Fiber Bundles with Singularities

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1. Introduction. Let M and N be connected topological spaces. A *singular fibering* is a map $f: (M, A_f) \rightarrow (N, f(A_f))$ such that

- (1) $A_f \neq M$ and is closed;
- (2) $f|A_f$ is a homeomorphism and $f^{-1}(f(A_f)) = A_f$;
- (3) for each component W of $N - f(A_f)$, $f|f^{-1}(W)$ is the projection map of a fiber bundle with base space W (and possibly empty fiber). Moreover, we may suppose that
- (4) A_f is a minimal set satisfying (1), (2), and (3).

This concept is a generalization of one introduced by Montgomery and Samelson [20], and related versions have been studied by Conner and Dyer [11], Mahowald [19], and Antonelli [1]. Other concepts of singular fibering have been investigated in [3], [15], [22], [26], [31], [32], and [33]. For a description of the relation between [32] and this paper, see (1.8).

Given maps $\psi: P \rightarrow Q$ and $\omega: R \rightarrow S$, define $\psi \times \omega: P \times R \rightarrow Q \times S$ by $\psi \times \omega(p, r) = (\psi(p), \omega(r))$. Define the open *cone* $c(P)$ as the identification space obtained from $P \times [0, 1)$ by identifying $P \times \{0\}$ to a point p^* , let ι be the identify map on $[0, 1)$, and let the *cone map* $c(\psi): c(P) \rightarrow c(Q)$ be the map induced by $\psi \times \iota$. Let ι_k be the identity map on E^k .

1.1. Examples. In each of (i), (ii), and (iii) below a map ψ is defined, and $c(\psi) \times \iota_k$ is a singular fibering with singular set A_f , a k -cell ($k = 0, 1, \dots$).

(i) Let $m = 2, 4$, or 8 , and let $\psi: S^{2m-1} \rightarrow S^m$ be a locally trivial fiber map with fiber S^{m-1} (or a homotopy 3-sphere in case $m = 4$); examples are the Hopf maps [28; p. 105–110].

(ii) Let T^m be a homotopy sphere such that $c(T^m)$ is an open $(m + 1)$ -manifold, and let $\psi: T^m \rightarrow E^0$ ($m = 0, 1, \dots$).

(iii) Let $\psi: S^1 \rightarrow S^1$ be a d -to-1 covering map with $d > 1$, e.g., $\psi(\theta) = d \cdot \theta \pmod{2\pi}$.

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1.2. Definitions. Consider a map $f : M \rightarrow N$. It is *proper* if, for each compact set $K \subset N$, $f^{-1}(K)$ is compact. It is *topologically equivalent* to $g : X \rightarrow Y$ if there exist homeomorphisms α of M onto X and β of N onto Y such that $\beta f \alpha^{-1} = g$. At $x \in M$, f is *locally topologically equivalent* to g if, for every neighborhood $W \subset M$ of x , there exist neighborhoods $U \subset W$ of x and V of $f(x)$ such that $f|U : U \rightarrow V$ is topologically equivalent to g .

The map $c(\psi) = c(\psi) \times \iota_0$ of (iii) is topologically equivalent to the complex analytic function defined by $g(z) = z^d$. Let $j : E^0 \times [0, 1] \times E^k \rightarrow E^{k+1}$ be the natural embedding; in the sequel we will not distinguish between $c(\psi) \times \iota_k$ in (ii) and $j \circ (c(\psi) \times \iota_k)$. Throughout this paper a symbol such as M^n denotes a separable n -manifold (without boundary, unless otherwise specified).

1.3. Theorem. Let $f : M^n \rightarrow N^p$ be a C^n singular fibering with $A_f \neq \emptyset$. (a) Then there is a closed set $Y \subset A_f$ such that $\dim Y < \dim A_f$ and, at each point x of $A_f - Y$, f is locally topologically equivalent to one of the maps $c(\psi) \times \iota_k$ of (1.1). (b) If, in addition, f is proper and onto, then only the maps of (1.1) (i) and (iii) occur.

In particular, if $q = \dim A_f$ ($q = 0, 1, \dots$), then the possible values of (n, p) are: $(q+4, q+3)$, $(q+8, q+5)$, $(q+16, q+9)$, $(n, q+1)$ with $n \geq q+1$, and $(q+2, q+2)$. If f is proper and onto, the complements of $f(A_f)$ and $f(Y)$ have nice connectivity properties (4.6), and, if in addition $n > p$, then A_f is a Z_2 -generalized manifold (4.8). See also (4.9).

1.4. Definition. Given a map $f : M \rightarrow N$ and $x \in M$, let F be the component of $f^{-1}(f(x))$ containing x . The *singular set* A_f is defined as follows: $x \in M - A_f$ if and only if there are neighborhoods U of F and V of $f(x)$ such that $f|U : U \rightarrow V$ is topologically equivalent to the projection map $\pi : V \times F \rightarrow V$. The set A_f is closed.

Suppose that $f : M^n \rightarrow N^p$ is proper. Then $f(A_f)$ is also closed, and, for any component W of $N^p - f(A_f)$ either $f^{-1}(W) = \emptyset$ or $f|f^{-1}(W) : f^{-1}(W) \rightarrow W$ is the projection map of a fiber bundle [32; (2.4)]. If f is C^1 , then (3.6) no regular value is in $f(A_f)$. For a singular fibering the two notions of singular set coincide.

1.5. Theorem. Let $f : M^n \rightarrow N^p$ be a C^n map with $\dim A_f \leq 0$, $n \neq 0$, and $(n, p) \neq (1, 1)$. Then at each point of A_f the map f is locally topologically equivalent to one of the maps $c(\psi) \times \iota_0$ of (1.1).

In particular, A_f is discrete, and, if $A_f \neq \emptyset$, then $(n, p) = (n, 1)$ with $n > 1$, $(2, 2)$, $(4, 3)$, $(8, 5)$, or $(16, 9)$.

In case $n = p = 1$, every point at which the derivative changes sign is in A_f ; thus the examples are elementary but messy. For A_f discrete Timourian [32; (1.6)] proved (1.5) without a differentiability hypothesis, and our proof uses his result.

1.6. Remark. Without some differentiability hypothesis (1.3) is false. There is a space P such that $P \times E^1$ is homeomorphic to E^4 , while P is not locally a manifold at any point [24]. Let ι be the identity map on P , let ψ be one of the

maps of (1.1), and let $f = c(\psi) \times \iota$; then f is a singular fibering with $A_f = P$, and its domain and range (except in case (ii)) are manifolds. (In case (iii) this example was described in [10; p. 620, (4.3)].)

While it is not true that P is a manifold except for a set of dimension less than $\dim P$, P (and any set P with $P \times E^m$ homeomorphic to E^n) is a generalized manifold [23; p. 17]. Perhaps a topological version of (1.3) is true if we expand the list (1.1) to include the above example for various different P . See (4.8).

1.7. Remark. The hypotheses in (1.3) and (1.5) are differential, while the conclusions are topological; it is natural to ask whether a differential conclusion is possible.

The maps $c(\psi) : K^n \rightarrow L^p$ of (1.1) are each C^∞ except at one point x of K^n , and it follows from the argument of [6; p. 95, (3.3)] that there is a C^∞ homeomorphism $h : L^p \rightarrow L^p$ such that $h \circ c(\psi)$ is C^∞ and the restriction map $h|_{(L^p - c(\psi)(x))}$ is a C^∞ diffeomorphism. Thus we may suppose that the maps $c(\psi) \times \iota_k$ of (1.1) are C^∞ .

We will say that a C^m map f in (1.2) is locally C^m diffeomorphically equivalent to a C^m map g if the maps α and β are C^m diffeomorphisms. Except at the points of Y (respectively, $f^{-1}(Y_f)$), is f in (1.3) (resp., (4.1)) locally diffeomorphically equivalent to one of the maps $c(\psi) \times \iota_k$ of (1.1)?

Let $\zeta : M^n \rightarrow N^p$ be any one of the maps $c(\psi) \times \iota_k$ of (1.1) with $k > 0$, let $y \in A_\zeta$, and let $\eta : M^n \rightarrow M^n$ be a C^∞ homeomorphism such that $\eta(y) = y$, $\eta|_{(M^n - \{y\})}$ is a C^∞ diffeomorphism, and the rank of the Jacobian matrix of η at y is 0. Since the rank of ζ at y is k , $\zeta \circ \eta$ is not locally C^1 diffeomorphically equivalent at y to ζ . From considerations of the dimensions n , p , and $\dim A_\zeta$, it could not be any of the other maps of (1.1), so the answer to our question is negative.

1.8. Relation of this paper to [32] and outline of the proofs. In [32] Timourian proves several lemmas, a theorem [32; (1.6)], and two propositions which we now use. While our Lemma (2.3) is a generalization of Timourian's main theorem [32; (1.6)], that theorem and several of its lemmas are used in proving the generalization. Specifically, the hypothesis in [32; (1.6)] is that A_f is discrete, and the weaker hypothesis in our (2.3) is that $\dim A_f \leq 0$ (together with other hypotheses). In the proof of (2.3) the spectral sequence of a fiber space (with twisted coefficients) is used to deduce that A_f is discrete, and the conclusion then follows from [32; (1.6)].

In Section 3 technical differential lemmas, most of them closely related to Thom's Transversality Theorem, are proved for use in Section 4. In Section 4 we prove *in effect* that if f satisfies all the hypotheses of (1.5), including the differentiability hypothesis, then it satisfies the additional hypotheses of (2.3), and the desired conclusion results.

Actually the situation is more complicated. We consider a lemma (4.1) sufficiently general to include (1.3)(a) and (1.5), and we prove (4.1) and another lemma (4.2) by simultaneous induction.

1.9. Remark. Let $f : M^n \rightarrow N^p$ be C^r where $r \geq \max(n - p + 1, 1)$, let $x \in M^n - A_f$, and let U , V , and F be as given in (1.4) the definition of A_f . There is [25] a regular value y of f in V , and $f^{-1}(y)$ is a $C^r(n - p)$ -manifold [30; p. 23]; thus F is an $(n - p)$ -manifold. In case f is not necessarily $C^{\max(n-p+1, 1)}$ (e.g., in (2.3)) we add to the definition of A_f the condition that F is a manifold.

