Entire Functions Satisfying a Linear Differential Equation

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1. Introduction. Let $f(z) = \sum_0^\infty a_n(z-a)^n$ be a transcendental entire function. Since the coefficients tend to zero, there exists a smallest non-negative integer N such that

$$(1.1) \quad \max \left\{ |f(a)|, \frac{|f^{(1)}(a)|}{1!}, \dots, \frac{|f^{(N)}(a)|}{N!} \right\} \geq \frac{|f^{(j)}(a)|}{j!}, \quad j = 0, 1, \dots.$$

Here $N = N_a$ and $f^{(j)}$ denotes f . If the integers N_a are all bounded above, then f is said to be of bounded index (cf: [2], [3]), and the smallest integer N such that for all complex numbers a , $N_a \leq N$ is called the index of f . It is known [3] that an entire function satisfying a linear differential equation with constant coefficients is of bounded index. We consider here functions $F(z)$ satisfying a linear differential equation and prove

Theorem 1. *Let $F(z)$ be a transcendental entire function of z and suppose that $F(z)$ satisfies a linear differential equation*

$$(1.2) \quad P_0(z)F^{(k)}(z) + P_1(z)F^{(k-1)}(z) + \dots + P_k(z)F(z) = Q(z),$$

where $P_j(z)$, $j = 0, 1, \dots, k$, $Q(z)$ are polynomials and $P_0(z) (\neq 0)$ is of degree not less than that of any $P_j(z)$. Then $F(z)$ is of bounded index.

Theorem 2. *If*

$$(1.3) \quad F(z) = F(z, k) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{(nk)!}, \quad k = 1, 2, \dots,$$

then $F(z)$ is of bounded index.

2. Proof of Theorem 1. We require the following

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Lemma. Let $f(z) \not\equiv 0$ be an entire function and T a given positive number. Then there exists an integer $n > 0$ such that for every z , $|z| \leq T$,

$$(2.1) \quad \max \left\{ |f(z)|, \frac{|f^{(1)}(z)|}{1!}, \dots, \frac{|f^{(n)}(z)|}{n!} \right\} \geq \frac{1}{2^n}.$$

Proof. Suppose the lemma is false. Then for each positive integer n , there exists a point z_n , $|z_n| \leq T$ such that

$$|f(z_n)| < \frac{1}{2^n}, \quad \frac{|f^{(1)}(z_n)|}{1!} < \frac{1}{2^n}, \dots, \frac{|f^{(n)}(z_n)|}{n!} < \frac{1}{2^n}.$$

Since this sequence of points $\{z_n\}_1^\infty$ lies in $|z| \leq T$, it has a limit point α such that $|\alpha| \leq T$. By choosing a subsequence (z_{n_p}) which tends to α we obtain

$$|f(\alpha)| \leq \frac{1}{2^{n_p}}, \quad p = 1, 2, \dots.$$

Hence $f(\alpha) = 0$. Similarly $f^{(k)}(\alpha) = 0$, $k = 1, 2, \dots$. Hence $f(z) \equiv 0$ and we have a contradiction, and the lemma is proved.

Proof of theorem. We may suppose without loss of generality that $P_0(z) = z^m + \dots$ where $m \geq k$; for otherwise we multiply both sides of (1.2) by a suitable expression of the form az^p . Write

$$P_i^{(0)}(z) = P_i(z), \quad M(r, P_i^{(i)}) = \max_{|z|=r} |P_i^{(i)}(z)|.$$

Let $c = 1/10$ and choose $T > 0$ such that for $|z| \geq T$,

$$(2.2) \quad |P_0(z)/z^m| \geq \frac{1}{2},$$

and

$$(2.3) \quad \frac{1}{|z^m|} \left\{ |P_0^{(1)}(z)| + \frac{|P_0^{(2)}(z)|}{2!} + \dots + \frac{|P_0^{(m)}(z)|}{m!} \right\} \leq \frac{1-c}{2}.$$

Then Cauchy's formula

$$\frac{F^{(p)}(z)}{p!} = \frac{1}{2\pi i} \int_{|t-z|=2} \frac{F(t) dt}{(t-z)^{p+1}}$$

gives for $|z| \leq T$ and $p = 1, 2, \dots$

$$(2.4) \quad \frac{|F^{(p)}(z)|}{p!} \leq \frac{M(T+2, F)}{2^p}.$$

Choose $n = n_0$ such that (2.1) holds when $|z| \leq T$ and then choose $P \geq n_0$ such that

$$(2.5) \quad M(T+2, F) \leq 2^{P+1-n_0}.$$

From (2.1), (2.4) and (2.5) we have for $|z| \leq T$, $j \geq P + 1$,

$$(2.6) \quad \max \left\{ |F(z)|, \dots, \frac{|F^{(P)}(z)|}{P!} \right\} \geq \frac{1}{2^{n_0}} \geq \frac{M(T+2, F)}{2^{P+1}} \\ \geq \frac{M(T+2, F)}{2^j} \geq \frac{|F^{(j)}(z)|}{j!};$$

and so

$$(2.7) \quad \max \left\{ |F(z)|, \dots, \frac{|F^{(P)}(z)|}{P!} \right\} \geq \frac{|F^{(j)}(z)|}{j!} \quad \text{for } |z| \leq T, \quad j = 1, 2, \dots.$$

