Dynamics of a Spinning Gas Cloud

FREEMAN J. DYSON

Communicated by ERNST BECKER

Abstract. A class of exact solutions of the inviscid hydrodynamical equations is found, representing motions of a cloud of compressible gas expanding freely in an otherwise empty space. These solutions have a symmetry-group $O(4)$ whose six generators are constants of the motion. The generators are linear combinations of components of angular momentum and of vorticity. The same symmetry occurs in the dynamics of the classical ellipsoids of Dirichlet, Dedekind and Riemann.

I. Introduction. 107 years ago there appeared three papers, under the names of Dirichlet [1], Dedekind [2] and Riemann [3], discussing the dynamics of an incompressible fluid ellipsoid held together by gravitational forces. The Dedekind and Riemann ellipsoids were then forgotten for a hundred years until Chandrasekhar [4] and Lebovitz [5] revived them. As Chandrasekhar stated in a recent historical review [6], "the reason for this total neglect must, in part, be attributed to a spectacular discovery by Poincaré which channeled all subsequent investigations along directions which appeared rich with possibilities. . . . The grand mental panorama that was thus created was so intoxicating that those who followed Poincaré were not to recover from its pursuit." However, the example of Poincaré can hardly be considered a sufficient excuse for consigning the Riemann ellipsoids to oblivion. After all, they occupy 30 pages of the single volume of Riemann's collected works [7], in which 55 pages are devoted to Riemann surfaces, 39 to Riemann integration, 20 to the Riemann invariants of compressible fluid dynamics, 16 to Riemannian geometry, and 9 to the Riemann hypothesis [8].

A central feature in the work of Dirichlet, Dedekind and Riemann is a principle of duality, first formulated explicitly by Dedekind [2]. The principle associates with every ellipsoid an adjoint ellipsoid having the same size and shape. The motions are related in such a way that the angular momentum of the original ellipsoid is equal to the vorticity of the adjoint and vice versa. The Dedekind ellipsoids, which have a fixed orientation in space, are the adjoints of the previously known Jacobi ellipsoids, which rotate without any internal motion. Riemann [3] determined the most general configuration of a spinning ellipsoid,

and made some statements concerning the stability of the motion which Chandrasekhar [4] and Lebovitz [5] have shown to be incorrect.

The present paper discusses a class of hydrodynamic motions which are in some sense a compressible analog to the incompressible Riemann ellipsoids. Instead of a sharp ellipsoidal surface, we have an unbounded Gaussian distribution of mass. Instead of gravitational forces, we have only the internal pressure forces. The neglect of gravitation is of course a serious defect, and it remains an open question whether any compressible analog to the Riemann ellipsoids including gravitation can exist.

The variety of known exact solutions of the equations of inviscid compressible hydrodynamics is so limited [9], that it seems worthwhile to place on record a new family of solutions, even if their practical utility is far from clear. These solutions were discovered in an attempt to describe the free expansion in space of an unsymmetrical mass of gas. In the original problem there was no angular momentum and no vorticity. However, the more interesting features of this class of solutions arise when angular momentum and vorticity are included. We present these rotating solutions, not as models for any particular application, but as new examples of the kind of dynamical symmetry which exists in the Riemann ellipsoids and which may perhaps exist under more general conditions.

Within our chosen class of solutions, the hydrodynamical equations possess an enlarged symmetry-group. The group is a direct product $[O(3) \times O(3)]$, where the first $O(3)$ is the usual group of space-rotations, and the second $O(3)$ is a group of internal motions of the gas. The generators of the first $O(3)$ are the operators of angular momentum, and the generators of the second $O(3)$ are operators of vorticity. All six generators are independent constants of the motion, in addition to the total energy and the linear momentum. There is also a discrete symmetry in the equations which allows the vorticity and the angular momentum to be interchanged. For example, any irrotational solution with non-zero angular momentum can be "reflected" into a rotational solution with zero angular momentum. This discrete symmetry is identical with the duality principle of Dedekind.

