A Characterization of Analyticity. III

KENNETH O. LELAND¹

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1. Introduction. This paper continues the development of the theory of families of *real* analytic functions characterized by Lipschitz type conditions initiated in [4] and continued in [6]. Handled here is the case of solution spaces of *elliptic* homogeneous constant coefficient partial differential equations. It will be shown that a suitably defined family of functions on a Euclidean space E into the reals R , either satisfies a uniform Lipschitz condition and consists of analytic functions, or contains arbitrary continuous functions satisfying no Lipschitz condition whatsoever.

2. Notation and definitions. Let R denote the real numbers and ω the positive integers. Throughout this paper E will denote a fixed Euclidean space of dimension p , $p \in \omega$. For $x \in E$, $\delta > 0$, set $U_x(\delta) = \{y \in E; \|y - x\| < \delta\}$, $B_x(\delta) = \{y \in E; \|y - x\| = \delta\}$, and set $U(\delta) = U_0(\delta)$, $B(\delta) = B_0(\delta)$, $U = (U1)$ and $V = B(1)$. Let G denote the rotation group (group of unitary linear transformations of E into E) of E . Let $F(E, R)$ be the family of continuous functions on open subsets of E into R . For $f \in F(E, R)$, and A an invertible map of E onto E , we set $(fA)(x) = f(A(x))$ for all $x \in \{t \in E; A(t) \in \text{dom } f\}$.

Let $F \subseteq F(E, R)$. Then F is called a T family if

- (1) For $f, g \in F$, F contains the function h such that $h(x) = f(x) + g(x)$ for $x \in \text{dom } f \cap \text{dom } g$.
- (2) For $f \in F$, $r \in R$, $rf \in F$.
- (3) For $f \in F$, S an open subset of $\text{dom } f$, the restriction $f|_S$ of f to S lies in F .
- (4) For $f \in F$, $x \in E$, F contains the translate fA_x , where $A_x(y) = x - y$ for $y \in E$.

F is called an R family if for $f \in F$, $r > 0$, F contains the function fA_r , where $A_r(x) = rx$ for $x \in E$. F is called a G family if for $f \in F$, $g \in G$, $fg \in F$. F is called an L family if there exists $N > 0$, such that for $f \in F$, $\bar{U} \subseteq \text{dom } f$, we have

$$|f(x) - f(0)| \leq N \sup \{|f(t)|; t \in U\} \|x\|$$

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for all $x \in U$. F is said to be strongly closed if F contains all functions f such that there exists a sequence $f_1, f_2, \dots$ of elements of F , such that $\text{dom } f = \bigcup_1^\infty \text{dom } f_i$, $\text{dom } f_i \subseteq \text{dom } f_{i+1}$, for $i \in \omega$, and such that for $n \in \omega$, H a compact subset of $\text{dom } f_n$, the sequence $f_n, f_{n+1}, \dots$ converges uniformly on H to $f|_H$.

3. Statement of results. From [4] the elements of TRL families are differentiable, indeed infinitely differentiable and analytic.

Theorem A. *Let F be a strongly closed TR family of $F(R, R)$. Then $F = F(R, R)$, OR there exists $n \in \omega$, such that all elements of F are polynomials, or the restrictions of polynomials to open subsets of R , of degree at most n .*

Such bounded degree polynomial families are [6] trivially TRL families, and from their finite dimensionality, must be strongly closed.

Theorem B. *Let F be a strongly closed TGR family of $F(E, R)$. Then $F = F(E, R)$, OR F is a TRL family and there exists $n \in \omega$, such that $\Delta^n f \equiv 0$ for all $f \in F$ (where Δ^n can be taken as the n -times iterated classical Laplacian operator).*

For the theory of TG families which are not R families, but which satisfy the Maximum Modulus Theorem, see [5].

Theorem C. *Let F be a strongly closed TR family of $F(E, R)$. Then there exists $e \in V$, such that F contains all functions of the form $g(x) = f[(x, e)]$, where $x \in E$, $f \in F(R, R)$, OR F is a TRL family and its elements satisfy an elliptic partial differential equation.*

For $n \in \omega$, let $C^n(E, R)$ be the family of n -times continuously differentiable functions on open subsets of E into R . We shall use the following results from [6].

Theorem 3.1. *Let F be a strongly closed T family of $F(E, R)$, $n \in \omega$, $f \in F$, and H a compact subset of $\text{dom } f$. Then there exists a sequence $f_1, f_2, \dots$ of $C^n(E, R) \cap F$, which converges uniformly on H to $f|_H$.*

Theorem 3.2. *Let F be a strongly closed TR family of $F(E, R)$, $n \in \omega$, $f \in C^n(E, R) \cap F$, $z \in \text{dom } f$, and $e \in E$. Then for $j = 1, \dots, n$, F contains the functions u, v , such that for $t \in E$, $u(t) = f_z^{(j)}(t, \dots, t)$, and for $x \in \text{dom } f$, $v(x) = f_x^{(j)}(e, \dots, e)$, where $f_x^{(j)}$ denotes the j -th (Fréchet) derivative of f at x . Moreover if F contains non-trivial elements, F contains the constant function $\bar{1}$.*

Our plan of attack starts with the showing that the space W of polynomials of the family F in question is either quite large or it is a TRL family. In the first case the Stone-Weierstrass Theorem is applied to the closure of W in F . In the second case a relation, $P(n)$, $n \in \omega$, is deduced for the elements of W . From Theorem 3.1, we have that the strong closure of $C = C^n(E, R) \cap F$ is F . Thus any Lipschitz conditions obtained for C transfer to F . For $f \in C$, we apply Theorem 3.2 to show that the derivatives of f lie in W and hence are subject to $P(n)$. We are thus able to interpret $P(n)$ as a differential equation acting on the elements of C . This equation is then manipulated to yield the desired uniform Lipschitz condition.

4. Proof of Theorem A. Let W be the family of all polynomials on R into R which lie in F . Let $n \in \omega$, and assume W contains a polynomial $P(x) \equiv \sum_0^n a_k x^k$ of degree n . We shall show that W contains all polynomials of degree at most n . Since W is a T family, W contains the polynomial Q such that $Q(x) = P(x+1)$ for $x \in R$. Since for $x \in R$, $Q(x) - P(x) = na_n x^{n-1} + \dots$, W contains an element of degree $n-1$. Continuing this process, we obtain that W contains the constant function $\bar{1}$. Now W contains an element $A(x) \equiv ax + b$ of degree 1, and thus W contains the polynomial $A - b\bar{1}$, where $(A - b\bar{1})(x) = ax$ for $x \in R$. Continuing we find that W contains the polynomial x^k for $k = 1, \dots, n$.

