Structure of Certain Nonnormal Operators*

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Introduction. Recently N. Suzuki [2] has studied operators with completely continuous imaginary parts, and he determined their type and gave a decomposition into a selfadjoint part and countably many primary components. We shall generalize this result and particularly prove that all operators A , which satisfy $AA^* - A^*A = C$ with C compact, are of type I , postliminal and have a decomposition by a unique countable family of mutually orthogonal projections $P_0, P_1, P_2, \dots, A = A_{P_0} \oplus \Sigma \oplus A_{P_i}$. The $A_{P_i}, i = 1, 2, \dots$, are primary of finite multiplicity and satisfy $A_{P_i}A_{P_i}^* - A_{P_i}^*A_{P_i} = C_i$ compact, and A_{P_0} is normal.

This result is then generalized to a larger class of operators, Theorem 2, and a further theorem on postliminal operators is presented. These theorems are then applied to some nontrivial examples of operators. The proofs are based on an investigation of concrete C^* -algebras on a separable Hilbert space. As basic reference, concerning C^* -algebras we shall use J. Dixmier's book [1] on C^* -algebras and quote by [D ...]. This is done for convenience mainly.

C^* -algebra considerations. The first three lemmas are essentially known and we therefore only indicate the proofs.

Lemma 1. *Let $\mathfrak{A}$ be a C^* -algebra with a two sided ideal I , then $\mathfrak{A}$ is postliminal iff $\mathfrak{A}/I$ and I are postliminal.*

This follows from [D5.7.4] and [D9.1], but can also be shown directly. The next lemma is a rewriting of [D2.10.3].

Lemma 2. *Let $\mathfrak{A}$ be a concrete C^* -algebra on a separable Hilbert space $\mathfrak{H}$ and let I be a two sided ideal of $\mathfrak{A}$, then any I -invariant, I -nondegenerate subspace is $\mathfrak{A}$ -invariant.*

Lemma 3. *Let $\mathfrak{H}$ be a separable Hilbert space and let $\mathfrak{A}$ be a C^* -algebra, which is contained in the compact operators $\mathfrak{K}(\mathfrak{H})$ of $\mathfrak{H}$. Then $\mathfrak{A}$ has a countable decomposition into primary parts $\mathfrak{A}_i$.*

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(1) $\mathfrak{A} = \Sigma' \oplus \mathfrak{A}_i \oplus 0_{\mathfrak{S}_0}$, where $\mathfrak{S}_0 = \{\psi \in \mathfrak{S} \mid \mathfrak{A}\psi = 0\}$ and $\Sigma' \oplus$ means restricted product.

(2) Each $\mathfrak{A}_i$ is isomorphic to a simple dual C^* -algebra and of multiplicity $n_i < \infty$, in other words $\mathfrak{A}_i$ is unitarily equivalent to $\mathfrak{A}_i \approx \mathfrak{C}(H_i) \oplus 1_{n_i}$, where H_i is a suitable Hilbert space.

(3) Corresponding to the decomposition of $\mathfrak{A}$ we have a decomposition of $\mathfrak{S}$

$$\mathfrak{S} = \sum_{i=0} \mathfrak{S}_i, \quad \mathfrak{S}_i = \sum_{k=1}^{n_i} \mathfrak{S}_i^k \quad i = 1, 2, \dots$$

(4) Let P_i, P_i^k be the projections on the subspaces $\mathfrak{S}_i, \mathfrak{S}_i^k$, then $P_i \in \mathfrak{A}'' \cap \mathfrak{A}', i = 0, 1, 2, \dots, ' denotes commutant, $P_i^k \in \mathfrak{A}', i = 1, 2, \dots, 1 \leq k \leq n_i$. This lemma follows directly from [D4.7.20], but can also be proved by using the representation theory [D ch. 8] for liminal C^* -algebras. The next lemma is a generalization of Lemma 3 and the key to the remainder.$

Lemma 4. *Let $\mathfrak{A}$ be a C^* -algebra of operators on the separable Hilbert space $\mathfrak{S}$. Then $\mathfrak{A}''$ has a decomposition by countably many orthogonal projections $P_0, P_1 \dots \in \mathfrak{A}'' \cap \mathfrak{A}'$ such that*

$$\mathfrak{A}'' = P_0 \mathfrak{A}'' \oplus \sum_{i=1} \oplus P_i \mathfrak{A}''$$