1.10. Remarks. If $f : M^n \rightarrow N^p$ with $A_f \neq M^n$, then $n \geq p$. (Some subset of M^n is homeomorphic (1.4) to $V \times F$ where $V \subset N^p$ is open, and we apply [16; p. 33].) If $n = p$, then (see (1.9)) A_f is the set of points at which f fails to be a local homeomorphism.

1.11. Remark. Let $f : E^n \rightarrow E^n$ be C^1 with $\dim A_f \leq n - 2$. Then its Jacobian determinant J is either nonnegative or nonpositive.

Proof. See (1.10). Let E_+ (resp., E_-) be the set of points $x \in E^n - A_f$ such that for some open n -cell U about x , f maps U homeomorphically onto $f(U)$ positive (resp., negative) orientation. Then E_+ and E_- are open and disjoint, and $E_+ \cup E_- = E^n - A_f$; since the latter space is connected, either E_+ or E_- is empty.

1.12. Definitions. A map $f : X \rightarrow Y$ is *light* if, for each $y \in Y$, $f^{-1}(y)$ is totally disconnected; it is *monotone* if each $f^{-1}(y)$ is connected. It is *quasi-monotone* if, for each connected open set $U \subset Y$ and component V of $f^{-1}(U)$, $f(V) = U$ [36; p. 151]. If $f : M^n \rightarrow N^p$ is C^1 , let $R_q(f)$ be the set of points $x \in M^n$ at which the Jacobian matrix (derivative map) of f has rank at most q .

1.13. Remarks. Let $f : M^n \rightarrow N^p$ be proper, and let $A \subset M^n$ be closed. Then $f(A)$ is closed. If f is light, then [16; pp. 91-92] $\dim A \leq \dim f(A)$.

1.14. Lemma. Let M and N be locally compact, locally connected metric spaces and let $f : M \rightarrow N$ be continuous with $f|A_f$ light. Then for any $x \in A_f$, there are connected open neighborhoods $U \subset M$ of x and $V \subset N$ of $f(x)$ such that $f|U : U \rightarrow V$ is proper.

Proof. If F_x is the component of $f^{-1}(f(x))$ containing x , then $F_x \subset A_f$ also; since $f|A_f$ is light, $F_x = \{x\}$. Let X be an open neighborhood of x with $\bar{X}$ compact, let V_k be the component of $S(f(x), 1/k)$ containing $f(x)$, and let U_k be the component of $X \cap f^{-1}(V_k)$ containing x ($k = 1, 2, \dots$).

Suppose that each $\bar{U}_k$ meets $\text{bdy } X$. Since $\bigcap_k \bar{U}_k$ is a compact connected subset of $f^{-1}(f(x))$ [36; p. 15, (9.4)] and it meets $\text{bdy } X$, a contradiction results. Thus for some k , $\bar{U}_k \subset X$; let $U = U_k$ and $V = V_k$.

Let $K \subset V$ be any compact set. For any $y \in \bar{U} \cap f^{-1}(K)$ there is a connected open neighborhood W of y such that $\bar{W} \subset X$ and $f(W) \subset V$; thus $W \subset X \cap f^{-1}(V)$, and since it meets U , $W \subset U$. As a result $\bar{U} \cap f^{-1}(K) = U \cap f^{-1}(K)$, implying that the latter set is compact. Since K is arbitrary, $f|U$ is proper.

1.15. Conventions. The symbol H refers to singular homology or cohomology, unless otherwise stated, $\tilde{H}$ refers to reduced singular homology or co-

homology, and $\tilde{H}$ to Alexander cohomology. Since all spaces considered in this paper are paracompact, $\tilde{H}$ agrees [27; p. 334] with Čech cohomology. The group of integers is denoted by Z , and the field of order 2 by Z_2 .

The boundary of a set X is denoted by $\text{bdy } X$ or ∂X (in case X is a manifold), the interior of X by $\text{int } X$, and the closure of X by $\bar{X}$ or $Cl[X]$. A map is a continuous function, the restriction of the function f to X is denoted by $f|X$, and the composition of two functions is denoted by gf or $g \circ f$. The distance between two points is $d(x, y)$, and $S(x, \epsilon) = \{y: d(x, y) < \epsilon\}$. Euclidean n -space is denoted by E^n , its origin by 0 , the closed ball $Cl[S(0, 1)]$ in E^n by D^n , and the sphere ∂D^n by S^{n-1} .

2. Topological results.

2.1. Definition. A space Y is *locally polyhedral* if each $y \in Y$ has a neighborhood homeomorphic to a finite polyhedron.

2.2. Lemma. Let (Y, L) be a compact pair of topological spaces, with Y and $L \neq \emptyset$ locally polyhedral and $Y - L$ an n -manifold; let A be a closed subset of $Y - L$ with $\dim A \leq r$; and let G be a finite dimensional vector space over Z_2 . Then $H^q(Y - (A \cup L); G)$ is finite ($q = 0, 1, \dots, n - r - 2$).

Proof. Let Y^* be the space obtained from Y by identifying L to a point ∞ , i.e., Y^* is the one point compactification of $Y - L$. From [27; p. 243, Theorem 3] $H^q(Y - (A \cup L); G)$ is the dual space of $H_q(Y - (A \cup L); G) \approx H_q(Y^* - (A \cup \{\infty\}); G)$, which by the Lefschetz Duality Theorem [27; p. 297] is isomorphic to the Alexander cohomology group $\tilde{H}^{n-q}(Y^*, (A \cup \{\infty\}); G)$. From the cohomology sequence of the triple $(Y^*, A \cup \{\infty\}, \{\infty\})$ [12; p. 29] and [16; p. 137, (F)] it follows that the latter group is isomorphic to $\tilde{H}^{n-q}(Y^*, \{\infty\}; G)$, which is isomorphic to $\tilde{H}^{n-q}(Y, L; G)$ [12; p. 266]. This group is isomorphic [27; p. 341, Corollary 6] to the singular group $H^{n-q}(Y, L; G)$. From [27; p. 342, Theorem 11 and p. 243, Theorem 3] $H^i(Y; G)$ and $H^i(L; G)$ are finite for all i . Thus, from the cohomology sequence of (Y, L) , $H^{n-q}(Y, L; G)$ is finite, yielding the desired conclusion.

2.3. Lemma. Let $f: M^n \rightarrow N^p$ be a proper map with $n \neq 0$, $(n, p) \neq (1, 1)$, and $\dim A_f = \dim f(A_f) \leq 0$. If $p \geq 3$ suppose in addition that for each $y \in f(A_f)$ and neighborhood W of y , there is an open p -cell U such that $y \in U \subset W$ and $U - f(A_f)$ is simply connected. Then at each point of A_f , f is locally topologically equivalent to one of the maps $c(\psi) \times \iota_0$ of (1.1).

In the special case that A_f is discrete the lemma was proved by Timourian in [32; (1.6)]. By (1.10) $n \geq p$. See (1.9) and (1.13).

Proof. We may suppose that N^p is connected, that $p \geq 2$ [32; (4.1)], and that A_f is not discrete; we will obtain a contradiction.

By [32; (2.5)] f is quasi-monotone; in fact there is a natural number k such that for each $y \in N^p - f(A_f)$, $f^{-1}(y)$ has exactly k components, and for each

$y \in N^p$, $f^{-1}(y)$ has at most k components. If $p = 2$, let $x \in A_f$ be such that $f^{-1}(f(x))$ has a maximal number j of components; if $p \geq 3$, let x be a limit point of A_f , and let j be the number of components of $f^{-1}(f(x))$. In each case choose an open p -cell W about $f(x)$ sufficiently small that $f^{-1}(W)$ has at least j components W_i ($i = 1, 2, \dots$), where $x \in W_1$.

If $p = 2$, let $U = W$; if $p \geq 3$, let U be the open p -cell given by the hypothesis for W and $f(x)$. Let $U_i = f^{-1}(U) \cap W_i$; since f is quasimonotone, $f(U_i) = U$, so there are exactly j sets U_i , each is connected, and each contains exactly one component of $f^{-1}(f(x))$. Let $e : U_1 \rightarrow U$ be $f|_{U_1}$. If $p = 2$ and x is an isolated point of A_f , we may suppose that W is chosen sufficiently small that $W_1 \cap A_f = \{x\}$; then e is a proper map with $A_* = \{x\}$, contradicting [32; (1.6)] if $n > 2$. Thus, if $n > p$, we may suppose that x is a limit point of A_* .

By the monotone light factorization theorem [36; p. 153, (8.4)] $e = hg$, where $g : U_1 \rightarrow X$ is monotone (onto), and $h : X \rightarrow U$ is light. In fact [32; (2.4)] $g|_{(U_1 - A_*)}$ is the projection map of a fiber bundle ξ with fiber a compact connected manifold F^{n-p} , and $h|_{(X - g(A_*))}$ is a k -to-1 covering map onto $U - e(A_*)$. In case $p \geq 3$ $U - f(A_f)$ is simply connected, so that $U - e(A_*)$ is also; hence [27; p. 81, Corollary 7] $k = 1$, so that h is a homeomorphism. Thus we may suppose that $X = U$ and $g = e$ if $p \geq 3$. If $p = 2$ it follows from the maximality property of x that $A_* = e^{-1}(e(A_*))$ and that e maps A_* homeomorphically onto $e(A_*)$, while if $p \geq 3$ the same conclusion is immediate from the monotonicity of e .

If $n = p$, then f is light; since it is quasi-monotone, it is open [36; p. 152, (8.2)]. For $n = p = 2$, the conclusion follows from [36; p. 198, (5.1)] applied to f . For $n = p \geq 3$, e is monotone and light, and so is a homeomorphism, contradicting the choice of x . Thus we may suppose that $n > p \geq 2$.

