*On Some Nonlinear Boundary Problems for the Equation of Minimal Surfaces**

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The purpose of this paper is to introduce certain nonlinear boundary value problems for the Euler minimal surfaces equation [3]; and, using only integral identity methods, to prove for each a uniqueness theorem. In Section III, a non-existence result is obtained for certain of these problems. Finally, for those problems which are solvable, the solutions are compared (Section IV) with harmonic functions solving analogous problems.

I. Introduction. The equation

$$(1) \quad (1 + u_y^2)u_{xx} - 2u_xu_yu_{xy} + (1 + u_x^2)u_{yy} = 0$$

can be thought of as the equation satisfied by the z -coördinate of a minimal surface $z = u(x, y)$, or as the equation satisfied by the potential in the "flow" of a Chaplygin gas. Although the latter correspondence is technically a fiction, since it presupposes an unreal density-speed relation $\rho = (1 + q^2)^{-1/2}$, it is in agreement up to terms of order q^4 with the precise relation in any subsonic region [1], where the character of the equation is very similar to that of the compressible flow equation. Hence it is logical to suspect that the results established herein carry over, with suitable speed restrictions, to compressible flows; and the realization of this will be, in fact, the substance of a sequel to this paper. Preliminary results may be found in [4].

We shall consider a domain D having a piecewise-smooth boundary Γ . We assume that u_x and u_y are $C'(D)$ and $C^0(D + \Gamma)$; and are the single-valued partial derivatives of a function $u(x, y)$ which, in general, is not itself required to be single-valued [1]. However, when we compare two functions u_1 and u_2 , it is understood that their difference is single-valued. In the case of potential flows, this amounts to fixing the circulation [4]. The outward unit normal to D will be denoted by " n ," the outward normal derivative of u by " u_n ," and arc length along Γ by " s ."

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The methods employed herein derive directly from a general procedure for elliptic problems developed earlier in [4], [5], [6], and [7].

II. Uniqueness. For simplicity, we set $\rho = 1/\sqrt{1+q^2}$, where $q^2 = u_x^2 + u_y^2$. Equation (1) may be rewritten simply as $\operatorname{div}(\rho \nabla u) = 0$. In place of standard Dirichlet-type data [2], we shall concern ourselves with a restriction on the direction of the unit normal vector $\mathbf{w} = (\rho u_x, \rho u_y, -\rho)$ to the minimal surface by specifying the cosine of the angle formed by $\mathbf{w}$ and the corresponding normal vector to the projection of the minimal surface on the x - y plane. More precisely, we have:

Theorem I. *For the boundary problem*

$$(2) \quad \begin{aligned} (1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} &= 0 \quad \text{in } D, \\ \rho u_n &= G(s, u) \quad \text{on } \Gamma, \end{aligned}$$

where G is a given function of s, u , non-increasing in u , any two solutions whose difference is single-valued must differ by at most a constant.

If u_1 and u_2 are two solutions of (1) with a single-valued difference, then following the methods in [4] we are led to the easily-verified identity

$$(3) \quad \int_{\Gamma} (u_1 - u_2) \left(\rho_1 \frac{\partial u_1}{\partial n} - \rho_2 \frac{\partial u_2}{\partial n} \right) ds = \int_D (\nabla u_1 - \nabla u_2) \cdot (\rho_1 \nabla u_1 - \rho_2 \nabla u_2) dS,$$

in which

$$\rho_1 = \left[1 + \left(\frac{\partial u_1}{\partial x} \right)^2 + \left(\frac{\partial u_1}{\partial y} \right)^2 \right]^{-1/2} \quad \text{and} \quad \rho_2 = \left[1 + \left(\frac{\partial u_2}{\partial x} \right)^2 + \left(\frac{\partial u_2}{\partial y} \right)^2 \right]^{-1/2}.$$

Now it is clear under the boundary conditions that the integral over Γ is not positive. In addition, by Schwarz's inequality and the definition of ρ , we obtain that

$$\begin{aligned} (\nabla u_1 - \nabla u_2) \cdot (\rho_1 \nabla u_1 - \rho_2 \nabla u_2) \\ = \rho_1 \|\nabla u_1\|^2 + \rho_2 \|\nabla u_2\|^2 - (\rho_1 + \rho_2) \nabla u_1 \cdot \nabla u_2 \\ \geq (\|\nabla u_1\| - \|\nabla u_2\|)(\rho_1 \|\nabla u_1\| - \rho_2 \|\nabla u_2\|) \geq 0. \end{aligned}$$

Thus the integrand over D is non-negative, and clearly can only equal zero provided $\|\nabla u_1\| = \|\nabla u_2\|$ and $\nabla u_1 \cdot \nabla u_2 = \|\nabla u_1\| \|\nabla u_2\|$; i.e., provided $\nabla u_1 = \nabla u_2$. Thus both integrands in (3) must vanish identically; and in particular, $\nabla u_1 = \nabla u_2$ in D , which proves the theorem.