Thus if W contains elements of arbitrarily large degree, W contains all polynomials on R to R , and from the Stone-Weierstrass Theorem, since F is strongly closed, $F = F(R, R)$.

We now consider the case when for some $n \in \omega$, all elements of W have degree at most n . Let $f \in F$. Without loss of generality, we take $\text{dom } f = R$. For $k \in \omega$, $x \in R$, set

$$g_k(x) = k^n \int_0^{1/k} \dots \int_0^{1/k} f(t_1 + \dots + t_n + x) dt_1 \dots dt_n.$$

Then the sequence $g_1, g_2, \dots$ is a sequence in F , which converges uniformly on compact subsets of R to f . Let $k \in \omega$, $z \in R$, and set for $q \in \omega$, $x \in R$,

$$w_q(x) = x^n \left\{ \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} g_k(z + jx/q) / [(x/q)^n n!] \right\}.$$

Then $w_1, w_2, \dots$ is a sequence in F , which converges uniformly on compact subsets of R to $g_k^{(n)}(z)x^n$. Since all elements of W have degree at most $n-1$, $g_k^{(n)}(z) = 0$. Since z is arbitrary, $g_k^{(n)} \equiv 0$, and $g_k \in W$. Letting $k \rightarrow \infty$, we conclude that $f \in W$.

5. The TGR case. Let $f \in F(E, R)$ such that $\text{dom } f = U(r)$ for some $r > 0$. We set

$$\tilde{f} = \int_G fg \, d\mu(g),$$

where μ denotes normalized invariant (Haar) measure on G . Clearly for $x \in E$, $x \neq 0$, $r = \|x\|$, we have $\tilde{f}(x) = \int_{B(r)} f \, d\mu$, where μ denotes normalized surface measure on $B(r)$.

Let $q, m \in \omega$, $q = 2m$, and set $a(x) = \int_V (x, t)^q \, d\mu(t)$ for $x \in E$. Then $a(x) = s_m(x, x)^m$ for $x \in E$ and some $s_m > 0$. For $f \in C^q(E, R)$, $x \in \text{dom } f$, we take as our working expression for $\Delta^m f(x)$, the expression

$$s_m^{-1} \int_V f_x^{(a)}(t, \dots, t) \, d\mu(t).$$

We observe for $t \in E$, $\|t\| = 1$, that $f_x^{(a)}(t, \dots, t)$ is the q -th directional derivative $\partial^q f / \partial t^q$ of f at x in the direction t . Thus the working expression is an unweighted average of q -th directional derivatives taken over all possible directions t . Since the space of homogeneous q -th order *rotation invariant* differential oper-

ators is one dimensional, the working expression must be a scalar multiple of the classical expression. In ¶6 the general elliptic operator shall be obtained by introducing suitable weighting functions into our averages.

We now define polynomial functions on E into R . For $n \in \omega$, let $H_n(E, R)$ be the space of all functions f on E into R , such that there exists a symmetric multilinear functional $\tilde{f}$ on $\prod_1^n E$ into R , such that $f(x) = \tilde{f}(x, \dots, x)$ for all $x \in E$. Thus for $n \in \omega$, $H_n(E, R)$ is the space of all homogeneous polynomials of degree n on E into R . Let H_0 be the family of all constant functions on E into R . Finally let $H(E, R)$ be the family of all functions (polynomials) of the form $f = \sum_0^\infty f_n$, where $f_n \in H_n(E, R)$ for $n = 0, 1, \dots$, and $f_n \equiv 0$ for n sufficiently large.

Theorem 5.1. *Let F be a TGR family of $H(E, R)$. Then $F = H(E, R)$, or there exists $q \in \omega$, such that for $n \geq q$, $n \in \omega$, and $f \in H_n(E, R) \cap F$, $\tilde{f} \equiv 0$.*

Proof. Let $n \in \omega$, and let F_n be the family of all $f \in F$ of degree at most n . Then F_n is a TGR family. For $f \in H = H(E, R)$, set $\|f\| = \sup \{|f(t)|; t \in U\}$. Since for $f \in H$, f vanishes identically on E if f vanishes on U , F is a normed linear space. Since F_n is finite dimensional, F_n is a Banach space and hence a closed subspace of H . Thus if $f_1, f_2, \dots$ is a sequence in F_n which converges on compact subsets of E to a limit function $f_0 \in H$, $\|f_n - f_0\| \rightarrow 0$ as $n \rightarrow \infty$, $n \in \omega$, and $f_0 \in F_n$. Thus F_n is strongly closed and we may apply Theorem 3.2 to F_n . Furthermore, for $f \in F_n$, since $\tilde{f} = \int_G fg \, d\mu(g)$ is the limit of sums of elements of F_n , $\tilde{f} \in F_n$. For $n \in \omega$, $x \in E$, set $\theta_n(x) = (x, x)^n$.

Let $n \in \omega$, and assume $\theta_n \in F$. For $e \in V$, from Theorem 3.2, F contains the function h such that $h(x) = (\theta_n)_x''(e, e) = n(n-1)(x, x)^{n-2}(x, e)^2 + n(x, x)^{n-1}(e, e)$ for $x \in E$. Now for $x \in E$, $\tilde{h}(x) = n(n-1)(x, x)^{n-2}[\rho(x, x)] + n(x, x)^{n-1} = [n(n-1)\rho + n]\theta_{n-1}(x)$, where $\rho = \int_V (t, e)^2 \, d\mu(t)$. Thus $\theta_{n-1} \in F$. Continuing we obtain $\theta_k \in F$ for $k = 0, 1, \dots, n$.

Let $k = 1, \dots, n$. Then for $a \in E$, F contains the function w_a such that $w_a(x) = k^{-1}(\theta_k)_x'(a) = (x, x)^{k-1}(x, a)$. Hence for $a, b \in E$, F contains the function $v(a, b)$ such that for $x \in E$, $v(a, b)(x) = (k-1)^{-1}(w_a)_x'(b) = (x, x)^{k-2}(x, b)(x, a) + (k-1)^{-1}(x, x)^{k-1}(a, b) = (x, x)^{k-2}(x, a)(x, b) + (k-1)^{-1}(a, b)\theta_{k-1}(x)$. Thus $h(a, b) = v(a, b) - (k-1)^{-1}(a, b)\theta_{k-1} \in F$, where $h(a, b)(x) = (x, x)^{k-2}(x, a)(x, b)$. Continuing this process we obtain that for $k = 1, \dots, n$, $a_1, \dots, a_k \in E$, that F contains all functions of the form $(x, a_1) \dots (x, a_k)$, where $x \in E$. Since the set of all such functions including $\bar{1}$ generates the family P_n of all elements of $H(E, R)$ of degree at most n , $P_n \subseteq F_n$, and $P_n = F_n$.