The $P_i, i > 0$, are minimal central projections and define a decomposition of $(1 - P_0)\mathfrak{A}''$ into primary parts $P_i \mathfrak{A}''$ of finite multiplicity n_i . Each $P_i \mathfrak{A}''$ is of type I and thus $\mathfrak{A}''$ is of type I iff $(P_0 \mathfrak{A})''$ is of type I. $\mathfrak{A}$ is postliminal iff $P_0 \mathfrak{A}$ and $P_i^k \mathfrak{A} / P_i^k \mathfrak{A}_c$ are postliminal. Here P_i^k are the irreducible components of $P_i, k = 1, \dots, n_i$ and $\mathfrak{A}_c = \mathfrak{A} \cap \mathfrak{C}(\mathfrak{S})$. Further $P_0 \mathfrak{A} \cap \mathfrak{C}(\mathfrak{S}) = (0)$.

Proof. To $\mathfrak{A}_c = \mathfrak{A} \cap \mathfrak{C}(\mathfrak{S})$ we apply Lemma 3 and find the decomposition

$$\mathfrak{A}_c = \sum' \oplus \mathfrak{A}_{c,i} \oplus 0_{\mathfrak{S}_i}$$

$$\mathfrak{S} = \sum_{i=0} \oplus \mathfrak{S}_i \quad \text{and for } i > 0 \quad \mathfrak{S}_i = \sum_{k=1}^{n_i} \mathfrak{S}_i^k.$$

Since $\mathfrak{A}_c$ is an ideal of $\mathfrak{A}$ and since the subspaces $\mathfrak{S}_i, \mathfrak{S}_i^k$ are $\mathfrak{A}_c$ invariant, Lemma 2 shows that they are also $\mathfrak{A}$ -invariant. This implies $P_i, P_i^k \in \mathfrak{A}'$ for the corresponding projections. By Lemma 3, $P_i \in \mathfrak{A}_c' \subset \mathfrak{A}''$, thus the P_i are central projections. $P_i \mathfrak{A}$ has the irreducible invariant subspaces $\mathfrak{S}_i^k$ for $i > 0$. Every operator $A \in P_i \mathfrak{A}$ can thus be written in matrix form

$$A = \begin{bmatrix} a_1 & & \\ & \ddots & 0 \\ 0 & & \ddots \\ & & & a_{n_i} \end{bmatrix} \quad \text{and} \quad \mathfrak{A}_{c,i} = \begin{bmatrix} c & & \\ & \ddots & 0 \\ 0 & & \ddots \\ & & & c \end{bmatrix}, c \in \mathfrak{C}(H_i)$$

since $\mathfrak{A}_{c,i}$ is primary of multiplicity n_i . Now

$$\begin{bmatrix} a_1 & & \\ & \ddots & 0 \\ 0 & & \ddots \\ & & & a_{n_i} \end{bmatrix} \begin{bmatrix} c \\ \ddots \\ 0 \\ c \end{bmatrix} = \begin{bmatrix} a_1 c & & \\ & \ddots & 0 \\ 0 & & \ddots \\ & & & a_{n_i} c \end{bmatrix} \in \mathfrak{A}_{c,i} \quad \text{thus} \quad = \begin{bmatrix} c' & & \\ & \ddots & 0 \\ 0 & & \ddots \\ & & & c' \end{bmatrix}$$

and hence $a_k c = a_i c$, $k = 1 \cdots n_i$, $\forall c \in \mathfrak{C}(H_i)$. This shows that $a_i = a = a_1$ and that also $P_i \mathfrak{A}$ is primary with multiplicity n_i for $i > 0$. Take the weak closure of $\mathfrak{A}$ now. Since $P_i = \sum_{k=1}^{n_i} P_i^k$ and since the P_i^k are irreducible, the central projections P_i , $i > 0$, are minimal central projections. By construction $P_0 \mathfrak{A} \cap \mathfrak{C}(\mathfrak{H}) = P_0 \mathfrak{A}_e = (0)$. The remainder follows easily by application of Lemma 1 and [D5.4.12] to $P_0 \mathfrak{A}$, $P_1 \mathfrak{A}$, $\dots$.