For any triangulation σ (lower case Greek letter) of U , let Σ (corresponding upper case Greek letter) be the union of those closed p -simplices of σ each of which meets $e(A_*)$; thus $e(A_*) \cap \text{bdy } \Sigma = \emptyset$. Since the 0-dimensional set $e(A_*)$ is closed (1.13), there exists a σ sufficiently fine that the component Σ' of Σ containing $f(x)$ is compact. Suppose that $\Sigma' \cap e(A_*)$ has (at least) t distinct points y_s , and let $V_s \subset \text{int } \Sigma'$ be mutually disjoint open p -cell neighborhoods of y_s ($s = 1, 2, \dots, t$). For any subdivision τ of σ , let T_s be the closure of the component of $\text{int } T$ containing y_s ; we may suppose that τ is sufficiently fine that each $T_s \subset V_s$, and so they are disjoint. Moreover there is a triangulation ν of $(\text{int } \Sigma') - e(A_*)$ which on $\bigcup \text{bdy } T_s$ is a subdivision of τ (use the italicized statement in [35; p. 318]). Let X' be $h^{-1}(\text{int } \Sigma')$; since $h|_{(X' - g(A_*))}$ is a k -to-1 covering map, ν induces a triangulation ϕ of $X' - g(A_*)$.

Since $\text{bdy } T_s$ is the carrier of a unique simplicial $(p-1)$ -cycle of $(\text{int } \Sigma') - e(A_*)$ (i.e. of ν) with $\mathbb{Z}_2$ coefficients, and since $h|_{(X' - g(A_*))}$ is a covering map, $h^{-1}(\text{bdy } T_s)$ is also the carrier of a cycle γ_s of $X' - g(A_*)$ (i.e. of ϕ). Now $h^{-1}(y_s)$ consists of a single point, since $A_* = e^{-1}(e(A_*))$, and h is quasimonotone, so that $\text{int } h^{-1}(T_s) = h^{-1}(\text{int } T_s)$ is connected. Let γ be any (nonzero) sum of the cycles γ_s , let Γ be the union of $h^{-1}(T_s) - g(A_*)$ for γ_s in γ , and let Δ be the

closure (in $X' - g(A_*)$) of a component of $X' - (g(A_*) \cup \Gamma)$ such that Δ meets both $\text{bdy } X'$ and Γ ; let Δ be a $(p - 1)$ -simplex of $\Gamma \cap \Delta$.

We now prove that γ is nonbounding—in fact, that there is no simplicial p -chain μ of the p -manifold $X' - g(A_*)$ such that the carrier of $\partial\mu$ contains Δ and is contained in the carrier of γ . Suppose that there were; then the carrier of μ contains a p -simplex either of Γ or of Δ . Since each component of $\bar{\Gamma}$ meets $g(A_*)$ and since Δ meets $\text{bdy } X'$, each component of Γ and Δ has an infinite number of p -simplices. Since simplicial homology uses finite chains, the carrier of $\partial\mu$ meets either $\text{int } \Gamma$ or $\text{int } \Delta$, yielding a contradiction of the choice of μ .

Thus the γ_* represent independent homology elements, so that ([12; p. 201], [27; p. 243, Theorem 3]) the singular cohomology group $H^{p-1}(X' - g(A_*); Z_2)$ has dimension at least t . Since x is a limit point of A_* , $X' \cap g(A_*)$ is infinite and $H^{p-1}(X' - g(A_*); Z_2)$ is also infinite.

Let U' be the manifold $g^{-1}(X')$. The cohomology spectral sequence [14; pp. 259–272 and p. 292] of the bundle ξ' with projection map $g: U' - A \rightarrow X' - g(A_*)$ has

$$E_2^{s,t} \approx H^s(X' - g(A_*); H^t(F; Z_2)),$$

where local coefficients are used; see [2; pp. 18–19] and (for the case of finite cell complexes) [28; §§30–31].

Since $A_* \cap \text{bdy } U' = \emptyset$, $e|_{\text{bdy } U'}$ is the projection map of a bundle with base space the polyhedron $\text{bdy } \Sigma'$ and fiber a manifold; thus $\text{bdy } U'$ is locally polyhedral. In (2.2) let $Y = \bar{U}'$, $L = \text{bdy } U'$, $A = U' \cap A_*$, $r = 0$ and $G = Z_2$; since $H^{p-1}(U' - A_*; Z_2)$ is finite, its subspace $E_\infty^{p-1,0}$ is also finite.

Since F is connected, $E_2^{p-1,0}$ has constant coefficients Z_2 , so that $E_2^{p-1,0}$ is (as we have seen) infinite.

Suppose that $p = 2$. From [2; p. 19, (2)] and the fact that [2; p. 25, (2)] is a resolution in our case, it follows [13; p. 36, (2.12.1)] that $E_2^{s,t}$ is the Čech sheaf cohomology group [13; pp. 25–27]. For any triangulation ω of $X' - g(A_*)$, the stars of the vertices form an open cover; by [35; p. 317] the set of such open covers is cofinal in the set of all open covers of $X' - g(A_*)$, and so is sufficient in defining the Čech cohomology group. Since $X' - g(A_*)$ is a 2-manifold, 0 is the only cochain unless $s = 0, 1$, or 2 , so that $E_2^{s,t} = 0$ except in these cases. Since $X' - g(A_*)$ is also nonclosed, an elementary geometric argument shows that the simplicial group $H^2(X' - g(A_*); Z_2) = 0$, and the same argument extends to the case of twisted coefficients to yield $E_2^{2,t} = 0$. As a result $E_2^{1,0} \approx E_\infty^{1,0}$, so that $E_\infty^{1,0}$ is infinite, yielding a contradiction.

Now suppose that $p \geq 3$. Then ξ' is a restriction of the bundle ξ (the bundle with projection map $g|_{(U_1 - A_*)}$), so that its bundle of coefficients $\mathfrak{B}(\xi')$ with fiber $H^t(F; Z_2)$ [28; pp. 151–4] is the restriction of $\mathfrak{B}(\xi)$. Since $X - g(A_*)$ is simply connected, the covering space $\mathfrak{B}(\xi)$ is a product bundle; as a result $\mathfrak{B}(\xi')$ is a product bundle, i.e., $E_2^{s,t}$ has constant coefficients.

Since $\text{bdy } X' = \text{bdy } \Sigma'$ in this case, it is locally polyhedral; from (2.2) with $Y = \bar{X}'$, $L = \text{bdy } X'$, $A = X' \cap g(A_*)$, and $r = 0$, it follows that $E_2^{s,t}$ is finite

for $s = 0, 1, \dots, p - 2$. Since $X' - g(A_s)$ is an open p -manifold, $E_2^{s,t} = 0$ if $s \geq p$. As a result $E_m^{p-1,0}$ is infinite ($m = 2, 3, \dots$), and since $E_p^{s,t} = E_\infty^{s,t}$, a contradiction results.

In case $p = 2$ the above proof can be simplified somewhat, and in case $p \geq 3$ there is an easier proof using the fact that x has arbitrarily small n -cell neighborhoods and the naturality of the exact sequence [14; p. 240] which arises since the spectral sequence of the bundle has only two non-zero rows. The above proof, however, is simpler than the two separate proofs.

The simple connectivity hypothesis is used twice: (1) at the beginning to show that $A_s = e^{-1}(e(A_s))$, so that the latter has dimension 0; (2) at the end to show that $E_2^{s,t}$ has constant coefficients ($s = 1, 2, \dots, p - 3$), so that these $E_2^{s,t}$ are finite. If instead of the simple connectivity hypothesis we suppose that f is a singular fibering, then (1) is true. In this case, however, $E_2^{s,t}$ has twisted coefficients, and it is not clear to the authors whether or not it is finite in the desired cases.

3. Differential lemmas.

3.1. Definition. A map $g: K^{n-p+m} \times E^{p-m} \rightarrow L^m \times E^{p-m}$ is called a *layer map* if for each $t \in E^{p-m}$ $g(K^{n-p+m} \times \{t\}) \subset L^m \times \{t\}$. The restriction map

$$g|_{(K^{n-p+m} \times \{t\})}: K^{n-p+m} \times \{t\} \rightarrow L^m \times \{t\}$$

is denoted by g_t , and its singular set by $A(g_t)$.

3.2. If g is a layer map, then $\bigcup_t A(g_t) \subset A_g$; if, in addition, $n > p$ and $g|_{A_g}$ is light, then $\bigcup_t A(g_t) = A_g$.

Proof. Let $x \in K^{n-p+m} \times \{t\}$, and let F be the component of $g^{-1}(g(x))$ containing x . If $x \notin A_g$, then (1.4) there are open neighborhoods U of F and V of $f(x)$ such that $g|_U: U \rightarrow V$ is topologically equivalent to the projection map $\pi: V \times F \rightarrow V$. Thus $g_t|_{(U \cap (K^{n-p+m} \times \{t\}))}$ is also topologically equivalent to a projection map, so that $x \notin A(g_t)$.

If $x \in A_g$, then, by definition of A_g , $F \subset A_g$. Since $g|_{A_g}$ is light, $F = \{x\}$; since $n > p$, $x \in A(g_t)$ also (see (1.9)).

3.3. Remark. We will make repeated use of the notion of transverse regularity [30; p. 23] of a differential map, and of Thom's Transversality Theorem [30; p. 26]. The differentiability hypothesis in the latter can be improved from C^n to C^r where $r = \max(n - q + 1, 1)$, if in the proof the theorem of A. P. Morse [30; p. 20] is replaced by Sard's Theorem [25]. In [30; p. 27] the hypothesis C^1 suffices, and V^n need not be compact if f is proper.

3.4. Lemma. Let $f: M^n \rightarrow N^p$ be a C^r proper map, let τ be a triangulation of N^p , let Γ be a subpolyhedron of dimension $p - q$, and let $r = \max(n - q + 1, 1)$. Let W be a neighborhood of Γ in N^p , let I be the identity map on N^p , and let $\mathcal{U}$ be a neighborhood of I in the fine C^r topology [21; p. 25]. Then there is a C^r dif-

feomorphism $A \in \mathcal{U}$ such that A agrees with I off W and f is transverse regular on $A(\sigma)$ for each open simplex σ of Γ .

Proof. We may suppose that N^p is a C^∞ manifold [21; p. 42, (4.9)], and that it is a complete Riemannian manifold [29; p. 212, (7.4)]; the distance function d induced by the Riemannian metric is compatible with the given topology on N^p [29; p. 203].