For $n \in \omega$, $h \in H_{2n}(E, R)$, we have $\tilde{h} \in F$ and for $x \in E$, $\tilde{h}(x) = a(x, x)^n = a\theta_n(x)$ for some $a \in R$. Thus if $\tilde{h} \neq 0$, $\theta_n \in F$, and $P_n \subseteq F$; and the theorem follows.

Theorem 5.2. *Let $q, m \in \omega$, $q = 2m$, and let F be a TGR family of $H(E, R)$ such that for $n \geq q$, $n \in \omega$, $\tilde{h} \equiv 0$, for all $h \in H_n(E, R) \cap F$. Then F is a TRL family. Moreover if F_0 is a TGR family of $C^q(E, R)$ such that for $f \in F_0$, $\Delta^m f \equiv 0$, then F_0 is a TRL family.*

We shall show for $J = F, F_0$, that $\tilde{J} = \{\tilde{f}; f \in J\}$ is finite dimensional. This enables us to express $f(0) = \tilde{f}(0)$ as an integral expression in $\tilde{f}$ and hence in f , for $f \in F$. These integral expressions can be interpreted as volume integrals from which the desired Lipschitz conditions may be extracted in a way similar to that employed on harmonic functions in [4] or the function families studied in [5].

Proof. For $f \in H(E, R)$, $r \in R$, $r \neq 0$, set

$$(Tf)(r) = r^{-p} \int_{U(1/r)} f \, dm,$$

and set $(Tf)(0) = f(0)$, where m is volume measure on E normalized so that $m(U) = 1$. We first show for $f \in F$, that $Tf \in W = \{h \in H(R, R); \text{degree } h \leq q-1\}$. Let $f \in F$, and for $n \in \omega$, set $a_n = p(p+n)^{-1} \int_V f_n \, d\mu$. Then for $r > 0$, $n \in \omega$,

$$\int_{B(r)} f_n \, d\mu = \int_V f_n(rx) \, d\mu(x) = r^n \int_V f_n \, d\mu = p^{-1}(p+n)a_n r^n,$$

and

$$(Tf_n)(r) = pr^{-p} \int_0^r s^{p-1} \int_{B(s)} f_n \, d\mu \, ds = pr^{-p} \int_0^r s^{p-1} p^{-1}(p+n)a_n s^n \, ds = a_n r^n.$$

Now for $n \geq q$, $n \in \omega$, by hypothesis, $a_n = p(p+n)^{-1} \int_V f_n \, d\mu = p(p+n)^{-1} \int_V \tilde{f}_n \, d\mu \equiv 0$, and thus $(Tf)(r) = \sum_{n=0}^{q-1} a_n r^n$ for $r \in R$, and $TF \in W$.

For $f \in W$, set $C(f) = \{f(1/q), f(2/q), \dots, f(1)\}$. Then for $f \in W$, if $C(f) \equiv 0$, f is a polynomial of degree at most $q-1$ with at least q roots and hence $f \equiv 0$. Thus C is one-to-one. Since W is of dimension q , the range of C is exactly $R^q = \prod_{i=1}^q R$. For $f \in W$, set $L(f) = f(0)$ and set $D = LC^{-1}$. Since D is linear, there exist $\rho_1, \dots, \rho_q \in R$ such that for $a = \{a_1, \dots, a_q\} \in R^q$, $D(a) = \sum_{i=1}^q \rho_i a_i$. Then for $f \in W$,

$$f(0) = L(f) = (LC^{-1})(Cf) = D(Cf) = \sum_{i=1}^q \rho_i (Cf)_i = \sum_{i=1}^q \rho_i f(k/q).$$

Then for $f \in F$, since $Tf \in W$,

$$f(0) = (Tf)(0) = \sum_{i=1}^q \rho_i (Tf)(k/q) = \sum_{i=1}^q \rho_i (k/q)^{-p} \int_{U(k/q)} f \, dm.$$

Thus the desired integral representation of elements of F is obtained. We now proceed to manipulate this representation to obtain a uniform Lipschitz condition.

For $A, B \subseteq E$, set $A + B = (A - B) \cup (B - A)$. Then for $x \in E$, $0 < r \leq 1$,

$$m[U_x(r) + U(r)] \leq m[U_x(1) + U] \leq 2||x|| \, c,$$

where $||x|| \, c$ is the volume of a "cylinder" with altitude $||x||$ and base c , and thus

$$\begin{aligned} \left| \int_{U_x(r)} f \, dm - \int_{U(r)} f \, dm \right| &\leq \int_{U_x(r) + U(r)} |f| \, dm \\ &\leq m[U_x(r) + U(r)] M(f) \leq 2cM(f) ||x||, \end{aligned}$$

where $M(f) = \sup \{|f(t)|; t \in U(3)\}$. Hence

$$|f(x) - f(0)| \leq \sum_1^q |\rho_k| (k/q)^{-p} 2cM(f) \|x\| \leq NM(f) \|x\|,$$

where $N = \sum_1^q |\rho_k| (k/q)^{-p} 2c$, and F is a TRL family.

We now consider F_0 . Let W_0 be the family of all continuous functions f on $[0, 1]$ into R such that $f|_{(0, 1)}$ lies in $C^q(R, R)$. Let $u \in W_0$ and set $f(x) = u(|x|)$ for $x \in \bar{U}$. Then for $x \in U$, $x \neq 0$, $t \in E$,

$$f_x^{(a)}(t, \dots, t) = u^{(a)}(|x|) \cdot (x, t)^a / \|x\|^a + \sum_0^{q-1} u^{(k)}(|x|) P[k, x](t),$$

where $P[k, x] \in H_k(E, R)$ for $k = 0, 1, \dots, q-1$, $x \in U$, $x \neq 0$. Let $e \in V$, and set for $r > 0$, $k = 0, 1, \dots, q-1$,

$$P_k(r) = s_m^{-1} \int_V P[k, re](t) d\mu(t).$$

Then for $0 < r < 1$,

$$\begin{aligned} \Delta^m f(re) &= s_m^{-1} \int_V f_{re}^{(a)}(t, \dots, t) d\mu(t) = u^{(a)}(r) s_m^{-1} \int_V (x, t)^a / \|x\|^a d\mu(t) \\ &+ \sum_0^{q-1} s_m^{-1} \int_V P[k, re](t) d\mu(t) = u^{(a)}(r) + \sum_0^{q-1} u^{(k)}(r) P_k(r). \end{aligned}$$

Set $W = \{u \in W_0; u^{(a)}(r) + \sum_0^{q-1} P_k(r) u^{(k)}(r) = 0 \text{ for } r \in (0, 1)\}$. Since the elements of W restricted to $(0, 1)$ are solutions of a linear ordinary differential equation with continuous coefficients, W is finite dimensional, and if any element of W vanishes on an open set in $(0, 1)$ it vanishes identically on $[0, 1]$.