Since the group $[O(3) \times O(3)]$ is isomorphic to $O(4)$, it is possible to simplify the equations of motion by writing them in explicitly $O(4)$ -invariant form. The equations then resemble those of a rigid body rotating in 4-dimensional Euclidean space. The generators of the $O(4)$ group are linear combinations of the 3-dimensional angular momentum and vorticity components, and it is just these linear combinations which appear in a simple way in the equations.

Except in special cases, the equations of motion will not be completely integrable in terms of known functions. In general, our analysis only succeeds in replacing the partial differential equations of hydrodynamics by a set of nine second-order ordinary differential equations which can be integrated numerically. Apart from an arbitrary function of temperature which appears in the equation of state, our solutions depend on 18 parameters which define the configuration and velocity of the gas at one instant of time.

II. Reduction of hydrodynamic equations to a mechanical problem. We use for independent variables the time t and three Lagrangian coordinates $\mathbf{a} = (a_1, a_2, a_3)$ which remain attached to a particular element of fluid as it moves. The position and velocity of the element are $\mathbf{r} = (r_1, r_2, r_3)$ and $\mathbf{u} = \dot{\mathbf{r}} = (u_1, u_2, u_3)$, the dot denoting differentiation with respect to time holding $\mathbf{a}$ fixed. The properties of the gas are described by the scalar quantities ρ density, P pressure, T temperature, U internal energy per gram, all of which are considered to be functions of $\mathbf{a}$ and t .

There are three dynamical equations, the momentum equation

$$(1) \quad \rho \dot{u}_i = -(\partial P / \partial r_i),$$

the energy equation

$$(2) \quad \rho \dot{U} = (P/\rho) \dot{\rho},$$

and the equation of continuity

$$(3) \quad \dot{\rho} = -\rho(\partial u_i / \partial r_i).$$

In addition we need the equations of state,

$$(4) \quad P = R\rho T,$$

with R constant in space and time, and

$$(5) \quad U = U(T).$$

Eq. (5) means that we assume U to be an arbitrary function of temperature but independent of density. This assumption is not always true for real gases, but it is less restrictive than the perfect-gas assumption,

$$(6) \quad U = nRT$$

with n constant.

The special class of solutions of Eq. (1)–(5) which we shall consider are characterized by the condition that $\mathbf{r}$ is a linear function of $\mathbf{a}$,

$$(7) \quad r_i = F_{ik} a_k,$$

with 9 coefficients F_{ik} which are functions of time alone. The entire volume of gas is assumed to flow by a continuous affine transformation of the space. Straight lines in the fluid remain straight, but lengths and angles in general change with time.

We denote by F the (3×3) matrix of coefficients F_{ik} , and by F_T the transposed matrix. The derivatives then transform by the rule contragredient to Eq. (7),

$$(8) \quad (\partial / \partial r_i) = G_{ik} (\partial / \partial a_k),$$

$$(9) \quad G = [F_T]^{-1}.$$

The velocity $\mathbf{u}$ is also a linear function of $\mathbf{a}$,

$$(10) \quad u_i = \dot{F}_{ik} a_k.$$

Eq. (8) and (10) give for the divergence,

$$(11) \quad (\partial u_i / \partial r_i) = G_{ik} \dot{F}_{ik} = Sp[F^{-1} \dot{F}] = (\dot{\varphi} / \varphi),$$

where

$$(12) \quad \varphi = \text{Det}[F_{ik}]$$

is a function of t alone. The general solution of Eq. (3) is then

$$(13) \quad \rho(\mathbf{a}, t) = f(\mathbf{a})[\varphi(t)]^{-1}.$$

By virtue of Eq. (4), the energy equation (2) may be written

$$(14) \quad \dot{U} + RT(\dot{\varphi} / \varphi) = 0.$$

The left side of Eq. (14) has no explicit dependence on $\mathbf{a}$. If we assume that the gas is isothermal (T independent of $\mathbf{a}$) at some initial time, then it will remain isothermal at all later times, the temperature being a function of t alone. We confine our attention to isothermal motions. Eq. (14) may also be written