The proof is somewhat similar to that of [9; Lemma 3]. First, we suppose that Γ is compact, and let Γ^j be the union of the open j -simplices of Γ . In this case we may suppose that $\bar{W}$ is compact also. Let $\mathfrak{X}$ be the set of C^r maps which agree with I off W ; there is a neighborhood $\mathfrak{X}$ of I in the coarse C^r topology such that $\mathfrak{X} \cap \mathfrak{Y} \subset \mathcal{U}$. We apply the Thom Transversality Theorem [30; p. 26, Theorem I.5], which uses the coarse C^r topology; there is a C^r diffeomorphism $A_0 \in \mathfrak{X} \cap \mathfrak{Y}$ such that f is transverse regular on $A_0(\Gamma^0)$ (let $A_0 = A^{-1}$ in the theorem). Call this property $\mathfrak{P}_0$.

Now suppose $(\mathfrak{P}_k)$ that there is a C^r diffeomorphism $A_k \in \mathfrak{X} \cap \mathfrak{Y}$ such that f is transverse regular on $A_k(\sigma^j)$ ($j = 0, 1, \dots, k$; σ^j a j -simplex of Γ). There is a neighborhood $\mathfrak{Y}$ of I in the coarse C^r topology sufficiently small that, for each C^r map $B \in \mathfrak{Y} \cap \mathfrak{Y}$, $B \circ A_k \in \mathcal{U}$ and f is transverse regular on $B \circ A_k(\sigma^j)$ for $j \leq k$ ([30; p. 27, Theorem I.6]; while each σ^j is a noncompact manifold, $\bar{\sigma}^j$ is compact, and the same argument applies). By the Transversality Theorem again there is a C^r diffeomorphism $B \in \mathfrak{Y} \cap \mathfrak{Y}$ such that f is transverse regular on $B \circ A_k(\Gamma^{k+1})$. Thus $A_{k+1} = B \circ A_k$ satisfies $\mathfrak{P}_{k+1}$, and the theorem for Γ compact follows by induction.

We now consider an infinite polyhedron Γ . Let $N_t \subset N^p$ be open sets ($t = 0, 1, \dots$; $N_0 = \emptyset$) such that $\bar{N}_t$ is compact, $\bar{N}_t \subset N_{t+1}$, and $\bigcup_t N_t = N^p$, and let $\Gamma_t \subset \Gamma$ be the union of those closed simplices of Γ each of which meets $\bar{N}_t$. We will define inductively a sequence of diffeomorphisms F_t of N^p ($t = 0, 1, \dots$; $F_0 = I$); G_t will be $F_t \circ G_{t-1}$, and W_t will be $W - G_{t-1}(\bar{N}_{t-1})$ ($t = 1, 2, \dots$; $G_0 = I$).

A set of C^r diffeomorphisms $F_1, F_2, \dots, F_t$ of N^p onto itself is said to satisfy Property $\mathfrak{Q}_t$ if (1) $G_t \in \mathcal{U}$, (2) f is transverse regular on $G_t(\sigma)$ for each open simplex σ of Γ_t , (3) $F_s \in \mathfrak{Y}_s$, and (4) the distance $d(F_s(x), x) < 2^{-s}$ for each $x \in N^p$ ($s = 1, 2, \dots, t$).

Given a set of diffeomorphisms satisfying $\mathfrak{Q}_t$, let V_{t+1} be an open neighborhood of $G_t(\Gamma_{t+1})$ with compact closure. Let $\mathfrak{X}$ be a neighborhood of I in the coarse C^r topology such that [30; p. 27, Theorem I.6] for each $F \in \mathfrak{X}$, f is transverse regular on $F(G_t(\sigma))$ for each open simplex σ of Γ_t , and, if $F \in \mathfrak{X} \cap \mathfrak{Y}_{t+1}$, then $F \circ G_t \in \mathcal{U}$ and $d(F(x), x) < 2^{-t-1}$ for each $x \in N^p$. Since $G_t(Cl(\Gamma_{t+1} - \Gamma_t))$ is a finite polyhedron, it follows from the first part of this proof that there is a C^r diffeomorphism $F_{t+1} \in \mathfrak{X} \cap \mathfrak{Y}_{t+1} \cap \mathfrak{Y}_{t+1}$ such that f is transverse regular on $F_{t+1}(G_t(\sigma))$ for each open simplex σ of that polyhedron. Thus there is a set of diffeomorphisms satisfying $\mathfrak{Q}_{t+1}$.

Given any $x \in N^p$, $Cl(S(x, 2))$ is compact [18; p. 172, (4.1)], so there is an

s sufficiently large that $Cl(S(x, 2)) \subset N_s$. From (4) $S(x, 1) \subset G_t(N_s)$ for every t , and from (3) G_t agrees with G_s on $S(x, 1)$ for every $t \geq s$. It follows that a limit diffeomorphism A is defined, and that it has the desired properties.

3.5. Lemma. *Let $f : M^n \rightarrow N^p$ be C^m and proper, let K^q be a compact C^m q -manifold ($m = 1, 2, \dots; q = 0, 1, \dots, p - 1$), and let ρ be a C^m diffeomorphism of a region in N^p onto $K^q \times E^{p-q}$. If f is transverse regular on $\rho^{-1}(K^q \times \{0\})$, then there are $\epsilon > 0$, a $C^m(n - p + q)$ -manifold (or the empty set) L , and a C^m diffeomorphism σ of $L \times S(0, \epsilon)$ onto $f^{-1}(\rho^{-1}(K^q \times S(0, \epsilon)))$ such that $\rho f \sigma$ is a layer map.*

The proof for $n = p$ was given in [7; p. 376, (2.7)], and it applies in the general case.

3.6. Corollary. *If $f : M^n \rightarrow N^p$ is a proper C^1 map, then $f(A_r) \subset f(R_{p-1}(f))$, i.e., no regular value is in $f(A_r)$.*

For $y \in N^p - f(R_{p-1}(f))$ apply (3.5) with $K^0 = \{y\}$; thus $y \notin f(A_r)$. If f is C^r with $r \geq \max(n - p + 1, 1)$, then [25] $\dim f(A_r) \leq p - 1$.

3.7. Lemma. *Let $f : M^n \rightarrow N^p$ be a C^r proper map, let $q = 1, 2, \dots$, let $r \geq \max(n - q + 1, 1)$, and let Y be a closed subset of N^p . Suppose either*

- (a) $\dim Y \geq q$, or
- (b) $q \leq p - 1$ and for some h ($h = 0, 1, \dots, p - q - 1$) and $x \in N^p - Y$ the homomorphism

$$\iota_{\#} : \pi_h(N^p - Y, x) \rightarrow \pi_h(N^p, x)$$

induced by inclusion ι is not a monomorphism.

Then for some m ($m = 0, 1, \dots, p - q$) there is a C^r embedding λ of $S^m \times E^{p-m}$ in N^p such that f is transverse regular on $\lambda(S^m \times \{t\})$ and $\lambda(S^m \times \{t\}) \cap Y \neq \emptyset$ for each $t \in E^{p-m}$.

Proof. First we consider hypothesis (a). The beginning of this proof is a modification of the proof of [9; Proposition 4]. Let $U \subset N^p$ be a C^r diffeomorph of E^p ; since it suffices to prove the theorem for the restriction map $f|_{f^{-1}(U)}$, we may suppose that $N^p = E^p$. Let X be a compact subset of Y such that $\dim X \geq q$, and let C be a compact subset of X such that the homomorphism

$$i^* : \tilde{H}^{q-1}(X; Z) \rightarrow \tilde{H}^{q-1}(C; Z)$$

induced by inclusion is not an epimorphism [16; p. 151]. By the naturality of the Alexander Duality Theorem [27; p. 296, Theorem 16] the homomorphism

$$j_* : \tilde{H}_{p-q}(E^p - X; Z) \rightarrow \tilde{H}_{p-q}(E^p - C; Z)$$

induced by inclusion is also not an epimorphism. Let z be a polyhedral cycle of $E^p - C$ whose homology class ψ is not in the image of j_* , let Ψ be the carrier of ψ , and let Ψ^s be the s -skeleton of Ψ ($s = -1, 0, \dots; \Psi^{-1} = \emptyset$).

Let I be the identity map on E^p , and let $\mathfrak{U}$ be a neighborhood (in the compact

open topology) of I sufficiently small that for each map $A \in \mathfrak{U}$, $A(\Psi) \cap C = \emptyset$ and $A(z)$ is homologous to z in $E^p - C$. By (3.4) there is a C^r diffeomorphism A of E^p onto itself such that (1) f is transverse regular on $A(\sigma^s)$ for each open simplex σ^s of Ψ ($s = 0, 1, \dots, p - q$), and (2) $A \in \mathfrak{U}$.

There exists a maximal integer $k \geq 0$ such that (3) $A(\Psi^{k-1}) \cap X = \emptyset$ for some C^r diffeomorphism A satisfying (1) and (2). Since $\psi \notin \text{imag } j_*$, $A(\Psi) \cap X \neq \emptyset$, so that $k \leq p - q$. For each A satisfying (1), (2) and (3) let $\Sigma(A)$ be the (possibly empty) set of (open) k -simplices σ^k of Ψ such that $A(\sigma^k) \cap X = \emptyset$; from the diffeomorphisms A satisfying (1), (2), and (3), we choose an A such that (4) $\Sigma(A)$ is maximal. Let $\mathfrak{B}$ be a neighborhood of A in the coarse C^r topology [21; p. 25] sufficiently small that each map in $\mathfrak{B}$ satisfies conditions (1), (2), (3), and (4). Let τ be an open k -simplex not in $\Sigma(A)$, and let W be an open neighborhood of τ in E^p such that $\bar{W}$ is compact and $\bar{W} \cap \Psi^k = \bar{\tau}$.

Now let $\alpha : D^k \rightarrow E^p$ be a C^r embedding such that $A(\bar{\tau}) \subset \alpha(\text{int } D^k)$ and f is transverse regular on $\alpha(D^k)$. According to [5; (3.5)] there is a C^r embedding $\beta : S^k \rightarrow E^p$ such that $\alpha(D^k) \subset \beta(S^k)$, the normal bundle of $\beta(S^k)$ is trivial, and f is transverse regular on $\beta(S^k)$. There is a C^r embedding λ of $S^k \times E^{p-k}$ into E^p such that $\lambda(s, 0) = \beta(s)$ for each $s \in S^k$; by [30; p. 27, Theorem I.6] we may suppose that f is transverse regular on $\lambda(S^k \times \{t\})$ for each $t \in E^{p-k}$. By (3.5) there is an $\epsilon > 0$, a $C^\infty(n - p + k)$ -manifold (or the empty set) L , and a C^r diffeomorphism σ of $L \times S(0, \epsilon)$ onto $f^{-1}(\lambda(S^k \times S(0, \epsilon)))$ such that $\lambda^{-1}f\sigma$ is a layer map g .