For $r \in (\frac{1}{2}, 1)$, set $\theta(r) = (r - \frac{1}{2})(1 - r)$, and for $r \in R - (\frac{1}{2}, 1)$, set $\theta(r) = 0$. For $u \in W$, $r \in [0, 1]$, set $(Tu)(r) = \theta(r)u(r)$. Then T is one-to-one. For $u, v \in Z = T(W)$, set $(u, v) = \int_0^1 u(r)v(r) dr$. Then Z is a finite dimensional inner product space and hence a Hilbert space. For $h \in Z$, set $L(h) = h(0)$, and set $D = LT^{-1}$. Then from the Riesz Theorem there exists $g_0 \in Z$, such that $D(h) = (h, g_0) = \int_0^1 h(r)g_0(r) dr$ for $h \in Z$. Thus for $u \in W$,

$$u(0) = (LT^{-1})(Tu) = D(Tu) = \int_0^1 g_0(r)(Tu)(r) dr = \int_0^1 g_0(r)\theta(r)u(r) dr.$$

Let $f \in F_0$, $\bar{U}(3) \subseteq \text{dom } f$. Then

$$\begin{aligned} f(0) &= \tilde{f}(0) = \int_0^1 g_0(r)\theta(r)\tilde{f}(re) dr = p \int_0^1 r^{p-1} \int_{B(r)} p^{-1}r^{1-p}g_0(r)\theta(r)f(x) d\mu(x) dr \\ &= p \int_0^1 r^{p-1} \int_{B(r)} \phi(x)f(x) d\mu(x) dr = \int_U \phi \cdot f dm = \int_E \phi \cdot f dm, \end{aligned}$$

where $\phi(x) = p^{-1} \|x\|^{1-p}g_0(\|x\|)\theta(\|x\|)$ for $x \in E$, $x \neq 0$, and $\phi(0) = 0$. Then

for $x \in U$,

$$\int_E f(t-x)\phi(t) dm(t) = \int_E f(t)\phi(t+x) dm(t)$$

and

$$\begin{aligned} |f(x) - f(0)| &= \left| \int_E f(t)\phi(t+x) dm(t) - \int_E f(t)\phi(t) dm(t) \right| \\ &\leq \int_E |f(t)| \cdot |\phi(t+x) - \phi(t)| dm(t) \leq \int_{U(2)} |f(t)| N_0 \|x\| dm(t) \\ &\leq M(f)N_0 \|x\|^{2^p} = NM(f) \|x\|, \end{aligned}$$

where $N = 2^{2^p}N_0$, and N_0 is such that $|\phi(s) - \phi(t)| \leq N_0 \|s - t\|$ for $s, t \in U(2)$. Thus F_0 is a *TRL* family.

Proof of Theorem B. From Theorem 5.1, for $W = H(E, R) \cap F$, either

- (1) $W = H(E, R)$; OR
- (2) There exists $q, m \in \omega$, $q = 2m$, such that for $n \geq q$, $n \in \omega$, and $h \in W \cap H_n(E, R)$, $\tilde{h} \equiv 0$.

If (1) holds, since F is strongly closed, it follows from the Stone-Weierstrass Theorem that $F = F(E, R)$.

Assume (2) holds. Set $Z = C^q(E, R) \cap F$, let $f \in Z$, and let $x \in \text{dom } f$. Then setting $h(t) = f_x^{(q)}(t, \dots, t)$ for $t \in E$, from Theorem 3.2, $h \in H_q(E, R) \cap F$ and $\tilde{h} \equiv 0$. Thus

$$\Delta^m f(x) = s_m^{-1} \int_V f_x^{(q)}(t, \dots, t) d\mu(t) = s_m^{-1} \int_V h d\mu = s_m^{-1} \int_V \tilde{h} d\mu = 0.$$

Then from Theorem 5.2, there exists $N > 0$, such that for $f \in Z$, $\bar{U} \subseteq \text{dom } f$, $x \in U$, we have

$$(3) \quad |f(x) - f(0)| \leq NM(f) \|x\|,$$

where $M(f) = \sup \{ |f(t)|; t \in U \}$. From Theorem 3.1, for $f \in F$, there exists a sequence $f_1, f_2, \dots$ of Z which converges uniformly on $\bar{U}$ to $f|_{\bar{U}}$, and thus f must satisfy (3), and F is a *TRL* family.

6. The general TR case. Let $Q \in H_q(E, R)$, $q \in \omega$, and $f \in C^q(E, R)$. For $x \in \text{dom } f$, set

$$[A(Q)f](x) = \int_V Q(t)f_x^{(q)}(t, \dots, t) d\mu(t).$$

Q and the differential operator $A = A(Q)$ are said to be elliptic if

$$\bar{Q}(x) = \int_V Q(t)(x, t)^q d\mu(t) > 0$$

for all $x \in E$, $x \neq 0$. For $x_1, \dots, x_q \in E$, set

$$Q_0(x_1, \dots, x_q) = \int_V Q(t)(x_1, t) \cdots (x_q, t) d\mu(t).$$

Then for $x \in E$, $\bar{Q}(x) = Q_0(x, \dots, x)$. Let $e_1, \dots, e_p$ be an orthonormal basis of E , let $z \in \text{dom } f$, and set $h = f_s^{(q)}$. Then

$$\begin{aligned} Af(z) &= \int_V Q \cdot h d\mu = \int_V Q(t) \sum \hat{h}(e_{i_1}, \dots, e_{i_q})(t, e_{i_1}) \cdots (t, e_{i_q}) d\mu(t) \\ &= \sum \hat{h}(e_{i_1}, \dots, e_{i_q}) \int_V Q(t)(t, e_{i_1}) \cdots (t, e_{i_q}) d\mu(t) \\ &= \sum \hat{h}(e_{i_1}, \dots, e_{i_q}) Q_0(e_{i_1}, \dots, e_{i_q}) = \sum \frac{\partial^q f(z)}{\partial x_{i_1} \cdots \partial x_{i_q}} a(i_1, \dots, i_q), \end{aligned}$$

where $a(i_1, \dots, i_q) = Q_0(e_{i_1}, \dots, e_{i_q})$, and the summation is taken over the family of all q -tuples $\{i_1, \dots, i_q\}$ formed from the integers $1, \dots, p$. Thus [1], [3]

$$\bar{Q}(x) = \sum a(i_1, \dots, i_q) x_{i_1} \cdots x_{i_q},$$

where $x_k = (x, e_k)$ for $x \in E$, $k = 1, \dots, p$, is the elliptic polynomial expression conventionally associated with A .