$$(15) \quad \varphi(dU/d\varphi) + RT = 0.$$

Since U is given as a function of T , Eq. (15) can be integrated to give a relation between U and φ ,

$$(16) \quad \varphi = \varphi_0 \exp \left[- \int (RT)^{-1} dU \right].$$

So from henceforth we may consider $U = U(\varphi)$ to be a known function of φ .

It remains only to satisfy the momentum equation (1). Because of the linearity of the motion expressed by Eq. (8) and (10), Eq. (1) gives

$$(17) \quad \ddot{F}_{ik} a_k = -RTG_{ik}[f^{-1}(\partial f / \partial a_k)].$$

This shows that $[\log f]$ must be a quadratic function of the a_k . Since the Lagrangian coordinates are nothing more than labels, we may without loss of generality make a linear transformation of the a_k so that $[\log f]$ reduces to a sum of squares. We have then a standard Gaussian density distribution

$$(18) \quad f(\mathbf{a}) = (2\pi)^{-3/2} \exp [-\frac{1}{2}(a_1^2 + a_2^2 + a_3^2)],$$

which describes an ellipsoidal mass-distribution moving in $\mathbf{r}$ -space according to Eq. (7). We have normalized $f(\mathbf{a})$ so that the total mass

$$(19) \quad m = \int \rho d_3 \mathbf{r} = \int f d_3 \mathbf{a}$$

is equal to unity. When Eq. (18) is substituted into Eq. (17), we find the equations of motion

$$(20) \quad \ddot{F} = RT[F_T]^{-1},$$

or

$$(21) \quad \ddot{F}_{ik} = RT\varphi^{-1}(\partial\varphi/\partial F_{ik}).$$

Now using Eq. (15) we find

$$(22) \quad \ddot{F}_{ik} + (\partial U/\partial F_{ik}) = 0,$$

where the energy U is given as a function of the F_{ik} through its known dependence on φ .

The equation of motion (22) is identical with that of a point particle in a 9-dimensional Euclidean space, the potential energy being a given function $U = U[\det F_{ik}]$ of the coordinates F_{ik} . The hydrodynamic equations are thus reduced to a problem in point mechanics.

III. Constants of the motion. From Eq. (22) we deduce immediately the energy integral

$$(23) \quad E = \frac{1}{2}\dot{F}_{ik}\dot{F}_{ik} + U = \text{constant}.$$

Also from Eq. (20) it follows that both of the matrices

$$(24) \quad J = F\dot{F}_T - \dot{F}F_T,$$

$$(25) \quad K = F_T\dot{F} - \dot{F}_TF,$$

are constant. Each of J , K is an antisymmetric matrix with 3 independent components. From Eq. (7), (10), (13) and (18) it is easy to verify that

$$(26) \quad J_{ki} = \int (r_k u_i - r_i u_k) \rho \, d_3\mathbf{r}$$

is a component of the total angular momentum of the system. The Eulerian vorticity tensor ζ has components

$$(27) \quad \zeta_{ki} = (\partial u_i/\partial r_k) - (\partial u_k/\partial r_i),$$

which are independent of position by virtue of Eq. (8)–(10). Thus

$$(28) \quad \zeta = (F_T)^{-1}\dot{F}_T - \dot{F}F^{-1},$$

and so

$$(29) \quad K = -F_T\zeta F.$$

The components of K are components of vorticity referred to the local Lagrangian coordinates $\mathbf{a}$ rather than to the Eulerian coordinates $\mathbf{r}$. Since K is a conserved quantity whereas ζ is not, we shall henceforth refer to K as the vorticity tensor.