The above proof can be generalized to an arbitrary ideal I instead of $\mathfrak{A}_e$. Lemma 4 also implies the known fact [D4.3.3] that continuous v. Neumann algebras on a separable Hilbert space can intersect the compact operators only in 0. Similarly a purely antiliminal C^* -algebra has only 0 in common with the compact operators.

The operators. We want to apply these results now to operators A of the form $A^*A - AA^* \in \mathfrak{C}(\mathfrak{H})$, which will be called *nearly normal operators*.

Definition. An operator A is postliminal (of type I) if the C^* (von Neumann) -algebra it generates is postliminal (of type I).

With the above definition we obtain:

Theorem 1. An operator A with $A^*A - AA^* \in \mathfrak{C}(\mathfrak{H})$ on a separable Hilbert space $\mathfrak{H}$ is postliminal and of type I and has a decomposition by a countable family of mutually orthogonal projections $P_0, P_1, P_2, \dots$ into its normal part A_{P_0} and its primary parts A_{P_i} . The primary parts A_{P_i} are of finite multiplicity n_i .

Proof. We can assume that A is nonnormal and thus $\mathfrak{A}_e = \mathfrak{A} \cap \mathfrak{C}(\mathfrak{H}) \neq (0)$, where $\mathfrak{A}$ is the C^* -algebra generated by A . Apply Lemma 4 to $\mathfrak{A}$. Since $(P_0 A)^* (P_0 A) - (P_0 A)(P_0 A)^* \in P_0 \mathfrak{A}_e = (0)$, $P_0 A$ is normal and hence A is of type I. By Lemma 1, A is postliminal if $\mathfrak{A}/\mathfrak{A}_e$ is postliminal. Let ψ be the canonical mapping $\psi: \mathfrak{A} \rightarrow \mathfrak{A}/\mathfrak{A}_e$ then $\psi(A)$ generates $\mathfrak{A}/\mathfrak{A}_e$. But $\psi(A)$ is normal. By Lemma 4 P_i^k commutes with A and $P_i^k A$ generates $P_i^k \mathfrak{A} \supset \mathfrak{C}(\mathfrak{H}_i^k)$; thus $P_i^k A$ is irreducible. The remainder of the theorem follows from Lemma 4.

Corollary 1. $A = N + C$, N normal, C compact, is of type I, postliminal and has the above decomposition.

Corollary 2. The left (right) shift operator is an irreducible postliminal operator of type I, which is not of the form $N + C$, N normal and C compact.

This reduces the study of nearly normal operators to the investigations of irreducible nearly normal operators.

Theorem 2. Let $p_i(z, \bar{z})$, $j = 1, \dots, n$, be a finite family of complex polynomials without constant terms such that $p_i(A, A^*) = 0$ implies A is of type I and postliminal. Then also $p_i(B, B^*) = C_i \in \mathfrak{C}(\mathfrak{H})$ implies B is of type I and postliminal and B has a countable decomposition by a countable family of mutually orthogonal projections $P_0, P_1, \dots$, $B = B_{P_0} \oplus \Sigma \oplus B_{P_i}$ where the B_{P_i} , $i > 0$, are the primary components of B with finite multiplicity n_i and $p_i(B_{P_0}, B_{P_0}^*) = 0$ and $p_i(B_{P_i}, B_{P_i}^*) \in \mathfrak{C}(\mathfrak{H}_i)$, $j = 1, \dots, n$, $i > 0$ and $\mathfrak{H}_i = P_i \mathfrak{H}$.

Proof. We can assume $p_1(B, B^*) \neq 0$ hence $\mathfrak{A}_\epsilon \neq (0)$ where $\mathfrak{A}$ is the C^* -algebra generated by B and $\mathfrak{A}_\epsilon = \mathfrak{A} \cap \mathfrak{C}(\mathfrak{H})$. Apply again Lemma 4 as in Theorem 1. $p_i(P_0B, P_0B^*) \in P_0\mathfrak{A}_\epsilon = (0) \Rightarrow P_0B$ is of type I and postliminal by assumption. Let again

$$\psi : \mathfrak{A} \rightarrow \mathfrak{A}/\mathfrak{A}_\epsilon \quad \text{then } 0 = \psi[p_i(B, B^*)] = p_i(\psi B, \psi B^*)$$

hence $\mathfrak{A}/\mathfrak{A}_\epsilon$ is postliminal.