For $x, t \in E$, $n \in \omega$, set $P[n, x](t) = (x, t)^n$ and set $P[0, x] = 1$.

Theorem 6.1. *Let F be a TR family of $H(E, R)$. Then either there exists $e \in V$, such that $P[n, e] \in F$ for all $n = 0, 1, \dots$, OR there exists an elliptic polynomial $Q \in H_q(E, R)$, $q \in \omega$, such that for all $n \geq q$, $n \in \omega$ and $h \in H_n(E, R) \cap F$,*

$$\int_V h(t) Q(t) d\mu(t) = 0.$$

We first show that either there exists $e \in V$ such that $P[n, e] \in F$ for $n = 0, 1, 2, \dots$, or that there exists $m \in \omega$ such that for $k \geq m$, $k \in \omega$, $P[k, x] \notin F$ for all $x \in V$. In the latter case a functional L can be found on $H_m(E, R)$ which vanishes on $W_m = H_m(E, R) \cap F$, but which annihilates no element of $\{P[n, x]; x \in V\}$. L then generates the desired polynomial Q .

Proof. Let $x \in V$, $n \in \omega$, and assume $P[n, x] \in F$. Then setting $h(t) = (P[n, x])'_t(x)$ for $t \in E$, as in the proof of Theorem 5.1, $h \in F$. Now $h(t) = n(t, x)^{n-1}(x, x) = n(t, x)^{n-1}$ for $t \in E$, and hence $P[n-1, x] \in F$.

We consider the case where for every $k \in \omega$, there exists $x_k \in V$, and $n_k \geq k$, $n_k \in \omega$, such that $P[n_k, x_k] \in F$. Then there exists $x_0 \in V$, and $k_1 < k_2 < \dots$ in ω , such that the sequence $x_{k_1}, x_{k_2}, \dots$ converges to x_0 . Let $n \in \omega$. Then $P[n_k, x_{n_k}]$, $P[n, x_{n_k+1}]$, $\dots$ is a sequence in $W_n = H_n(E, R) \cap F$, which converges uniformly on V to $P[n, x_0] \in V$. For $f, g \in Z = W_n$, $H_n(E, R)$, set $(f, g) = \int_V f(t)g(t) d\mu(t)$. Then Z is a finite dimensional inner product space and hence a Hilbert space. Thus $P[n, x_0] \in W_n$ for all $n = 0, 1, \dots$

We now consider the case where there exists $m \in \omega$, such that for all $k \geq m$, $k \in \omega$, $x \in V$, $P[k, x] \notin F$. Let $x \in V$. Then there may be found a linear functional L_x on $H_m = H_m(E, R)$ into R , such that $L_x(h) = 0$ for all $h \in W_m$, and $L_x(P[m, x]) = 1$. From the Reisz Theorem there exists $Q_x \in H_m$, such that $L_x(h) = (h, Q_x) = \int_V h(t) Q_x(t) d\mu(t)$ for all $h \in H_m$. Since the mapping $T_x : t \rightarrow (P[m, t], Q_x)$, where $t \in E$, is continuous, and $T(x) = 1$, there exists $\delta_x > 0$, such that for $t \in U_x(\delta_x)$, $T_x(t) > 0$. Since V is compact, there exist $x_1, \dots, x_k \in V$, $k \in \omega$, such that $V \subseteq \bigcup_1^k U_{x_i}(\delta_{x_i})$.

For $s, t \in E$, set $Q_0(s, t) = \sum_1^k Q_{x_i}(s) Q_{x_i}(t)$. Set $q = 2m$. For $h \in H_q(E, R)$, set $h_s(t) = \hat{h}(s, \dots, s, t, \dots, t)$, where the expression $h_s(t)$ is of degree m in each of its entries, and set

$$L(h) = \int_V \int_V Q_0(s, t) h_s(t) d\mu(t) d\mu(s)$$

for $h \in H_q$. Then there exists $Q \in H_q$ such that $L(h) = (h, Q) = \int_V h \cdot Q d\mu$ for all $h \in H_q$. Let $h \in H_q \cap F$, and $s \in E$. Then $h_s \in W_m$ and

$$\begin{aligned} \int_V Q_0(s, t) h_s(t) d\mu(t) &= \int_V \sum_1^k Q_{x_i}(s) Q_{x_i}(t) h_s(t) d\mu(t) \\ &= \sum_1^k Q_{x_i}(s) \int_V Q_{x_i}(t) h_s(t) d\mu(t) = 0. \end{aligned}$$

Thus $L(h) = \int_V Q \cdot h d\mu = 0$.

Let $x \in V$ and set $h = P[q, x]$. Then $h_s(t) = (x, s)^m (x, t)^m$ for $s, t \in E$, and

$$\begin{aligned} \int_V Q \cdot P[q, x] d\mu &= \int_V Q(t) (x, t)^q d\mu(t) = \int_V \int_V Q_0(s, t) h_s(t) d\mu(s) d\mu(t) \\ &= \sum_1^k \int_V \int_V Q_{x_i}(s) Q_{x_i}(t) (x, s)^m (x, t)^m d\mu(s) d\mu(t) \\ &= \sum_1^k \int_V Q_{x_i}(s) (x, s)^m d\mu(s) \int_V Q_{x_i}(t) (x, t)^m d\mu(t) \\ &= \sum_1^k L_{x_i}(P[m, x])^2 > 0, \end{aligned}$$

since for some $i = 1, \dots, k$, $x \in U_{x_i}(\delta_{x_i})$ and $L_{x_i}(P[m, x]) > 0$.

In the proof of the next theorem we shall need the following version of integration by parts.