The constancy of the 3 components of J is due to the invariance of the hydrodynamical equations under the group $O(3)$ of rotations of the Eulerian

coordinate-system (r_1, r_2, r_3) . In the same way, the constancy of K reflects the invariance of the equations under an independent group $O(3)$ of rotations of the Lagrangian coordinates (a_1, a_2, a_3) . The second $O(3)$ -invariance is of course dependent upon our special choice of ellipsoidal density-distribution given by Eq. (18). The total invariance group $[O(3) \times O(3)]$ allows independent rotations of the Eulerian and Lagrangian coordinates.

The transformation $F \rightarrow F_T$ leaves the equations of motion (21) invariant and interchanges J and K . This discrete symmetry of the theory, reversing the roles of Eulerian and Lagrangian coordinates, is the Dedekind duality principle [2]. It is also related to the symmetry between space-fixed and body-fixed coordinates in the dynamics of a rigid top.

In order to describe the motion of the gas in detail, it is convenient to write the matrix F in a canonical form

$$(30) \quad F = O_1 D O_2 ,$$

where O_1 and O_2 are orthogonal and D is diagonal. The representation (30) always exists, and is in general ambiguous only to the extent of sign-changes and permutations of the three elements (D_1, D_2, D_3) of D . Physically speaking, O_1 defines the orientation of the gas with respect to the Eulerian coordinates $\mathbf{r}$, O_2 defines the orientation with respect to the Lagrangian coordinates $\mathbf{a}$, and D defines the shape of the mass-distribution. The principal moments of inertia are in fact

$$(31) \quad I_1 = D_2^2 + D_3^2, \text{ etc.}$$

The time-derivatives of O_1 and O_2 define two angular velocity matrices A and B according to

$$(32) \quad \dot{O}_1 = -O_1 A, \quad \dot{O}_2 = B O_2 .$$

A and B are antisymmetric and so can be represented as axial vectors ω and φ by

$$(33) \quad A_{kl} = \epsilon_{jkl} \omega_j, \quad B_{kl} = \epsilon_{jkl} \varphi_j .$$

Eq. (30) gives

$$(34) \quad \dot{F} = O_1 [\dot{D} - A D + D B] O_2 ,$$

and the equation of motion (22) becomes

$$(35) \quad \ddot{D} - 2A\dot{D} + 2\dot{D}B - \dot{A}D + D\dot{B} \\ + A^2 D - 2ADB + DB^2 + [\partial U / \partial D] = 0 .$$

Here $[\partial U / \partial D]$ means the diagonal matrix whose elements are $[\partial U / \partial D_i]$, since U is a known function of

$$(36) \quad \varphi = \text{Det } F_{ik} = D_1 D_2 D_3 .$$

It is easy to separate the diagonal and non-diagonal components of Eq. (35). The diagonal components give three dynamical equations for the D_i , determining the rate of expansion of the gas-cloud under the influence of the pressure-force $[\partial U/\partial D]$. The non-diagonal components give six kinematical equations for the angular velocities ω and φ , involving only the inertial properties of the gas-cloud but not the pressure-force. We do not write out these equations explicitly, since they appear in a simpler and clearer form in the 4-dimensional notations which we shall introduce in the following section.

IV. Four-dimensional invariance. The isomorphism between $[O(3) \times O(3)]$ and $O(4)$ can be represented as a one-to-one correspondence between the real asymmetric (3×3) matrix F_{ik} and a symmetric traceless (4×4) matrix $\Phi_{\alpha\beta}$, $\alpha, \beta = 0, 1, 2, 3$. The elements of $\Phi_{\alpha\beta}$ are

$$(37) \quad \Phi_{\alpha\beta} = \frac{1}{2} \left\{ \begin{array}{cccc} F_{11} + F_{22} + F_{33} & F_{23} - F_{32} & F_{31} - F_{13} & F_{12} - F_{21} \\ F_{23} - F_{32} & F_{11} - F_{22} - F_{33} & F_{12} + F_{21} & F_{31} + F_{13} \\ F_{31} - F_{13} & F_{12} + F_{21} & F_{22} - F_{33} - F_{11} & F_{23} + F_{32} \\ F_{12} - F_{21} & F_{31} + F_{13} & F_{23} + F_{32} & F_{33} - F_{11} - F_{22} \end{array} \right\}.$$