Again the problem is reduced to the study of the irreducible operators P_i^*B , which satisfy $p_i(P_i^*B, P_i^*B^*) = P_i^*C_i$, and the operator P_0B .

Obviously Theorem 1 follows from Theorem 2 for $p(B, B^*) = BB^* - B^*B$. Using further results on C^* -algebras, [D5.52], it is obvious that a postliminal operator is of type I . The converse of this is not true in general as we see from the following example.

Let A be a primary operator of type II or type III . Then $\mathfrak{A}$, the C^* -algebra generated by A , is not postliminal, and by [D9.1.V] there exist two inequivalent irreducible representations π_1, π_2 with the same kernel $I = \ker \pi_i$. Then $\pi_i(\mathfrak{A}) \simeq \mathfrak{A}/I$ and $\mathfrak{A}/I$ is not postliminal since it has at least two inequivalent irreducible faithful representations, [D9.1.V]. Hence $\pi_i(\mathfrak{A})$ are not postliminal $i = 1, 2$, but $\pi_i(\mathfrak{A})'' = B(\mathfrak{H}_{\pi_i})$. Therefore $\pi_i(A)$ is of type I but not postliminal.

Another application of Lemma 1 leads to the following theorem:

Theorem 3. *Let A be a postliminal operator on a separable Hilbert space $\mathfrak{H}$. Let $B = A + C$, $C \in \mathfrak{C}(\mathfrak{H})$ then B is postliminal.*

Proof. Let $\mathfrak{A}, \mathfrak{B}$ be the C^* -algebras generated by A respective B . By Lemma 1 and 4 we have to show that $\mathfrak{B}/\mathfrak{B}_\epsilon$ is postliminal. By [D1.8.4]. $\mathfrak{B}/\mathfrak{B}_\epsilon \simeq (\mathfrak{B} + \mathfrak{C})/\mathfrak{C} = (\mathfrak{A} + \mathfrak{C})/\mathfrak{C} \simeq \mathfrak{A}/\mathfrak{A}_\epsilon$.

By assumption $\mathfrak{A}/\mathfrak{A}_\epsilon$ is postliminal hence so is $\mathfrak{B}/\mathfrak{B}_\epsilon$.

Applications. We want to consider now some nontrivial examples of postliminal operators. Let $\mathfrak{H}$ be a separable Hilbert space then construct $\tilde{\mathfrak{H}} = \mathfrak{H} \oplus \dots \oplus \mathfrak{H}$, n -copies of $\mathfrak{H}$. Then any operator $A \in B(\tilde{\mathfrak{H}})$ can be written as $A = (A_{ij})$ with $A_{ij} \in B(\mathfrak{H})$. In general A is not of type I , even if all A_{ij} $i, j = 1, \dots, n$ are selfadjoint. But it is easy to see that A is postliminal if $A_{ij} \in \mathfrak{B}$ for all i, j , where $\mathfrak{B}$ is a commutative C^* -algebra in $B(\mathfrak{H})$. Using Theorem 3, this generalizes to: $A = (A_{ij})$ is postliminal, if for all $i, j = 1, \dots, n$, there exist a compact operator C_{ij} such that $A_{ij} - C_{ij} \in \mathfrak{B}$ where $\mathfrak{B}$ is a commutative C^* -algebra in $B(\mathfrak{H})$.

Another example for a nontrivial postliminal operator is $A = (A_{ij})$ with $A_{ii}A_{ii}^* - A_{ii}^*A_{ii} \in \mathfrak{C}(\mathfrak{H})$ $i = 1 \dots n$ and $A_{ij} \in \mathfrak{C}(\mathfrak{H})$ for $i \neq j$, $i, j = 1, \dots, n$. In this case we can write $A = A_0 + C$ with $A_0 = (\delta_{ij}A_{ii})$ and $C \in \mathfrak{C}(\tilde{\mathfrak{H}})$. And obviously $AA^* - A^*A \in \mathfrak{C}(\tilde{\mathfrak{H}})$ in this case.

Operators of this kind have been considered recently by N. Dunford [3].

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