E) = \int \lambda(dx) \int P(x, dy) 1_E(x, y)$$

where $1_E(x, y) = 1$ if $(x, y) \in E$, $= 0$ if $(x, y) \notin E$. η may be decomposed into two measures η_c and η_s ,

$$\eta = \eta_c + \eta_s,$$

where η_c is absolutely continuous to $\lambda \times \lambda$ while η_s is singular to $\lambda \times \lambda$. Let $p_1(x, y)$ be a derivative of η_c with respect to $\lambda \times \lambda$ and let us define Markov operator P_1 by

$$(5) \quad P_1 f(x) = \int p_1(x, y)f(y)\lambda(dy).$$

Then P_1 is the largest λ -continuous Markov operator such that $P_1 \leq P$, and is called the λ -continuous part of P ([8], Section III). Decomposition of probability transition functions with respect to a fixed measure has been used in the past by W. Doeblin [1] and J. L. Doob [3]. Here it is put in a slightly different setting.

Lemma 1. *Let P be a Markov operator with a probability transition function $P(x, A)$. If for (λ) almost all x , $P(x, \cdot)$ is absolutely continuous to λ , then P is λ -continuous. The converse is also true if $\mathfrak{B}$ is generated by a countable collection.*

Proof. Let E_s be a set in $\mathfrak{B} \times \mathfrak{B}$ such that

$$\lambda \times \lambda(E_s) = 0 \quad \text{and} \quad \eta_s(E) = \eta(E \cap E_s)$$

for all $E \in \mathfrak{B} \times \mathfrak{B}$. Let

$$[E_s]^x = \{y: (x, y) \in E_s\}.$$

Then for (λ) almost all x , $\lambda([E_s]^x) = 0$. If for (λ) almost all x $P(x, \cdot)$ is absolutely continuous to λ . Then $P(x, [E_s]^x) = 0$ for (λ) almost all x so that $\eta_s(X \times X) = \eta(E_s) = \int \lambda(dx) P(x, [E_s]^x) = 0$. Hence $\eta_c = \eta$ and λ -continuity of P follows immediately.

The converse follows from Theorem 5 of [8].

Lemma 2. *Let P and P_1 be Markov operators with probability transition functions. If P_1 is λ -continuous then PP_1 and P_1P are λ -continuous.*

Proof. Let $P(x, A)$ be a probability transition function of P and P_1 be given by (5). Let $R = PP_1$. Then, for all $f \in L_\infty(\lambda)$,

$$Rf(x) = \int P(x, dz) \int p_1(z, y)f(y)\lambda(dy)$$

$$= \int \left[\int P(x, dz) p_1(z, y) \right] f(y) \lambda(dy)$$

so that R is λ -continuous. The operator P_1P has probability transition function

$$Q(x, A) = \int p_1(x, y) P1_A(y) \lambda(dy).$$

If $p_1(x, \cdot)$ is λ -integrable (which is true for (λ) almost all x) then $Q(x, \cdot)$ is absolutely continuous to λ . The λ -continuity of P_1P follows from Lemma 1.

Theorem 2. *If P is a conservative, ergodic Markov operator with a probability transition function and P_n is the λ -continuous part of P^n , then either $P_n = 0$ for all n or $P_n 1 \uparrow 1$ a.e. (λ) .*

Proof. By Lemma 2, P_nP is λ -continuous. Since $P_nP \leq P^{n-1}$, we have $P_nP \leq P_{n+1}$. It follows that $P_n 1 = P_n P 1 \leq P_{n+1} 1$. Hence the sequence $\{P_n 1\}$ is nondecreasing. Let the limit of the sequence be l . We also have $PP_n \leq P_{n+1}$ for PP_n is λ -continuous by Lemma 2. Hence

$$(6) \quad PP_n 1 \leq P_{n+1} 1.$$

Letting $n \rightarrow \infty$ in (6), we obtain

$$Pl \leq l.$$

Since P is conservative and ergodic, l is a constant function. If $l = 0$, then $P_n 1 = 0$ for all n so that $P_n = 0$ for all n . If $l \neq 0$, then there is a positive integer r such that $P_r \neq 0$. Consider the Markov operator P^r . It is conservative and there is a positive integer δ such that the whole space X is partitioned into δ indecomposably P^r -closed sets (cf. Section II of [7]). Among these δ indecomposably P^r -closed sets, there is one set C such that $P_r I_C \neq 0$. Define measure λ_C on $\mathcal{B}$ by $\lambda_C(A) = \lambda(C \cap A)$. Since the values of the function $P^r f$ on C are not affected by the values of f outside C , P^r may be considered as a Markov operator on $L_\infty(\lambda_C)$ to $L_\infty(\lambda_C)$ which is then conservative and ergodic. For the same reason, P_r and $P^r - P_r$ may also be considered as Markov operators on $L_\infty(\lambda_C)$ to $L_\infty(\lambda_C)$. Applying Theorem 1 to $P^r - P_r$, we conclude that $(P^r - P_r)^n 1 \rightarrow 0$ a.e. (λ) on C . Write

$$(7) \quad P^{nr} = [P_r + (P^r - P_r)]^n.$$

If we expand the right side of (7), $(P^r - P_r)^n$ would be one term among $(n+1)$ terms in the expansion, and is the only term which does not contain P_r as a factor. Hence $P^{nr} - (P^r - P_r)^n$ is λ -continuous by Lemma 2 so that

$$(8) \quad P_{nr} \geq P^{nr} - (P^r - P_r)^n.$$

Since $P^{nr} 1 - (P^r - P_r)^n 1 \rightarrow 1$ a.e. (λ) on C , $l = 1$ a.e. (λ) on C by (8). Since l is a constant function, we have $l = 1$.

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