Suppose now that $|z| \geq T$. Let $n \geq n_1 = \max(m+k, k+q+1, P+1)$ where q denotes the degree of $Q(z)$. Differentiating the equation (1.2), $(n-k)$ times we have

$$F^{(n)}P_0 + F^{(n-1)}\{(n-k)P_0^{(1)} + P_1\} + \dots \\ + F^{(n-k)}\left\{\frac{(n-k) \dots (n-2k+1)}{k!} P_0^{(k)} \right. \\ \left. + \frac{(n-k) \dots (n-2k+2)}{(k-1)!} P_1^{(k-1)} + \dots \right. \\ \left. + \frac{(n-k) \dots (n-2k+1+s)}{(k-s)!} P_s^{(k-s)} + \dots + P_k\right\} + \dots \\ + F^{(n-m-1)}\left\{\frac{(n-k) \dots (n-k-m+1)}{m!} P_1^{(m)} + \dots \right. \\ \left. + \frac{(n-k) \dots (n-m)}{(m+1-k)!} P_k^{(m+1-k)}\right\} + \dots \\ + F^{(n-k-m)}\left\{\frac{(n-k) \dots (n-k-m+1)}{m!} P_k^{(m)}\right\} = 0.$$

Hence we have for $|z| \geq T$,

$$(2.8) \quad \frac{F^{(n)}}{n!} + \frac{F^{(n-1)}}{(n-1)!} \left\{ \frac{(n-k)P_0^{(1)} + P_1}{nP_0} \right\} + \dots + \frac{F^{(n-k)}}{(n-k)!} \\ \cdot \left\{ \left(\frac{(n-k) \dots (n-2k+1)}{k!} P_0^{(k)} + \dots + P_k \right) / n(n-1) \dots (n-k+1)P_0 \right\} + \dots \\ + \frac{F^{(n-m-k)}}{(n-m-k)!} \left\{ \frac{(n-k) \dots (n-k-m+1)}{m!} \right. \\ \left. \cdot P_k^{(m)} / n(n-1) \dots (n-k-m+1)P_0 \right\} = 0.$$

Write

$$(2.9) \quad \begin{aligned} \gamma_1 &= \left| \frac{P_0^{(1)}}{P_0} \right| + \frac{|P_1|}{n |P_0|}, \dots, \\ \gamma_k &= \left| \frac{P_0^{(k)}}{k! P_0} \right| + \frac{1}{n |P_0|} (|P_1^{(k-1)}| + |P_2^{(k-2)}| + \dots + |P_k|), \dots, \\ \gamma_{m+k} &= \frac{1}{n} \left| \frac{P_k^{(m)}}{P_0} \right|. \end{aligned}$$

Then from (2.8) we obtain

$$(2.10) \quad \frac{|F^{(n)}|}{n!} \leq \frac{|F^{(n-1)}|}{(n-1)!} \gamma_1 + \dots + \frac{|F^{(n-m-k)}|}{(n-m-k)!} \gamma_{m+k}.$$

Choose $n_2 = n_2(k, m) \geq n_1$ so large that for $n \geq n_2$,

$$(2.11) \quad \frac{1}{n} \sum_{j=0}^m \sum_{i=1}^k \frac{M(T, P_i^{(j)})}{T^m} \leq \frac{c}{2}.$$

Then for $|z| \geq T$ and $n \geq n_2$,

$$(2.12) \quad \frac{1}{n} \sum_{j=0}^m \sum_{i=1}^k \frac{M(|z|, P_i^{(j)})}{|z|^m} \leq \frac{c}{2}.$$

The inequalities (2.2) and (2.3) now give,

$$\begin{aligned} \gamma_1 + \gamma_2 + \dots + \gamma_{m+k} &\leq 1 - c + \frac{1}{n} \sum_{j=0}^m \sum_{i=1}^k \frac{M(|z|, P_i^{(j)})}{|P_0(z)|} \\ &\leq 1 - c + \frac{2}{n} \sum_{j=0}^m \sum_{i=1}^k \frac{M(|z|, P_i^{(j)})}{|z|^m} \\ &\leq 1. \end{aligned}$$

Write $n_2 = l$. Then for $n \geq l$ and $|z| \geq T$, (2.10) gives

$$\max \left\{ |F(z)|, \dots, \frac{|F^{(n-1)}(z)|}{(n-1)!} \right\} \geq \frac{|F^{(n)}(z)|}{n!}.$$

Taking $n = l, l+1, \dots$ successively we obtain

$$(2.13) \quad \max \left\{ |F(z)|, \dots, \frac{|F^{(l-1)}(z)|}{(l-1)!} \right\} \geq \frac{|F^{(l)}(z)|}{l!},$$

for $j = 1, 2, \dots$ and $|z| \geq T$. The theorem now follows from (2.7) and (2.13). Furthermore the index $N \leq l-1$.