When F undergoes a transformation

$$(38) \quad F \rightarrow O_1 F O_2,$$

with O_1 and O_2 in $O(3)$, Φ will undergo a transformation

$$(39) \quad \Phi \rightarrow \Omega_T \Phi \Omega,$$

with $\Omega = \Omega(O_1, O_2)$ an orthogonal (4×4) matrix in $O(4)$. The relation between Ω , O_1 and O_2 is simple when expressed in differential form. Suppose that O_1 and O_2 are varying with time according to Eq. (32), (33). Then Ω will vary with time according to

$$(40) \quad \dot{\Omega} = \Lambda \Omega,$$

with Λ an antisymmetric (4×4) matrix whose elements are given by

$$(41) \quad \Lambda_{\alpha\beta} = \begin{Bmatrix} 0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & 0 & \alpha_3 & -\alpha_2 \\ \beta_2 & -\alpha_3 & 0 & \alpha_1 \\ \beta_3 & \alpha_2 & -\alpha_1 & 0 \end{Bmatrix},$$

with

$$(42) \quad \alpha = \frac{1}{2}(\omega + \varphi), \quad \beta = \frac{1}{2}(\omega - \varphi).$$

When F is expressed in the canonical form (30), Φ takes the form

$$(43) \quad \Phi = \Omega_T \Delta \Omega,$$

where Δ is a diagonal (4×4) matrix with the elements

$$(44) \quad \Delta_0 = \frac{1}{2}(D_1 + D_2 + D_3), \quad \Delta_j = D_j - \Delta_0, \quad j = 1, 2, 3.$$

The 4-dimensional angular-velocity matrix Λ is then defined by Eq. (40) and (41). If we introduce the antisymmetric matrix Γ by

$$(45) \quad \Gamma_{\alpha\beta} = \Delta_\alpha - \Delta_\beta,$$

or

$$(46) \quad \Gamma_{\alpha\beta} = \begin{Bmatrix} 0 & D_2 + D_3 & D_3 + D_1 & D_1 + D_2 \\ -D_2 - D_3 & 0 & D_1 - D_2 & D_1 - D_3 \\ -D_3 - D_1 & D_2 - D_1 & 0 & D_2 - D_3 \\ -D_1 - D_2 & D_3 - D_1 & D_3 - D_2 & 0 \end{Bmatrix},$$

the equations of motion (22) become

$$(47) \quad \ddot{\Delta}_\alpha - 2 \sum_\beta \Gamma_{\alpha\beta} \Lambda_{\alpha\beta}^2 + (\partial U / \partial \Delta_\alpha) = 0,$$

$$(48) \quad (d/dt)[\Gamma_{\alpha\beta}^2 \Lambda_{\alpha\beta}] + \Gamma_{\alpha\beta} \sum_\gamma (\Gamma_{\alpha\gamma} - \Gamma_{\gamma\beta}) \Lambda_{\alpha\gamma} \Lambda_{\gamma\beta} = 0.$$

Eq. (48) are the six kinematical equations which determine the behavior of the angular velocities α and β when the inertial coefficients D_i are given. Eq. (47) are the three dynamical equations determining the time-development of the D_i . Since U is given explicitly as a function of $\varphi = D_1 D_2 D_3$, it is more convenient to write Eq. (47) in the form

$$(49) \quad \ddot{D}_i - 2 \sum_{\alpha=0,j} \sum_{\beta=k,l} \Gamma_{\alpha\beta} \Lambda_{\alpha\beta}^2 + (\partial U / \partial D_i) = 0.$$