Lemma 6.1. *Let $Q \in H_q(E, R)$, $q \in \omega$, q even, and set $A = A(Q)$. Let $f, g \in C^q(E, R)$, $\bar{U} \subseteq \text{dom } f$, $\text{dom } g$, such that $g_x^{(k)} = 0$ for all $x \in V$, $k = 0, 1, \dots, q - 1$. Then*

$$\int_U (Af) \cdot g dm = \int_U f \cdot (Ag) dm.$$

Proof. Let $a, t \in V$. Then there exists $\rho \geq 0$, such that $b = a + \rho t \in V$. For $r \in [0, \rho]$, set $u(r) = f(a + rt)$ and $v(r) = g(a + rt)$. Then by repeated application of integration by parts on $[0, \rho]$,

$$\begin{aligned} \int_a^b f_x^{(q)}(t, \dots, t) g(x) d\pi(x) &= \int_0^\rho u^{(q)}(r) v(r) dr \\ &= - \int_0^\rho u^{(q-1)}(r) v'(r) dr + \big|_0^\rho u^{(q-1)}(r) v(r) = - \int_0^\rho u^{(q-1)}(r) v'(r) dr \\ &= \dots = \int_0^\rho u(r) v^{(q)}(r) dr = \int_a^b f(x) g_x^{(q)}(t, \dots, t) d\pi(x), \end{aligned}$$

where π is linear measure on the interval $[a, b]$. Thus

$$\int_U f_x^{(q)}(t, \dots, t) g(x) dm(x) = \int_U f(x) g_x^{(q)}(t, \dots, t) dm(x)$$

for $t \in V$, and hence

$$\begin{aligned} \int_U A f(x) g(x) dm(x) &= \int_U \int_V f_x^{(q)}(t, \dots, t) Q(t) g(x) d\mu(t) dm(x) \\ &= \int_V Q(t) \left\{ \int_U f_x^{(q)}(t, \dots, t) g(x) dm(x) \right\} d\mu(t) \\ &= \int_V Q(t) \left\{ \int_U f(x) g_x^{(q)}(t, \dots, t) dm(x) \right\} d\mu(t) \\ &= \int_U f(x) A g(x) dm(x). \end{aligned}$$

Theorem 6.2. Let Q be an elliptic element of $H_q(E, R)$, $q \in \omega$, and let F be a TR family of $C^{q+p}(E, R)$ such that for $f \in F$, $A = A(Q)$, $Af \equiv 0$. Then F is a TRL family.

Our approach in the proof of this theorem is based on the method of John [3] for showing the analyticity of solutions of arbitrary elliptic equations with analytic coefficients. First the elements f of $C^{q+p}(E, R)$ are expressed in terms of surface means and derivatives of surface means of f taken over spheres with radii bounded away from zero. The version of this argument most suited to this paper and used here is found in John [2, p. 171]. Then for $f \in F$, one shows that surface means of f and derivatives thereof can be expressed in terms of weighted volume means of f . Thus for $f \in F$, f can be expressed in terms of weighted volume means of f , and hence as in the proof of Theorem 5.2, the desired uniform Lipschitz condition is obtained.

Proof. Set $F_0 = \{f \in F; \bar{U} \subseteq \text{dom } f\}$, and set for $f \in F_0$, $0 < r, s \leq 1$,

$$M(f, r, s) = \int_{B(r)} \int_{B_s(x)} f(t) d\mu(t) d\mu(x).$$

Let $f \in F_0$. Then

$$(1) \quad \begin{aligned} M(f, r, s) &= \int_V \int_V f(rx + sy) d\mu(y) d\mu(x) \\ &= \int_V \int_V f(sy + rx) d\mu(x) d\mu(y) = M(f, s, r). \end{aligned}$$

Now for $g \in G$, $0 < r, s \leq 1$, $x, y \in V$, $y = g(x)$,

$$\int_{B_x(r)} \tilde{f} d\mu = \int_{B_y(r)} \tilde{f} d\mu$$

and $M(fg, r, s) = M(f, r, s)$, and thus

$$M(f, r, s) = M(\tilde{f}, r, s) = \int_{B(r)} \int_{B_s(s)} \tilde{f}(y) d\mu(y) d\mu(x) = \int_{B_s(s)} \tilde{f} d\mu.$$

Let $e \in V$, and set $u(r) = \tilde{f}(re)$ for $0 \leq r \leq 2$. Then for $0 < r, s \leq 1$

$$\begin{aligned} M(f, r, s) &= \int_{B_{re}(s)} u(|x|) d\mu(x) = a \int_0^\pi u(\rho(\theta)) s^{p-1} \sin^{p-2} \theta d\theta \\ &= a \int_{r-s}^{r+s} u(\rho) \{1 - (r^2 + s^2 - \rho^2)^2 / 4r^2 s^2\}^{(p-3)/2} (\rho/rs) d\rho, \end{aligned}$$

where $\cos \theta = [r^2 + s^2 - \rho(\theta)^2] / 2rs$ and $as^{p-2} \sin^{p-2} \theta$ represents the "volume" of the sphere $B[\rho(\theta)] \cap B_{re}(s)$ for $\theta \in [0, \pi]$, $0 < r, s \leq 1$. Then for $-2 < x < 2$, setting $r = 1 + x/2$ and $s = 1 - x/2$, we have

$$\begin{aligned} A(x) &= M(f, 1 + x/2, 1 - x/2) \\ &= a2^{p-1}(4 - x^2)^{2-p} \int_x^2 u(\rho) [(4 - \rho^2)(\rho^2 - x^2)]^{(p-3)/2} \rho d\rho. \end{aligned}$$

For $-2 < x < 2$, set

$$B(x) = (4 - x^2)^{p-2} a^{-1} 2^{1-p} A(x),$$

and set $C(x) = B(\sqrt{x})$ for $0 \leq x < 4$. Then for $0 \leq x < 4$,

$$(2) \quad C(x) = \int_{x^{1/2}}^2 u(\rho) \{(4 - \rho^2)(\rho^2 - x)\}^{(p-3)/2} \rho d\rho.$$

We now restrict ourselves to the case when p is odd. Then $(p - 3)/2$ is an integer and the integrand of (2) and its first $(p - 5)/2$ derivatives with respect to x vanish at the lower limit, and thus

$$\begin{aligned} (3) \quad C^{(p-1)/2}(x) &= \frac{d}{dx} \int_{x^{1/2}}^2 u(\rho) [4 - \rho^2]^{(p-3)/2} [(p - 3)/2]! d\rho \\ &= x^{-1/2} \{u(\sqrt{x})(4 - x)^{(p-3)/2} [(p - 3)/2]! \sqrt{x}\} \\ &= u(\sqrt{x})(4 - x)^{(p-3)/2} [(p - 3)/2]!. \end{aligned}$$