3. Remarks. (i) A transcendental entire function satisfying equation (1.2) is of rational order ρ such that $1/k \leq \rho \leq 1$ and of perfectly regular growth [4; pp. 106–9], [5; pp. 71–73].

(ii) Let F be a transcendental entire function and let F satisfy an equation of the form (1.2). In order that F be of bounded index it is sufficient that the degree of P_0 be not less than that of any P_i . The following examples show that when this condition is not satisfied F may or may not be of bounded index.

(a) $F^{(1)} - 2zF = 0$. $F(z) = \exp(z^2)$ is not of bounded index.

(b) $F^{(2)} + (z-1)F^{(1)} - zF = 0$. $F(z) = \exp(z)$ is of bounded index.

(iii) Each of the following functions satisfies a differential equation of the form (1.2), and the condition on the degree of P_0 , and is of bounded index.

(a) $\cos z, \sin z, \cos \sqrt{z}, (\sin \sqrt{z})/\sqrt{z}$.

(b) The confluent hypergeometric function

$$1 + \sum_{n=1}^{\infty} \frac{a(a+1) \cdots (a+n-1)}{b(b+1) \cdots (b+n-1)} \frac{z^n}{n!} \cdot (b \neq 0, -1, -2, \dots), \quad (a \neq 0, -1, -2, \dots).$$

(c) Bessel functions of the first kind and order p , where p is a nonnegative integer,

$$J_p(z) = \left(\frac{1}{2}z\right)^p \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{1}{2}z\right)^{2r}}{r! \Gamma(p+r+1)}.$$

(iv) For entire functions satisfying differential equations of infinite order see [1; pp. 247–248], [2].

4. Proof of Theorem 2. We will show that the conditions of Theorem 1 are satisfied by

$$(4.1) \quad F(z) = F(z, k) = 1 + \sum_1^{\infty} \frac{z^n}{(nk)!}.$$

It is easily seen that (a) $F(z, k)$ is an entire function for $k = 1, 2, \dots$, and (b) $F(z, 1) = e^z$ is of index $N = 0$. Suppose now $k \geq 2$ and let $w (\neq 1)$ be a k^{th} root of unity. Then

$$(4.2) \quad f(t) = \frac{\exp(t) + \exp(wt) + \cdots + \exp(w^{k-1}t)}{k} = F(t^k).$$

Write $z = t^k$. Then $f(t) = F(z)$ and we have

$$\frac{df}{dt} = kz^{(k-1)/k} \frac{dF}{dz}.$$

It can be proved, by induction argument that

$$\begin{aligned} \frac{d^p f}{dt^p} &= c_{1,p} z^{p((k-1)/k) - (p-1)} \frac{d^p F}{dz^p} \\ &+ c_{2,p} z^{p((k-1)/k) - (p-2)} \frac{d^2 F}{dz^2} + \cdots + c_{p,p} z^{p((k-1)/k)} \frac{d^p F}{dz^p} \quad \text{for } p = 1, 2, \dots \end{aligned}$$

Here $c_{i,p}$ are constants and $c_{p,p} \neq 0$. Taking $p = k$ we have

$$(4.3) \quad F(z) = f(t) = \frac{d^k f(t)}{dt^k} = c_{1,k} \frac{dF}{dz} + c_{2,k} z \frac{d^2 F}{dz^2} + \cdots + c_{k,k} z^{k-1} \frac{d^k F}{dz^k},$$

and so $F(z)$ satisfies a linear differential equation of the form (1.2). Hence $F(z)$ is of bounded index.

Remarks. (i) If $k = 3$, the equation (4.3) becomes

$$z^2 \frac{d^3 F}{dz^3} + 2z \frac{d^2 F}{dz^2} + \frac{2}{9} \frac{dF}{dz} - \frac{F}{27} = 0.$$

It can be proved, by the argument of Theorem 1, that $N \leq 2$.

(ii) If $k = 4$, the equation becomes

$$z^3 \frac{d^4 F}{dz^4} + \frac{9}{2} z^2 \frac{d^3 F}{dz^3} + \frac{51}{16} z \frac{d^2 F}{dz^2} + \frac{3}{32} \frac{dF}{dz} - \frac{F}{256} = 0,$$

and $N \leq 3$.

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