Let ν be $\beta^{-1}(A(\tau))$; since $A(\tau) = \lambda(\nu \times \{0\})$, it follows from (3) (i.e., the maximality of k) that $\lambda(\partial\nu \times \{0\}) \cap X = \emptyset$. There is an open subset ω of ν such that $\bar{\omega}$ (relative to ν) is compact and $\lambda((\bar{\nu} - \omega) \times \{0\}) \cap X = \emptyset$, and there is a $\zeta < \epsilon$ such that $\lambda((\bar{\nu} - \omega) \times S(0, \zeta)) \cap X = \emptyset$.

Suppose that there exists $t_i \rightarrow 0$ such that $\lambda(\nu \times \{t_i\}) \cap X = \emptyset$. Then there is a C^r diffeomorphism B of E^p such that B agrees with the identity I off $A(W)$, B is sufficiently near I that $B \circ A \in \mathfrak{B}$,

$$B(\lambda((\bar{\nu} - \omega) \times \{0\})) \subset \lambda((\bar{\nu} - \omega) \times S(0, \zeta)),$$

and $B(\lambda(\omega \times \{0\})) = \lambda(\omega \times \{t_i\})$ for some t_i . As a result $B \circ A(\tau) \cap X = \emptyset$, contradicting the choice of τ and of $\mathfrak{B}$. Thus for some $\eta < \zeta$, $\lambda(\omega \times \{t\}) \cap X \neq \emptyset$ for every $t \in S(0, \eta)$, so that the embedding $\lambda|_{(S^k \times S(0, \eta))}$ is the desired embedding.

We now suppose hypothesis (b). There is $\xi > 0$ such that $S(x, \xi) \cap Y = \emptyset$, and by Sard's Theorem [25] there is a regular value $y \in S(x, \xi)$; the homomorphism

$$\iota_{\#} : \pi_h(N^p - Y, y) \rightarrow \pi_h(N^p, y)$$

induced by inclusion is also not a monomorphism. The manifold N^p has a C^r triangulation [21; p. 101, (10.6)], and we may suppose that y is a vertex. Let $\rho^h \subset E^h$ denote a closed geometric h -simplex, and let u be a vertex of ρ^h .

There exists a map $\gamma : (\partial\rho^{h+1}, u) \rightarrow (N^p - Y, y)$ which can be extended to

a map $\Gamma : (\rho^{h+1}, u) \rightarrow (N^p, y)$, but not to a map into $N^p - Y$. Choose $\epsilon > 0$ sufficiently small that, if the uniform distance $d(\gamma, \delta) < \epsilon$, then γ is homotopic in $N^p - Y$ to δ . Let $\Delta : (\rho^{h+1}, u) \rightarrow (N^p, y)$ be a simplicial approximation to Γ such that $d(\Gamma, \Delta) < \epsilon$, let $\delta = \Delta|_{\partial\rho^{h+1}}$, and let Ψ be the (subdivided!) polyhedron $\Delta(\rho^{h+1})$.

By (3.4) there is a C^r diffeomorphism A of N^p onto itself such that (1) f is transverse regular on $A(\sigma^s)$ for each open simplex σ^s of Ψ ($s = 0, 1, \dots, p - q$), and (2) $d(\gamma, A \circ \delta) < \epsilon$ and $A(y) = y$. There exists a maximal integer k such that (3) $A(\Psi^{k-1}) \cap Y = \emptyset$ for some C^r diffeomorphism A satisfying (1) and (2). Since γ is homotopic to $A \circ \delta$, $0 \leq k \leq h + 1 \leq p - q$. The proof concludes as in case (a).

In this paper a weaker conclusion than that of (3.7) suffices, namely that $\lambda(S^m \times E^{p-m}) \cap Y \neq \emptyset$; however, in a subsequent paper Church will use the stronger conclusion given above.

3.8. Lemma. *Let $g : K^{n-p+m} \times E^{p-m} \rightarrow L^m \times E^{p-m}$ be a proper C^r layer map, let $r \geq \max(n - p + m, 1)$, and let $Y \subset L^m \times E^{p-m}$ be closed with $\dim(Y \cap (L^m \times \{0\})) > 0$. Then there are a C^r layer map*

$$h : J^{n-p+k} \times E^{p-k} \rightarrow S^k \times E^{p-k}$$

with J^{n-p+k} compact (or empty), a C^r embedding α of $J^{n-p+k} \times E^{p-k}$ in $K^{n-p+m} \times E^{p-m}$, and a C^r embedding β of $S^k \times E^{p-k}$ in $L^m \times E^{p-m}$ such that:

- (1) $k < m$,
- (2) $\beta h \alpha^{-1}$ is a restriction of g ,
- (3) $Y \cap \beta(S^k \times E^{p-k}) \neq \emptyset$,
- (4) $\dim(Y \cap \beta(S^k \times \{t\})) \leq 0$ for every $t \in E^{p-k}$, and
- (5) g is transverse regular on $\beta(S^k \times \{t\})$ for every $t \in E^{p-k}$.

In fact, if E^{p-k} is written as $E^{m-k} \times E^{p-m}$ suitably, then α , β , and h are all layer maps with respect to E^{p-m} .

Proof. We apply (3.7) to $g_0 : K^{n-p+m} \times \{0\} \rightarrow L^m \times \{0\}$ and $Y \cap (L^m \times \{0\})$; there is a C^r embedding λ of $S^k \times E^{m-k}$ in $L^m \times \{0\}$ ($k = 0, 1, \dots, m - 1$) such that g_0 is transverse regular on $\lambda(S^k \times \{s\})$ for each $s \in E^{m-k}$ and $Y \cap \text{imag } \lambda \neq \emptyset$. Among the embeddings satisfying these conditions we may suppose that the chosen λ has minimal k .

Suppose that $\dim(Y \cap \lambda(S^k \times \{s'\})) > 0$ for some $s' \in E^{m-k}$; we may suppose that $s' = 0$. Since g_0 is transverse regular on $\lambda(S^k \times \{0\})$, $g_0^{-1}(\lambda(S^k \times \{0\}))$ is a C^r manifold M [30; p. 23]; let $f : M \rightarrow \lambda(S^k \times \{0\})$ be the restriction of g_0 . From (3.7) applied to f and $Y \cap \lambda(S^k \times \{0\})$ there is a C^r embedding μ of $S^j \times E^{k-j}$ into $S^k \times \{0\}$ ($j = 0, 1, \dots, k - 1$) such that $Y \cap \lambda(\mu(S^j \times \{0\})) \neq \emptyset$ and f (and thus g) is transverse regular on $\lambda(\mu(S^j \times \{s\}))$ for each $s \in E^{k-j}$. Write E^{m-i} as $E^{k-i} \times E^{m-k}$, let $\pi : S^k \times \{0\} \rightarrow S^k$ be projection, let ι be the identity map on E^{m-k} , and let $\nu = \lambda \circ ((\pi \circ \mu) \times \iota)$. Then ν is a C^r embedding of $S^j \times E^{m-i}$ into $L^m \times \{0\}$ and $Y \cap \nu(S^j \times \{0\}) \neq \emptyset$. From [30; p. 27, Theorem I.6] there is a $\delta > 0$ such that g is transverse regular on $\nu(S^j \times \{u\})$

for each $u \in S(0, \delta)$, so that the minimality property of λ is contradicted. Thus $\dim(Y \cap \lambda(S^k \times \{s\})) \leq 0$ for each $s \in E^{p-k}$.

Let σ be the identity map on E^{p-m} , let $\rho : L^m \times E^{p-m} \rightarrow L^m$ be projection, and write E^{p-k} as $E^{m-k} \times E^{p-m}$; then $(\rho \circ \lambda) \times \sigma$ is a C^r embedding of $S^k \times E^{p-k}$ into $L^m \times E^{p-m}$. The conclusion follows from an application of (3.5).

4. Proofs of the theorems.

4.1. Lemma. *Let M^n and N^p be connected, and let $f : M^n \rightarrow N^p$ be a C^n proper map such that $\emptyset \neq A_f \neq M^n$ and $f|_{A_f}$ is light. If $n = p$, suppose in addition that one of the following conditions holds:*

(I) *For every $x \in M^n$, there are open n -cells $U \subset M^n$ and $V \subset N^n$ such that $x \in U$, $f(U) \subset V$, and the Jacobian determinant of $f|_U : U \rightarrow V$ does not change sign.*

(II) *The map f is a singular fibering.*

Then there is a closed subset $Y_f \subset f(A_f)$ such that $\dim Y_f < \dim f(A_f)$ and, for each $x \in A_f - f^{-1}(Y_f)$, f at x is locally topologically equivalent to one of the maps $c(\psi) \times \iota_k$ of (1.1).

Suppose that f satisfies the hypotheses of (4.1) with $n = p$ and condition (I) satisfied; then (1.10) f is light, and from the Jacobian condition it is also open [34; p. 91] (see also [6; p. 89, (1.7)]). Since $A_f \neq \emptyset$, $n \geq 2$, and we may apply [6; p. 91, (2.1)], where $B_f = A_f$. Thus $\dim A_f = n - 2$, and there is a closed set E with $\dim E \leq n - 3$ such that at each $x \in A_f - E$ f is locally topologically equivalent to one of the maps $c(\psi) \times \iota_{n-2}$ of (1.1)(iii). An inspection of the proof shows that $\dim f(E) \leq n - 3$ also, and with Y_f as the closed (1.13) set $f(E)$, (4.1) results in case $n = p$ and (I) holds.