As a corollary, we get

Corollary I. *For the boundary problem*

$$(4) \quad \begin{aligned} (1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} &= 0 \quad \text{in } D, \\ \rho u_n &= h(s) \quad \text{on } \Gamma, \end{aligned}$$

where h is a given function of arc length, any two solutions whose difference is single-valued must differ by at most a constant. Moreover, (4) can have a solution only for those h for which $\int_{\Gamma} h(s) ds = 0$ and $-1 < h(s) < 1$.

The second statement follows most easily from the identity

$$\int_{\Gamma} \rho u_n ds = \int_D \operatorname{div} (\rho \nabla u) dS = 0,$$

valid for any solution u of (1), and by direct consideration of the boundary condition, since $|\rho u_n| = |\rho \nabla u \cdot n| \leq \rho \|\nabla u\| < 1$.

The existence question for this problem is not yet completely settled, and will be the subject of a subsequent paper.

Clearly the results we have obtained can be extended to allow, on part or all of the boundary, specification of u itself rather than ρu_n ; or they can be extended to the case where the right-hand side of (1) is not identically zero. But these extensions, which are little more than immediate observations, will not be pressed further here. Results of this nature for Laplace's equation appear in [5].

With G linear, Theorem 1 yields the following corollary:

Corollary II. *For the "Stekloff-type" boundary problem [8]*

$$(5) \quad \begin{aligned} (1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} &= 0 \quad \text{in } D, \\ \rho u_n &= h(s)[u - u_0] \quad \text{on } \Gamma, \end{aligned}$$

in which $h(s) \leq 0$ and u_0 is a given constant, any two solutions whose difference is single-valued must differ by at most a constant.

A similar result to the above is of course obtained at once for the boundary condition $\rho u_n = h(s)e^{k^2 u}$; and here, as in problem (5), the restriction $h(s) \leq 0$ can be removed only at the cost of obtaining a much weaker result. This can in fact be done for (5) by applying the divergence theorem to the integral

$$\int_{\Gamma} \frac{f(u_1)}{f(u_2)} \left[f(u_2) \rho_1 \frac{\partial u_1}{\partial n} - f(u_1) \rho_2 \frac{\partial u_2}{\partial n} \right] ds,$$

in which $f(u) = u - u_0$ (see for example Theorem 4.2 in [5]). We shall instead consider the exponential case just mentioned, which provides somewhat stronger results than does (5) while remaining just as illustrative of the type of restrictions that do occur.

Theorem II. *For the boundary problem*

$$(6) \quad \begin{aligned} (1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} &= 0 \quad \text{in } D, \\ \rho u_n &= h(s)e^{k^2 u} \quad \text{on } \Gamma, \end{aligned}$$

if the difference between any two single-valued solutions is never zero, then it is a constant.

Notice that the restriction on the sign of $h(s)$ has been removed, but that we have only claimed uniqueness among a restricted set of functions.

The proof is almost immediate from the following identity, which is valid for any two single-valued solutions u_1, u_2 to (1), provided they are never equal:

$$\begin{aligned} \int_{\Gamma} \frac{1}{e^{k^2 u_1} - e^{k^2 u_2}} \left(e^{k^2 u_2} \rho_1 \frac{\partial u_1}{\partial n} - e^{k^2 u_1} \rho_2 \frac{\partial u_2}{\partial n} \right) ds \\ = \int_D \frac{-k^2 e^{k^2 u_1} e^{k^2 u_2}}{(e^{k^2 u_1} - e^{k^2 u_2})^2} (\nabla u_1 - \nabla u_2) \cdot (\rho_1 \nabla u_1 - \rho_2 \nabla u_2) dS. \end{aligned}$$

We may assume in the proof that $k^2 \neq 0$, since for $k = 0$ the stated result follows from Corollary I. Now if u_1 and u_2 solve (6), the integral over Γ is zero. Hence the integral over D also vanishes, and so $\nabla u_1 \equiv \nabla u_2$ as claimed.

III. Non-existence. As was made clear by the second half of Corollary I, we cannot always expect (2) to have a solution. This fact is underlined by the theorem which we now prove, in which we show that certain quite general problems of the type (2) cannot have solutions. The real effect is a compatibility requirement.

Theorem III (Non-existence). *If $\int_{\Gamma} h(s) ds > 0$ ($\int_{\Gamma} h(s) ds < 0$), then there exists no solution to the boundary problem*

$$\begin{aligned} (1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} &= 0 \quad \text{in } D, \\ \rho u_n &= h(s)K(u) \quad \text{on } \Gamma, \end{aligned}$$

with $K(u) \neq 0$ and $K'(u) \geq 0$ ($K'(u) \leq 0$) on $D + \Gamma$. If $\int_{\Gamma} h(s) ds = 0$, then there can only exist such a solution u provided $K(u)$ is actually constant on $D + \Gamma$.