In the 4-dimensional notation we have also a unified representation of the angular momentum and vorticity operators J and K . The 4-dimensional angular momentum matrix is

$$(50) \quad \Psi = \Phi \dot{\Phi} - \dot{\Phi} \Phi.$$

The 6 components of Ψ are constants of the motion by virtue of Eq. (22). If we write Ψ in the canonical form

$$(51) \quad \Psi = \Omega_T M \Omega,$$

then the components of M are the quantities

$$(52) \quad M_{\alpha\beta} = \Gamma_{\alpha\beta}^2 \Lambda_{\alpha\beta},$$

whose time-derivatives are given by Eq. (48). Now J and K can also be written in canonical form as

$$(53) \quad J = O_1 \hat{J} O_{1T}, \quad K = O_{2T} \hat{K} O_2,$$

and we find from Eq. (24), (25) and (34)

$$(54) \quad \hat{J}_{ij} = M_{ij} - M_{0k}, \quad \hat{K}_{ij} = M_{ij} + M_{0k}.$$

But Ψ , J and K are constants of the motion, and therefore the same linear relation must hold between Ψ , J , K as between M , $\hat{J}$, $\hat{K}$; we could suppose that at some time t_0 we have instantaneously $\Omega = O_1 = O_2 = I$, and then the relation between Ψ , J , K established at $t = t_0$ will be preserved for all t . Therefore the components of the 4-dimensional angular momentum are

$$(55) \quad \Psi_{\alpha\beta} = \frac{1}{2} \begin{pmatrix} 0 & K_{23} - J_{23} & K_{31} - J_{31} & K_{12} - J_{12} \\ J_{23} - K_{23} & 0 & J_{12} + K_{12} & J_{13} + K_{13} \\ J_{31} - K_{31} & J_{21} + K_{21} & 0 & J_{23} + K_{23} \\ J_{12} - K_{12} & J_{31} + K_{31} & J_{32} + K_{32} & 0 \end{pmatrix}.$$

Eq. (55) is the standard group-theoretical relation between the generators of $O(4)$ and $[O(3) \times O(3)]$. This derivation of Eq. (55) just verifies that our definition (50) of the 4-dimensional angular momentum is consistent with the usual conventions of group theory. In terms of 4-dimensional quantities, the total energy E is

$$(56) \quad \begin{aligned} E &= \frac{1}{2} \sum_{\alpha,\beta} \dot{\Phi}_{\alpha\beta} \dot{\Phi}_{\alpha\beta} + U \\ &= \frac{1}{2} \sum_{\alpha} \dot{\Delta}_{\alpha}^2 + \frac{1}{2} \sum_{\alpha,\beta} \Gamma_{\alpha\beta}^2 \Lambda_{\alpha\beta}^2 + U, \end{aligned}$$

showing the precise analogy between the dynamics of our system and that of a 4-dimensional top.

The Dedekind symmetry which interchanges J with K , or F with F_T , appears in the 4-dimensional picture as a reversal in the signs of the $(0j)$ and $(j0)$ elements of the matrices Φ , Λ and Ψ . This symmetry is therefore a reflection of the 4-dimensional space in the 0-hyperplane. The total symmetry-group is $O(4)$ augmented by reflections, *i.e.* the group of all orthogonal (4×4) matrices with determinant either $+1$ or -1 .

V. Special solutions. An immediate corollary of Eq. (50) is the following statement. In order that a fixed pair of Lagrangian and Eulerian coordinate systems exist such that

$$(57) \quad r_i = D_i(t) a_i, \quad j = 1, 2, 3,$$

it is necessary and sufficient that

$$(58) \quad J = K = 0.$$

The necessity is obvious from the definitions (24) and (25). To prove sufficiency, suppose that Eq. (58) holds. Then $\Psi = 0$, and Eq. (50) shows that Φ and $\dot{\Phi}$ commute. At a given time t_0 we can find a coordinate system in which both Φ and $\dot{\Phi}$ are instantaneously diagonal. But then the equations of motion (22) imply that Φ remains diagonal in the same coordinate system for all t . In this coordinate system F is also diagonal and so Eq. (57) holds.