Suppose that $f : M^n \rightarrow N^n$ is a C^n proper map with $A_f \neq \emptyset$, $\dim A_f \leq 0$, and $n \geq 2$. By (1.11) f satisfies the hypotheses of (4.1)(I), so that $\dim A_f = n - 2$; hence $n = 2$, and at each point of A_f the map f is locally topologically equivalent to one of the maps $c(\psi) \times \iota_0$ of (1.1)(iii). From (1.14), (1.5) is true for $n = p$.

Suppose now that (4.1) is true. Let f be a map satisfying the hypotheses of (1.3)(a) (resp., (1.5) with $A_f \neq \emptyset$ and $n \neq p$), and let X be the nonempty set $\text{bdy } A_f$ (resp., let $X = A_f$). By (1.14) for each $x \in X$ there are connected open neighborhoods $U(x)$ of x and V of $f(x)$ such that $f|_{U(x)} : U(x) \rightarrow V$ is proper. From (4.1) applied to $f|_{U(x)}$, $\dim(U(x) \cap A_f) < n$, and since $x \in X$ is arbitrary, $X = A_f$. For each $x \in A_f$ let $W(x)$ be an open neighborhood of x with $\bar{W}(x) \subset U(x)$, and let $Y(x)$ be $U(x) \cap f^{-1}(Y_{f|_{U(x)}})$ (resp., $Y(x) = \emptyset$). Then $Y(x)$ is closed in $U(x)$, $\dim Y(x) < \dim A_f$ (1.13), and at each point of $(U(x) \cap A_f) - Y(x)$ f is locally topologically equivalent to one of the maps $c(\psi) \times \iota_k$ of (1.1). There exist x_k ($k = 1, 2, \dots$) such that the $W(x_k)$ cover M^n and the $\bar{W}(x_k)$ are locally finite; thus the set $Y = \bigcup_k (\bar{W}(x_k) \cap Y(x_k))$ is closed. As a result (1.3)(a) and (1.5) are consequences of (4.1). For (1.3)(b) see (4.5).

Given $p = 1, 2, \dots$ let (4.1_p) denote the statement of (4.1) for all $n \neq p$, and for $n = p$ in case condition (II) is satisfied but (I) is not; it thus suffices to prove (4.1_p) for all p . If $p = 1$, then $(3.6) \dim f(A_f) \leq 0$. By (1.13) $\dim A_f \leq 0$, so that (4.1_1) follows from [32; (4.1)]. We will now prove that (4.1_p) implies (4.2_p) (below), and that (4.2_m) for all $m \leq p$ implies (4.1_{p+1}) ; thus (4.1) and (4.2) follow by induction.

4.2._m Lemma. Let K^{n-p+m} and L^m be compact connected C^{n-p+m} manifolds, let $L^m = S^m$ if $m \geq 3$, and let

$$g : K^{n-p+m} \times E^{p-m} \rightarrow L^m \times E^{p-m}$$

be a C^{n-p+m} layer map. Suppose either that $n > p$ and $m \geq 1$; or that $n = p$, g is a singular fibering, and $m = 1$. If $g|_{A_g}$ is light, $A_g \neq \emptyset$, and $\dim g_t(A(g_t)) \leq 0$ for each $t \in E^{p-m}$, then A_g is the disjoint union of a finite number of $(p-m)$ -cells.

In fact, at each point of A_g , g is locally topologically equivalent to one of the maps $c(\psi) \times \iota_{p-m}$ of (1.1)(i) and (ii).

4.3. (4.1_m) implies (4.2_m) .

Proof. If $n > p$, then $(3.2) A_g = \bigcup_i A(g_i)$; since $g|_{A_g}$ is light, $g_t|_{A(g_t)}$ is also light, so that $(1.13) \dim A(g_t) \leq 0$. By (4.1_m) $A(g_t)$ is finite, and at each point of $A(g_t)$ the map g_t is locally topologically equivalent to one of the maps $c(\psi) \times \iota_0$ of (1.1)(i) and (ii). Let t be any point in E^{p-m} . If $m = 1$ and $A(g_t) \neq \emptyset$, then [32; (4.1)] $A(g_t)$ has exactly 2 points. If $m = 2$ then $A(g_t) = \emptyset$ (4.1_m) . If $m \geq 3$, then $L^m = S^m$ by hypothesis, so that $(S^m \times \{t\}) - g_t(A(g_t))$ is simply connected; if $A(g_t) \neq \emptyset$, then [32; (4.2)] the cardinality of $A(g_t)$ is the mod 2 Euler characteristic of K^{n-p+m} .

If $n = p$, then $K = L = S^1$, and each g_t is a singular fibering by hypothesis; it follows readily as in [32; (4.1)] that $A(g_t)$ has either 2 points or none, and, at each point of $A(g_t)$, g_t is locally topologically equivalent to a folding map, i.e., to $c(\psi) \times \iota_0$ with $m = 0$ in (1.1)(ii).

Consider any $t \in E^{p-m}$ such that $A(g_t) \neq \emptyset$; for convenience we suppose that this t is 0. Let y_i ($i = 1, 2, \dots, J$) be the points of $g_0(A(g_0))$, and let $D_i \subset L^m$ be mutually disjoint C^{n-p+m} diffeomorphs of the closed unit m -ball D^m such that $y_i \in (\text{int } D_i) \times \{0\}$ and g is transverse regular on $\partial D_i \times \{0\}$. Let $\Gamma_i \subset K^{n-p+m}$ be the manifold with boundary [30; p. 23] defined by $g^{-1}(D_i \times \{0\}) = \Gamma_i \times \{0\}$, and let $\Gamma_{i,j}$ ($j = 1, 2, \dots$) be the components of Γ_i . Choose $\epsilon > 0$ such that g is transverse regular on $\partial D_i \times \{t\}$ for each $t \in S(0, \epsilon)$, and let $\Lambda_i = g^{-1}(D_i \times S(0, \epsilon))$; then the manifold $\partial \Lambda_i$ is transverse to $K^{n-p+m} \times \{t\}$ for each $t \in S(0, \epsilon)$.

By [5; (3.3)] there is a C^{n-p+m} layer diffeomorphism η of $K^{n-p+m} \times S(0, \epsilon)$ onto itself such that $\eta(\Gamma_i \times S(0, \epsilon)) = \Lambda_i$; if $\Lambda_{i,j}$ denotes the component of Λ_i containing $\Gamma_{i,j} \times \{0\}$, and if $\Gamma_{i,j}(t)$ is $\Lambda_{i,j} \cap (K^{n-p+m} \times \{t\})$, then $\Gamma_{i,j}(t)$ is diffeomorphic to $\Gamma_{i,j}$ for each $t \in S(0, \epsilon)$.

Suppose that $\Gamma_{i,j}(t) \cap A(g_t) = \emptyset$ for some $t \in S(0, \epsilon)$ and (i, j) ; then $g_t|_{\Gamma_{i,j}(t)}$ is the projection map of a fiber bundle [32; (2.4)] with contractible base space

$D_i \times \{t\}$, so that [28; p. 53, (11.6)] $\Gamma_{i,j}$ is homeomorphic to $D_i \times F$ for some compact connected manifold F with $\partial F = \emptyset$. In particular if $n > p$, then $\Gamma_{i,j}$ is not contractible; in either case g_i maps $\Gamma_{i,j}(t)$ onto $D_i \times \{t\}$.

Let Δ denote the set of (i, j) such that $A(g_0) \cap \Gamma_{i,j}(0) \neq \emptyset$. Then (by (4.1_m)) for $(i, j) \in \Delta$, $A(g_0) \cap \Gamma_{i,j}(0)$ consists of a single point, and $\Gamma_{i,j}$ is the cone of a homotopy $(n - p + m - 1)$ -sphere and so is contractible. Suppose that $n > p$; from the previous paragraph, for each $t \in S(0, \epsilon)$, $A(g_t) \cap \Gamma_{i,j}(t) \neq \emptyset$. Since the cardinality of $A(g_0)$ agrees with that of $A(g_t)$ for each $t \in S(0, \epsilon)$, $A(g_t) \cap \Gamma_{i,j}(t)$ is a single point $y_{i,j}(t)$ for $(i, j) \in \Delta$, and $A(g_t) \cap \Gamma_{i,j}(t) = \emptyset$ for $(i, j) \notin \Delta$. Let $\zeta = \epsilon$.

Now suppose that $n = p$ (so that $m = 1$); $A(g_0)$ consists of two points, and at each of them g_0 is topologically equivalent to a folding map. For each $(i, j) \in \Delta$, $g_0(\Gamma_{i,j}(0)) = \bar{P}$, where P is one of the two component intervals of $(D_i \times \{0\}) - \{y_i\}$, so there is a ζ , $0 < \zeta < \epsilon$, such that $g_t(\Gamma_{i,j}(t))$ is also not all of $D_i \times \{t\}$ for each $t \in S(0, \zeta)$. From the paragraph before last, it follows that $A(g_t) \cap \Gamma_{i,j}(t) \neq \emptyset$ for each $t \in S(0, \zeta)$. Since each $A(g_t)$ consists of either two points or none, for each $t \in S(0, \zeta)$, $A(g_t) \cap \Gamma_{i,j}(t)$ consists of a single point $y_{i,j}(t)$ if $(i, j) \in \Delta$, and is empty otherwise.

For $(i, j) \in \Delta$, $A_\sigma \cap \Lambda_{i,j}$ is closed, so that the function $y_{i,j} : S(0, \zeta) \rightarrow A_\sigma \cap \Lambda_{i,j}$ is continuous, and thus is a homeomorphism; as a result $A_\sigma \cap \Lambda_{i,j}$ is a $(p - m)$ -cell. Since the layer we started with—and indexed as 0—was arbitrary, A_σ is the union of a finite number of $(p - m)$ -cells. In fact the number of cells equals the number of elements of Δ : 2 if $m = 1$, and the mod 2 Euler characteristic of K^{n-p+m} if $m > 1$.

Consider a fixed $(i, j) \in \Delta$. Let $h : D_i \rightarrow D^m$ be a homeomorphism onto, let $U(x, \delta) \subset E^m$ be the set of z such that $d(x, z) < \delta$, and let $\pi : D_i \times E^{p-m} \rightarrow D_i$ be projection. There exists $\delta > 0$ sufficiently small that

$$\bar{U}(h(\pi(g_t(y_{i,j}(t)))), \delta) \subset \text{int } D^m;$$

the layer map $\xi : \bar{U}(0, \delta) \times S(0, \zeta) \rightarrow D^m \times S(0, \zeta)$ defined by

$$\xi(x, t) = (x + h(\pi(g_t(y_{i,j}(t)))), t)$$

is a homeomorphism into.