From (1), we have $B(x) = B(-x)$ for $-2 < x < 2$, and thus $B'(0) = 0$. Now

$$C'(x) = -B'(\sqrt{x})/2\sqrt{x} \\ = -[B'(\sqrt{x}) - B'(0)]/2\sqrt{x} \rightarrow B''(0)/2 \text{ as } x \rightarrow 0, \text{ for } -2 < x < 2.$$

Thus $C'(0)$ exists and $C'(0) = B''(0)/2$. Continuing we find that $C^{(k)}(0)$ exists and can be written as a linear combination of $B(0), \dots, B^{(2k)}(0)$ for $k \leq (p-1)/2$, $k \neq \omega$. Thus we may write, from (3), $C^{((p-1)/2)}(0) = bu(0)$, where $b = 4^{(p-3)/2}[(p-3)/2]!$, and

$$(4) \quad f(0) = u(0) = b^{-1} \sum_0^{p-1} \sigma_k B^{(k)}(0),$$

where $\sigma_1, \sigma_2, \dots, \sigma_k \in R$.

Equation (4) gives the desired representation of f in terms of iterated surface means taken over spheres with radii bounded away from 0. We now employ integration by parts (Lemma 6.1) to express the surface means of (4) as volume means.

Set $\epsilon = \frac{1}{4}$. For $x \in E$, set $w(x) = 0$ if $\|x\| < \epsilon$, set $w(x) = (\|x\| - \epsilon)^{q+1}[(\|x\| - 2\epsilon)^{q+1} + 1]/\epsilon^{q+1}$ if $\epsilon \leq \|x\| \leq 2\epsilon$, and set $w(x) = 1$ if $\|x\| > 2\epsilon$. For $r > 0$, $x \in E$, set $\phi_r(x) = (r - \|x\|)^q$. Let $\theta \in C^q(E, R)$, $\text{dom } \theta = E$. Then $(\theta\phi_r w)^{(k)}(x) = 0$ for $x \in B(r)$, $k = 0, 1, \dots, q-1$, $r > 2\epsilon$, and thus from Lemma 6.1, for $2\epsilon < r \leq 1$,

$$(5) \quad 0 = \int_{U(r)} (Af) \cdot \theta\phi_r w \, dm = \int_{U(r)} f \cdot A(\theta\phi_r w) \, dm \\ = \int_{U(r)} f \theta \bar{Q} q! \rho^{-q} w \, dm + \sum_1^q \int_{U(r)} f(x) (r - \|x\|)^k J_k(x) \, dm(x),$$

where $J_1, \dots, J_k \in F(E, R)$ and $\rho(x) = \|x\|$ for $x \in E$. Differentiating, we obtain from (5), for $2\epsilon < r \leq 1$,

$$(6) \quad pr^p \int_{B(r)} f \theta \bar{Q} \rho^{-q} \, d\mu = \sum_1^q \int_{U(r)} f(x) J_k(x) \frac{d}{dr} (r - \|x\|)^k \, dm(x).$$

Choosing θ so that $(\theta \bar{Q} q! \rho^{-q})(x) = 1$, for $\|x\| > 2\epsilon$, $x \in E$, we have from (6), for $r > 2\epsilon$, $\int_{B(r)} f \, d\mu$ expressed as a sum of weighted volume integrals of f . If we differentiate again we will obtain an expression for $d/dr \int_{B(r)} f \, d\mu$ which contains weighted surface means of f . By suitable choice of θ these too can be reduced to volume means. Thus from (4), we may write

$$(7) \quad f(0) = \int_V \int_{U_v(1)} f(t) \phi(t-y) \, dm(t) \, d\mu(y),$$

where $\phi \in C^q(E, R)$, $\text{dom } \phi = E$. Then as in the proof of Theorem 5.2, for $x \in U$, $A = U_{x+u}(1)$, $B = U_v(1)$,

$$|f(x) - f(0)| = \left| \int_V \left\{ \int_A f(t) \phi(t-x-y) \, dm(t) - \int_B f(t) \phi(t-y) \, dm(t) \right\} d\mu(y) \right|$$

$$\begin{aligned}
&\leq \int_V \left\{ \int_{A+B} |f(t)| \cdot |\phi(t-x-y) - \phi(t-y)| dm(t) \right. \\
&\quad \left. + \int_{A \cap B} |f(t)| \cdot |\phi(t-x-y) - \phi(t-y)| dm(t) \right\} d\mu(y) \\
&\leq \int_V \left\{ 2M(f)M(\phi)m(A+B) + M(f)N_1 \|x\| m(A \cap B) \right\} d\mu(y) \\
&\leq M(f)M(\phi)N_2 \|x\| + M(f)N_1 \|x\| = NM(f) \|x\|,
\end{aligned}$$

where $N = M(\phi)N_2 + N_1$, $N_2 > 0$, N_1 is such that $|\phi(s) - \phi(t)| \leq N_1 \|s - t\|$ for $s, t \in U(3)$, and $M(f) = \sup \{|f(t)|; t \in U(3)\}$.

To handle the case when p is even, set $E' = E \times R$ and $f'(x, r) = f(x)$ for $x \in \text{dom } f$, $r \in R$, and define Q' and A' as follows. For $h \in H'_q = H_q(E', R)$, set $S(h) = \int_V h \cdot Q d\mu + h(e)$, where $e = (0, 1) \in E'$. Let Q' be the element of H'_q such that $S(h) = \int_V Q' \cdot h d\mu'$ for all $h \in H'_q$. Let $z \in \text{dom } f'$ and set $h' = (f')_z^{(q)}$. Then $h'(e) = (\partial^q / \partial e^q) f(z) = 0$, and $h'(t) = h(t)$ for $t \in E \subseteq E'$. Thus

$$A'f(z) = \int_V h' \cdot Q' d\mu' = S(h') = \int_V h' \cdot Q d\mu + h'(e) = \int_V h \cdot Q d\mu = Af(z).$$

Furthermore for $x \in E'$, $x \neq 0$,

$$\bar{Q}'(x) = \int_V Q'(t)(x, t)^q d\mu' = \int_V Q(t)(x, t)^q d\mu + (x, e)^q.$$

Since $(x, t)^q > 0$ for some $t \in E$, or $(x, e)^q > 0$, $\bar{Q}'(x) > 0$.

Proof of Theorem C. The proof of Theorem C is similar to that of Theorem B, using Theorems 6.1 and 6.2 in place of Theorems 5.1 and 5.2.