The motions of the special form (57), having zero angular momentum and zero vorticity, describe the free expansion of an ellipsoidal gas-cloud with fixed orientation. This is the problem from which our investigation started. As a very special example, suppose that we are dealing with a perfect gas satisfying Eq. (6) with $n = [\gamma - 1]^{-1}$, where γ is the ratio of specific heats, and suppose that the cloud begins its expansion from rest at $t = 0$ with the spheroidal distribution

$$(59) \quad D_1 = D_2 = a, \quad D_3 = b.$$

The equations of motion (22) are

$$(60) \quad \begin{aligned} \ddot{D}_1 &= 2 D_1 D_3 U_0 (\gamma - 1) (D_1^2 D_3)^{-\gamma}, \\ \ddot{D}_3 &= D_1^2 U_0 (\gamma - 1) (D_1^2 D_3)^{-\gamma}. \end{aligned}$$

These equations are easy to integrate numerically and give for large t the behavior

$$(61) \quad D_1 = D_2 \sim a't, \quad D_3 \sim b't,$$

where a', b' are constants related to a and b . The final anisotropy-ratio (a'/b') will depend only on the initial anisotropy (a/b) and on the gas-constant γ . A cloud which is initially cigar-shaped will expand into a disc-shape, and vice versa.

Acknowledgements. I would like to thank the General Atomic Division of General Dynamics Corporation for hospitality during the year 1958–59. The initial impetus for this work arose out of U. S. Air Force Contract AF29(601)-6214, under which I was then working at General Atomic.

I am also grateful to Professor Ernst Becker of Darmstadt for correcting some errors in an earlier version of this paper, and to Professor Chandrasekhar of Chicago for pointing out the connection with the classical work of Dirichlet, Dedekind and Riemann.

REFERENCES

- [1] G. L. DIRICHLET, *J. Reine Angew. Math.*, **58** (1860) 181.
- [2] R. DEDEKIND, *J. Reine Angew. Math.*, **58** (1860) 217.
- [3] B. RIEMANN, *Abh. Königl. Ges. Wiss. Göttingen* **9** (1861) 3.
- [4] S. CHANDRASEKHAR, *Ap. J.*, **141** (1965) 1043; **142** (1965) 890; **145** (1966) 842.
- [5] N. R. LEBOVITZ, *Ap. J.*, **145** (1966) 878. I am grateful to Dr. Chandrasekhar for sending me in advance of publication a chapter of a forthcoming book by him, in which his work on the stability of fluid ellipsoids is collected and perfected.
- [6] S. CHANDRASEKHAR, *Comm. Pure Appl. Math.*, **20** (1967) 251.
- [7] BERNHARD RIEMANN's *Gesammelte Mathematische Werke und Wissenschaftlicher Nachlass*, ed. H. Weber, B. G. Teubner, Leipzig, 1876.
- [8] Here I have merely counted the number of pages in each of the five papers in which Riemann announced one of his fundamental discoveries. In no case did he publish again in the same field.

- [9] I have been able to find in the literature only one set of solutions which overlap those presented here. H. Grad, *Comm. Pure Appl. Math.*, **2** (1949) 331, Appendix 2, obtains all solutions of the Boltzmann equation for a rarified gas in which the velocity distribution of particles at any point in space is Maxwellian. In the continuum limit these become solutions of the hydrodynamic equations, and some of them then coincide with some of mine. For this reference I am indebted to private communications from H. Grad and A. Lenard.

Institute for Advanced Study
Princeton, New Jersey
Communication date: DECEMBER 21, 1967