Let I be the identity map on $S(0, \zeta)$, and let

$$\Lambda = \Lambda_{i,j} \cap g^{-1}(h^{-1} \times I(\text{imag } \xi)) \quad \text{and} \quad G = \xi^{-1} \circ (h \times I) \circ g \mid \Lambda.$$

Then G is obtained from g by restriction to a neighborhood of $A_\sigma \cap \Lambda_{i,j}$ in $\Lambda_{i,j}$ and (layer) change of coordinates (so that $A_\sigma = A_g$), $\text{imag } G = \bar{U}(0, \delta) \times S(0, \zeta)$ (unless $m = 1$, in which case $\bar{U}(0, \delta)$ is replaced by either $[0, \delta]$ or $[-\delta, 0]$), and $G(A_g) = \{0\} \times S(0, \zeta)$.

If $\omega : \bar{U}(0, \delta) \times S(0, \zeta) \rightarrow (S^{m-1} \times (0, 1]) \times S(0, \zeta)$ denotes a layer homeomorphism (onto), then $\omega \circ G \mid (\Lambda - A_g)$ is the projection map of a bundle λ over the base space $S^{m-1} \times (0, 1] \times S(0, \zeta)$ (in case $m = 1$, S^0 is replaced by one of its points). The restriction bundle $\mu = \lambda \mid (S^{m-1} \times \{0\} \times \{0\})$ is

equivalent to one of the bundles $\psi : X \rightarrow S^{m-1}$ of (1.1)(i) and (ii). Since the map σ of $S^{m-1} \times (0, 1] \times S(0, \zeta)$ into itself defined by $\sigma(x, s, t) = (x, 0, 0)$ is homotopic to the identity map τ , the induced bundles $\sigma^*(\lambda)$ and $\tau^*(\lambda) = \lambda$ are equivalent. Thus, if ι denotes the bundle whose projection map is the identity map on $(0, 1] \times S(0, \zeta)$, then λ is equivalent to $\mu \times \iota$ [17; p. 15, (4.1)]; in particular, its projection map is

$$\rho : X \times (0, 1] \times S(0, \zeta) \rightarrow S^{m-1} \times (0, 1] \times S(0, \zeta),$$

where $\rho(x, s, t) = (\psi(x), s, t)$. That G has the desired form at $y_{i,j}(0)$ now follows; since the layer indexed as 0 is arbitrary, the theorem is proved.

4.4. Lemma. (4.2_m) for $m = 1, 2, \dots, p - 1$ implies (4.1_p).

Proof. We consider a C^n proper map $f : M^n \rightarrow N^p$ such that $\emptyset \neq A_f \neq M^n$, $f|A_f$ is light, and $p \geq 2$. By (1.10) $n \geq p$. If $n = p$, we also suppose (see (4.1_p)) that f is a singular fibering, and that, at some point x of M^n , the Jacobian determinant of f locally changes sign; thus (1.11) $\dim A_f \geq p - 1$. Let $q = \dim f(A_f)$. By (1.13) $f(A_f)$ is closed, and $\dim A_f \leq q$.

Let V be the set of points in A_f at which f is not locally topologically equivalent to one of the maps listed in (1.1), and let Y_f be the closed (1.13) set $f(V)$. It suffices to prove that $\dim Y_f < q$; suppose the contrary.

Consider any C^n embedding λ of $S^m \times E^{p-m}$ into N^p such that (a) $m \leq p - 1$, and if $n = p$, then $m \leq 1$; (b) $f(A_f) \cap \lambda(S^m \times \{0\}) \neq \emptyset$; and (c) f is transverse regular on $\lambda(S^m \times \{t\})$ for each $t \in E^{p-m}$. By (3.5) there is an $\epsilon > 0$, a C^n $(n - p + m)$ -manifold J , and a C^r diffeomorphism σ of $J \times S(0, \epsilon)$ onto $f^{-1}(\lambda(S^m \times S(0, \epsilon)))$ such that $\lambda^{-1}f\sigma$ is a layer map g ; we may suppose that $S(0, \epsilon) = E^{p-m}$. Note that $\lambda^{-1}(f(A_f)) = g(A_g)$. Thus from (b), we have (d) $1 \leq m$. Suppose that (e) $g(A_g) \cap (S^m \times \{t\})$ has dimension at most 0 for each $t \in E^{p-m}$. From (3.2) it follows that the hypothesis of (4.2_m) is satisfied, so that g at each point of A_g is locally topologically equivalent to one of the maps $c(\psi) \times \iota_{p-m}$ of (1.1)(i) and (ii). As a result, $Y_f \cap \text{imag } \lambda = \emptyset$, and $\dim A_f \geq p - m$.

Suppose $q = 0$ (then $n > p$). Let $x \in A_f$, and let U be any open p -cell neighborhood of $f(x)$. If $p = 2$, or if $p \geq 3$ and $U - f(A_f)$ is simply connected, then the desired conclusion is a consequence of (2.3). If $p \geq 3$ and $U - f(A_f)$ is not simply connected, then by (3.7) applied to $f|f^{-1}(U)$ and $Y = f(A_f)$ there is an embedding λ satisfying (a), (b), and (c), and there is a corresponding layer map g . By (3.8) we may suppose that (e) is also satisfied, so that $\dim A_f \geq 1$; a contradiction results.

Now suppose that $q > 0$; we apply (3.7) to f and Y_f and obtain a C^n embedding λ satisfying (a') $m \leq p - q$, (b') $Y_f \cap \lambda(S^m \times \{0\}) \neq \emptyset$, and (c). In case $n = p$, $q \geq p - 1$, so that $m = 0$ or 1; by (d) $m = 1$. In either case (a), (b), and (c) are satisfied, and if (e) is also satisfied, a contradiction of (b') results. If (e) is not satisfied, then we apply (3.8) with $Y = g(A_g)$; by restriction of g and change of coordinates we obtain a layer map h satisfying (a'') $m \leq p - q - 1$, and if $n = p$, $m \leq 1$, (b), (c), and (e). As a result, $\dim A_f \geq q + 1$, yielding a contradiction.

4.5. Proof of (1.3)(b). Suppose that $f : M^n \rightarrow N^p$ is a proper C^n singular fibering with $f(M^n) = N^p$, and that f at $x \in A_f$ is locally topologically equivalent to one of the maps $c(\psi) \times \iota_{p-1}$ of (1.1)(ii) ($p \geq 1$). There are a p -cell $V \subset N^p$, a $(p-1)$ -cell Γ with $f(x) \in \Gamma \subset V \cap f(A_f)$, and a homeomorphism h of (V, Γ) onto (E^p, E^{p-1}) , where E^{p-1} is a $(p-1)$ -plane in E^p .

Let $\mathfrak{E}^n$ be the space of C^n maps sending E^1 into V , with the course C^n topology [21; p. 25]. There are an embedding $\lambda \in \mathfrak{E}^n$ with $\lambda(E^1) \cap \Gamma \neq \emptyset$, and a neighborhood $\mathfrak{U} \subset \mathfrak{E}^n$ of λ , such that $\delta(E^1) \cap \Gamma \neq \emptyset$ for each $\delta \in \mathfrak{U}$. (Roughly speaking, $\lambda(E^1)$ crosses Γ , and if δ is sufficiently near λ , then $\delta(E^1)$ also crosses Γ .) By [30; p. 26] there is a C^n diffeomorphism H of V onto itself such that $H^{-1} \circ \lambda \in \mathfrak{U}$ and f is transverse regular on $H^{-1}(\lambda(E^1))$; thus $H^{-1}(\lambda(y)) \in \Gamma$ for some $y \in E^1$.

Now $f^{-1}(H^{-1}(\lambda(E^1)))$ is a manifold K^{n-p+1} , and we apply [32; (4.1)] to $f|K^{n-p+1}$; it follows readily that, for some $\epsilon > 0$, either $H^{-1}(\lambda((y-\epsilon, y)))$ or $H^{-1}(\lambda((y, y+\epsilon)))$ is not in the image of $f|K^{n-p+1}$. Thus it is not in the image of f , contradicting the onto-ness of f .

4.6. Remark. Let $f : M^n \rightarrow N^p$ be a proper C^n singular fibering with $f(M^n) = N^p$, and let $X = f(A_f)$ (resp., $f(Y)$, where Y is the smallest set satisfying the properties in (1.3)). Then for each j ($j = 0, 1, \dots, n-p$ (resp., $n-p+1$)) the homomorphism

$$\iota_{\#} : \pi_j(N^p - X, x) \rightarrow \pi_j(N^p, x)$$

induced by inclusion is a monomorphism.

Note that the result applies equally well if N^p is replaced by an open p -cell $U \subset N^p$ and if f is replaced by $f|f^{-1}(U)$. In this case $U - f(A_f)$ is $(n-p)$ -connected, so the result gives information about the local character of the embeddings of $f(A_f)$ and $f(Y)$ in N^p .

If $Y = \emptyset$, the conclusion is an immediate consequence of (1.3) and general position arguments. The examples of (1.1)(ii) show that the onto-ness hypothesis is required.

Proof. We suppose, more generally, that f satisfies the hypothesis of (4.1) with $\dim A_f \leq p-2$ (see (1.3); $f(Y)$ is replaced by Y_f). For $n = p$, f satisfies the hypotheses of (4.1)(I) (see (1.11)), and so is light open; the conclusion is given in [8; p. 413, (1.2)]. Thus we may suppose that $n > p$. Let $q = \dim A_f$. At each point of $A_f - f^{-1}(Y_f)$, f is locally topologically equivalent (4.1) to one of the maps $c(\psi) \times \iota_k$ of (1.1)(i), so that $q = 2p - n - 2$. By (1.1) we may suppose that $q > 0$.

The argument now is similar to that of the last paragraph of the proof of (4.4). Suppose that $\iota_{\#}$ is not a monomorphism; we apply (3.7) to f and $f(A_f)$, and obtain a C^n embedding λ satisfying (a') $m \leq n - p + 1$, (b), and (c). If (e) is not satisfied, we apply (3.8) to obtain a new layer map for which (e) is satisfied. Thus $q \geq 2p - n - 1$, and a contradiction results. For Y_f the argument is closely parallel to that of the last paragraph of (4.4).