The proof is immediate from the identity

$$\int_{\Gamma} \frac{\rho u_n}{K(u)} ds = \int_D \frac{-K'(u)}{K^2(u)} \rho ||\nabla u||^2 dS,$$

valid for any solution u to (1). Note that without the restriction $K(u) \neq 0$, the theorem is false, as evidenced by the problem

$$\begin{aligned} (1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} &= 0 \quad \text{for } x^2 + y^2 < \frac{1}{3}, \\ \rho u_n &= u \quad \text{for } x^2 + y^2 = \frac{1}{3}, \end{aligned}$$

which has the very simple solution $u = x + y$.

IV. Comparison. To assume a more optimistic stance, when a solution to problem (2) does exist we would like to have some way of comparing it *a priori* with the solution to a simpler problem, *e.g.* a boundary problem for Laplace's equation. To this end, we introduce the solution v of the corresponding Neumann

problem

$$(7) \quad \begin{aligned} \Delta v &= 0 \quad \text{in } D, \\ v_n &= G(s, v) \quad \text{on } \Gamma, \end{aligned}$$

provided it exists. Some of the results that we shall derive stem from suggestions made by L. E. Payne in a personal communication.

To begin, we make a direct application of the divergence theorem to obtain the identity

$$(8) \quad \int_{\Gamma} (u - v)(\rho u_n - v_n) ds = \int_D (\nabla u - \nabla v) \cdot (\rho \nabla u - \nabla v) dS.$$

Under the specified boundary conditions, the left-hand side is non-positive. Thus if the integrand of the right-hand side is non-negative, it is identically zero. But

$$\begin{aligned} (\nabla u - \nabla v) \cdot (\rho \nabla u - \nabla v) &= \rho \|\nabla u\|^2 - (1 + \rho) \nabla u \cdot \nabla v + \|\nabla v\|^2 \\ &\geq (\|\nabla u\| - \|\nabla v\|)(\rho \|\nabla u\| - \|\nabla v\|); \end{aligned}$$

and since $\rho < 1$, this is certainly positive at any point where $\|\nabla u\| < \|\nabla v\|$ or $\rho \|\nabla u\| > \|\nabla v\|$. Thus we get the following theorem:

Theorem IV (Comparison). *If u solves (2) and v solves (7), and if $u - v$ is single valued, then there is some subdomain of D in which*

$$\|\nabla u\| \geq \|\nabla v\| \geq \rho \|\nabla u\|.$$

If G is actually a function of s alone, then this result can be reinforced and amplified. In this case, in addition to the interior equations $\operatorname{div}(\nabla v) = \operatorname{div}(\rho \nabla u) = 0$ in D , we have the boundary condition $v_n = \rho u_n$ satisfied on Γ .

We can, first of all, again infer that there is some subdomain in which $\rho \|\nabla u\| \leq \|\nabla v\|$ by proving the integral inequality

$$(9) \quad \int_D \frac{1}{\rho} \|\nabla v\|^2 dS \geq \int_D \rho \|\nabla u\|^2 dS.$$

In order to prove this, we need only employ the following self-evident identity:

$$0 = \int_{\Gamma} u(\rho u_n - v_n) ds = \int_D \nabla u \cdot (\rho \nabla u - \nabla v) dS.$$

Now with this identity and Schwarz's inequality, it is clear that

$$\left(\int_D \rho \|\nabla u\|^2 dS \right)^2 = \left(\int_D \nabla u \cdot \nabla v dS \right)^2 \leq \int_D \rho \|\nabla u\|^2 dS \int_D \frac{1}{\rho} \|\nabla v\|^2 dS,$$

from which (9) follows at once.

Except in the degenerate case, the subdomain just referred to cannot be extended to all of D , because we have the complementary inequality

$$(10) \quad \int_D \|\nabla v\|^2 dS \leq \int_D \rho^2 \|\nabla u\|^2 dS.$$

This inequality not only implies that there is some subdomain in which $\rho \|\nabla u\| \geq \|\nabla v\|$, and where hence $\|\nabla u\| \geq \|\nabla v\|$ as in Theorem IV; but it also, by virtue of the fact that $\rho \|\nabla u\| < 1$, yields as a bonus the bound

$$\int_D \|\nabla v\|^2 dS < \text{Area}_D.$$

All that remains to be shown is the derivation of (10), and to this end we again derive an auxiliary identity:

$$0 = \int_{\Gamma} v(\rho u_n - v_n) ds = \int_D \nabla v \cdot (\rho \nabla u - \nabla v) dS.$$

As before, we use this with Schwarz's inequality to obtain that

$$\left(\int_D \|\nabla v\|^2 dS \right)^2 = \left(\int_D \nabla v \cdot \rho \nabla u dS \right)^2 \leq \int_D \|\nabla v\|^2 dS \int_D \rho^2 \|\nabla u\|^2 dS,$$

which is equivalent to (10).

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