7. Applications. The following theorem relates the work of this paper with the theory of distributions [1], [3].

Theorem 7.1. *Let Q be an elliptic element of $H_q(E, R)$, $q \in \omega$, and set $A = A(Q)$. Let F be the family of all $f \in F(E, R)$ such that $\int_E f(t)Ah(t) dm(t) = 0$ for all elements h of the family $Z(f)$ of infinitely differentiable functions on E into R supported on compact subsets of domain f . Then F is an TRL family.*

The elements of F may be considered as “weak” solutions of the equation

$$(1) \quad Af = 0.$$

Proof. Clearly F is a strongly closed TR family. Let $n \in \omega$, n even, $n \geq q$, $z \in E$, $z \neq 0$, and set $h(t) = P[n, z](t) = (z, t)^n$ for $t \in E$. Then for $x, t \in E$, $h_x^{(q)}(t, \dots, t) = n(n-1) \cdots (n-q+1)h(x, \dots, x, t, \dots, t) = (z, x)^{n-q}(z, t)^q$ and

$$\begin{aligned}
Ah(x) &= \int_V h_x^{(q)}(t, \dots, t)Q(t) d\mu(t) = (z, x)^{n-q} \int_V (z, t)^q Q(t) d\mu(t) \\
&= (z, x)^{n-q} \bar{Q}(x).
\end{aligned}$$

Thus $Ah(x) > 0$ for all $x \in E$ such that $(x, z) \neq 0$. But from Lemma 6.1, for all $w \in Z(h)$,

$$\int_E h \cdot (Aw) \, dm = \int_E (Ah) \cdot w \, dm = 0,$$

and thus $Ah \equiv 0$. Hence $h \notin F$, and from Theorem C, F is a *TRL* family. Thus the elements of F are infinitely differentiable and analytic. Applying Lemma 6.1 we see that all weak solutions of (1) are "strong" classical solutions of (1).

Theorem 7.2. (*mean value families*) *Let m_0 be a non-negative normalized Borel measure with compact support $K \subseteq E$, such that K does not lie in any proper subspace of E . Let F be the family of all $f \in F(E, R)$, such that for $x \in \text{dom } f$, $\delta > 0$, such that $x + \delta K \subseteq \text{dom } f$, we have*

$$f(x) = \int_K f(x + \delta t) \, dm_0(t).$$

Proof. Clearly F is a strongly closed *TR* family. Furthermore, for $x \in E$, $x \neq 0$, $k \in \omega$, k even,

$$\int_K P[n, x] \, dm_0 = \int_K (x, t)^k \, dm_0(t) > 0 = P[k, x](0),$$

and $P[n, x] \notin F$. Thus from Theorem C, F is a *TRL* family.

In [6] it is shown that with respect to a suitably chosen inner product on E , the elements of F are harmonic functions.

Theorem 7.3. *A necessary and sufficient condition that a *TR* family F of $F(E, R)$ be a *TRL* family is that there exist $N > 0$, such that for $f \in F$, $\bar{U} \subseteq \text{dom } f$,*

$$(1) \quad |f(0)| \leq N \int_U |f(t)| \, dm(t).$$

Proof. Assume F is a *TRL* family. Then from equation (7) of the proof of Theorem 6.2, there exists $\phi \in F(E, R)$, $\text{dom } \phi = E$, such that for $f \in F$, $\bar{U} \subseteq \text{dom } f$, $\epsilon = \frac{1}{2}$,

$$\begin{aligned} |f(0)| &\leq \int_{B(\epsilon)} \int_{U_\nu(\epsilon)} |f(t)| \cdot |\phi(t - y)| \, dm(t) \, d\mu(y) \\ &\leq \int_{B(\epsilon)} \int_{U_\nu(\epsilon)} |f(t)| \, N \, dm(t) \, d\mu(y) \leq N \int_{B(\epsilon)} \int_U |f| \, dm \, d\mu = N \int_U |f| \, dm, \end{aligned}$$

where $N = \sup \{|\phi(t)|; t \in U(3)\}$.

Assume F is not a *TRL* family. Then from Theorem C, there exists $e \in V$, such that F contains all functions of the form $h[(x, e)]$, where $x \in E$, $h \in F(R, R)$. Let $N > 0$. Then there exists $n \in \omega$, such that $\int_{-1}^{+1} (1 - r^2)^n \, dr < 1/Nc$, where c is the volume of a cube of side 1 in E . For $x \in E$, set $f_n(x) = [1 - (x, e)^2]^n$. Then

$f_n \in F$, and

$$N \int_U |f_n| dm = N \int_U [1 - (x, e)^2]^n dm \leq cN \int_{-1}^{+1} (1 - r^2)^n dr \\ < cN(1/cN) = 1 = f_n(1).$$

Thus there exists no $N > 0$, such that (1) is satisfied.

8. Concluding remark. Basic to the proof of Theorems B and C was the fact that if the families F in question were TRL families, they were solution spaces of elliptic differential equations. If we replace R by an arbitrary Banach space B and consider TRL families of $F(E, B)$ rather than $F(E, R)$, this no longer need be the case, as the following example shows. Let E be the complex plane K . For $z = x + iy \in E$, set $R(z) = x$ and $I(z) = y$. For $f(x) \equiv \sum_0^\infty a_k x^k \in H(R, R)$, and $z \in E$, set

$$(Tf)(z) = f[R(z)] + \left[R \left(\sum_0^\infty a_k z^k \right) \right]_i$$

and $F = \{Tf; f \in H(R, R)\}$. Now for $f \in F$, $I(f)$ is a harmonic function. Thus there exists $N > 0$, such that for $f \in F$, $h = I(f)$, and $z \in U$,

$$|h(z) - h(0)| \leq NM(h) \|z\|,$$

where $M(h) = \sup \{|h(t)|; t \in U\}$. Then for $f \in F$, $z = x + iy \in U$, $h = I(f)$, we have $Rf(z) = h(x)$, and

$$\begin{aligned} \|f(z) - f(0)\| &\leq \|Rf(z) - Rf(0)\| + \|If(z) - If(0)\| \\ &= |h(x) - h(0)| + |h(z) - h(0)| \leq NM(h) \|x\| + NM(h) \|z\| \\ &\leq 2NM(h) \|z\| \leq 2NM(f) \|z\|. \end{aligned}$$

Thus F is a TRL family. However, since $\{R(f); f \in F\} \cong H(R, R)$, there can exist no $Q \in H_q(E, R)$, $q \in \omega$, Q elliptic, such that $A(f) \equiv 0$ for $A = A(Q)$ and $f \in F$.

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Illinois Institute of Technology

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