4.7. Remarks. For each $n \geq 3$ there is a C^n 3-to-1 open map $f : E^n \rightarrow E^n$

(thus satisfying condition (I) in (4.1)) such that $1 + \dim Y_f = \dim f(A_f)$; f is $\phi \times \iota_{n-3}$, where $\phi : E^3 \rightarrow E^3$ is the map of [10; p. 614, (3.3)] and [6; p. 98, (3.4)].

In case $n = p$ (1.3) and (4.1) are somewhat related to [4; p. 704, Theorem 3], since a light quasi-monotone map is open [36; p. 152, (8.2)].

4.8. Theorem. *Let $f : M^n \rightarrow N^p$ be a proper onto C^n singular fibering with $n > p$ and $A_f \neq \emptyset$. Then $n - p = 1, 3$, or 7 , and A_f is a locally orientable Z_2 -generalized $(2p - n - 2)$ -manifold.*

Proof. For each $x \in M^n - A_f$, each component of $f^{-1}(f(x))$ is a homotopy sphere of dimension $1, 3$, or 7 , and $\dim A_f = 2p - n - 2$ (see (1.3)). Since $f|_{(M^n - A_f)}$ is open and $\dim A_f \leq p - 2$, it follows from [36; p. 149, (7.81)] that f is open.

Let $y \in A_f$ and let L be a C^n diffeomorph of D^n such that $f(x) \in \text{int } L$ and f is transverse regular on ∂L [30; p. 26]. Let K be the component of $f^{-1}(L)$ containing x ; then K is a compact C^n manifold with boundary, and since f is open, $f(K) = L$. By [32; (2.4)] $f|_{(K - A_f)}$ has a factorization $f = h \circ g$, where g is the projection map of a fiber bundle with connected fiber, and h is a covering map. Since $\pi_1((\text{int } L) - f(A_f)) = 0$ (4.6), h is a homeomorphism [27; p. 81, Corollary 7], so that $f|_{(K - A_f)}$ is the projection map of a fiber bundle with base space $L - f(A_f)$ and fiber a homotopy sphere of dimension $1, 3$, or 7 .

Let $D(K)$ and $D(L)$ be the doubles [21; p. 52] of K and L , respectively, and let $e : D(K) \rightarrow D(L)$ be f on each copy of K . Then e is a fiber space with singularities in the sense of Conner and Dyer [11; p. 303], and it follows from [11; p. 309] that each component of A_e is a locally orientable Z_2 -generalized $(n - k \cdot (n - p + 1))$ -manifold for some integer k . That $k = 2$ follows from the fact (1.3) that $A_f - Y$ is a $(2p - n - 2)$ -manifold, where $\dim Y < \dim A_f$.

Suppose that f is not necessarily proper, but that $\dim A_f \leq p - 2$; then (1.3) $n - p = 1, 3$, or 7 . For each $x \in A_f$, there are (1.14) connected open neighborhoods U of x and V of $f(x)$ such that $f|_U : U \rightarrow V$ is proper, so that $f(U) = V$ [32; (2.5)]. We now apply (4.8) to $f|_U$ to deduce the same conclusion.

For a map satisfying the above hypotheses except for differentiability, A_f need not be a manifold (1.6). Whether it is necessarily a manifold under the C^n hypothesis is unknown to the authors.

4.9. Remark. *Let $f : M^n \rightarrow N^p$ be an onto C^n singular fibering with $n > p$, let G be a field, let M^n and N^p be compact G -cohomology spheres, and let N^p be simply connected. Then either A_f is empty or it is a G -cohomology (Alexander-Spanier) $(2p - n - 2)$ -sphere.*

Proof. By (1.3) $n - p = 1, 3$, or 7 , and by (4.6) $N^p - f(A_f)$ is simply connected. As in the proof of (4.8) it follows that the fiber is connected, and so (1.3) is a homotopy sphere. The proof of [11; p. 306, (2.1)] with Z_2 replaced by any field G implies that A_f is a G -cohomology $(n - k \cdot (n - p + 1))$ -sphere

for some integer k . From the Lefschetz Duality Theorem [27; p. 297] and the cohomology sequence of (M^n, A_f) (resp., $(N^p, f(A_f))$) $M^n - A_f$ (resp., $N^p - f(A_f)$) is an homology $((k \cdot (n - p + 1)) - 1)$ -sphere (resp., $((k - 1) \cdot (n - p + 1))$ -sphere)). From the Gysin sequence [27; p. 483] it follows that $k = 2$.

Under certain hypotheses P. L. Antonelli actually obtains the global structure of f in (4.9). Specifically, let $f : S^n \rightarrow S^p$, $n > p > 1$, be a singular fibering with fiber S^{n-p} , let A_f and $f(A_f)$ be locally flat simply connected q -manifolds, let $S^p - f(A_f)$ be connected, and let $\pi_1(S^p - f(A_f))$ act simply on $H_*(S^{n-p}; Z)$. Assume the Poincaré conjecture in dimensions 3 and 4. Then Antonelli proves [1; p. 38] that f is topologically equivalent to the $(q + 1)$ -suspension of a map ψ from 1.1 (i).

REFERENCES

- [1] P. L. ANTONELLI, *Structure Theory for Montgomery-Samelson Fiberings between Manifolds*, Ph.D. dissertation, Syracuse University, 1967.
- [2] G. E. BREDON, *Sheaf Theory*, McGraw-Hill, New York, 1967.
- [3] E. J. BRODY, On the fibered spaces of Seifert, *Quart. J. Math. Oxford*, Ser. (2), **13** (1962) 161-171.
- [4] P. T. CHURCH, Differentiable maps with non-negative Jacobian, *J. Math. Mech.*, **16** (1967) 703-708.
- [5] ———, Differentiable monotone maps on manifolds, *Trans. Amer. Math. Soc.*, (to appear).
- [6] ———, Differentiable open maps on manifolds, *Trans. Amer. Math. Soc.*, **109** (1963) 87-100.
- [7] ———, Factorization of differentiable maps with branch set dimension at most $n - 3$, *Trans. Amer. Math. Soc.*, **115** (1965) 370-387.
- [8] ———, On points of Jacobian rank k , *Trans. Amer. Math. Soc.* **110** (1964) 413-423.
- [9] ———, On points of Jacobian rank k . II, *Proc. Amer. Math. Soc.*, **16** (1965) 1035-1038.
- [10] ———, & E. HEMMINGSEN, Light open maps on n -manifolds. II, *Duke Math. J.*, **28** (1961) 607-624.
- [11] P. E. CONNER & E. DYER, On singular fiberings by spheres, *Mich. Math. J.*, **6** (1959) 303-311.
- [12] S. EILENBERG & N. STEENROD, *Foundations of Algebraic Topology*, Princeton University Press, Princeton, 1952.
- [13] F. HIRZEBRUCH, *Topological Methods in Algebraic Geometry*, Springer-Verlag, New York, 1966.
- [14] S. T. HU, *Homotopy Theory*, Academic Press, New York, 1959.
- [15] ———, On fiberings with singularities, *Mich. Math. J.*, **6** (1959) 131-149.
- [16] W. HUREWICZ & H. WALLMAN, *Dimension Theory*, 2nd ed., Princeton University Press, Princeton, 1948.
- [17] D. HUSEMOLLER, *Fibre Bundles*, McGraw-Hill, New York, 1966.
- [18] S. KOBAYASHI & K. NOMIZU, *Foundations of Differential Geometry, Volume I*, Interscience, New York, 1963.
- [19] M. MAHOWALD, On singular fiberings, *Canadian J. Math.*, **15** (1963) 214-219.
- [20] D. MONTGOMERY & H. SAMELSON, Fiberings with singularities, *Duke Math. J.*, **13** (1946) 15-56.
- [21] J. R. MUNKRES, *Elementary Differential Topology*, Princeton University Press, Princeton, 1963.
- [22] P. ORLIK, E. VOGT, & H. ZIESCHANG, Zur Topologie gefaseter dreidimensionaler Mannigfaltigkeiten, *Topology*, **6** (1967) 49-64.

- [23] F. A. RAYMOND, Separation and union theorems for generalized manifolds with boundary, *Mich. Math. J.*, **7** (1960) 7–21.
- [24] R. H. ROSEN, E^4 is the cartesian product of a totally non-euclidean space and E^1 , *Ann. of Math.*, **73** (1961) 349–361.
- [25] A. SARD, The measure of the critical values of differentiable maps, *Bull. Amer. Math. Soc.*, **48** (1942) 883–890.
- [26] H. SEIFERT, Topologie dreidimensionaler gefaster Räume, *Acta Math.*, **60** (1933), 147–238.
- [27] E. H. SPANIER, *Algebraic Topology*, McGraw-Hill, New York, 1966.
- [28] N. STEENROD, *The Topology of Fiber Bundles*, Princeton University Press, Princeton, 1951.
- [29] S. STERNBERG, *Lectures on Differential Geometry*, Prentice-Hall, Englewood Cliffs, 1964.
- [30] R. THOM, Quelques propriétés globales des variétés différentiables, *Comment. Math. Helv.*, **28**, (1954) 17–86.
- [31] M. C. THORNTON, Singularly fibered manifolds, *Ill. J. Math.*, **11** (1967) 189–201.
- [32] J. G. TIMOURIAN, Fiber bundles with discrete singular set, *J. Math. Mech.*, **18** (1968) 61–70.
- [33] ———, *Singular Fiberings of Manifolds*, Ph.D. dissertation, Syracuse University, 1967.
- [34] C. J. TITUS & G. S. YOUNG, A Jacobian condition for interiority, *Mich. Math. J.*, **1** (1952) 89–94.
- [35] J. H. C. WHITEHEAD, Simplicial spaces, nuclei, and m -groups, *Proc. London Math. Soc.*, Series (2), **45** (1939) 243–327.
- [36] G. T. WHYBURN, *Analytic Topology*, 2nd ed., Amer. Math. Soc. Colloq. Publ. Vol. 28, Amer. Math. Soc., Providence, 1963.

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