*State Diagrams for Hilbert Space Operators**

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1. Introduction. A large amount of information has been recently accumulated for both (i) specific types of Hilbert space operators, especially the physically important ones (*e.g.*, symmetric operators), and (ii) general mathematical state diagrams (*i.e.*, existence, uniqueness, and *a priori* estimate behavior for abstract operators and their adjoints). It is the purpose of this paper to fill a gap by providing (ii) for (i); specifically, we derive (new) state diagrams for the symmetric, closed symmetric, essentially self-adjoint, maximal symmetric, normal, Cayley Transforms, Fredholm, normally solvable, subnormal, hyponormal, seminormal, normaloid, and related operator classes (Theorem 3.1).

In Section 2 we will establish the notation, definitions, and some preliminary facts. In Section 3 the state diagrams are restricted by operator-theoretic methods; the remainder of the proof of the state diagrams (*i.e.*, that they have been minimalized) is completed in Section 4 by the exhibition of examples affirming the existence of all remaining states.

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2. Preliminaries. Throughout this paper T is always a linear operator with domain $D(T)$ a dense subspace of a normed linear space X and range $R(T)$ in a normed linear space Y ; often we will take X and Y complete (*i.e.*, Banach spaces), and moreover, usually as $X = Y = H$, a Hilbert space. We shall make the usual assumption that the scalars are complex, although much of the following holds for real scalars also. $N(T)$ will denote the kernel of T , braces $\{\cdot\}$ will denote a class (*e.g.*, of operators), the symbol $\subseteq$ means restriction, and $\subset$ means proper restriction. The (algebraic) dual operator $T' : D(T') \subset Y' \rightarrow X'$ mapping continuous linear functionals on Y into continuous linear functionals on X will be denoted as T' , while the usual Hilbert space adjoint (conjugate) dual operator will be denoted as T^* .

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For any T consider the three mutually exclusive range possibilities:

- I. $R(T) = Y$;
- II. $R(T)$ properly dense in Y ;
- III. $R(T)$ not dense in Y .

To each of these “range states” now subdivide by the three mutually exclusive “inverse state” possibilities:

- 1. T^{-1} continuous;
- 2. T^{-1} not continuous, but T is 1 – 1;
- 3. T does not have an inverse.

Then any given T is in one of nine *states*, e.g., $T \in I_1$ means that T is surjective and possesses a bounded inverse. Similarly T' is in one of the nine states; plotting T horizontally and T' vertically, there are conceivably 81 possible (T, T') state combinations, e.g., $T \in I_1 I_1$ means that both T and T' are surjective with bounded inverses. The power of the *state diagram* shown in Fig. 1 is that only 16 states are possible; for closed T in a Hilbert space, only 7 are possible. The state diagram thus serves as a useful summary of all existence, uniqueness, and *a priori* estimate (i.e., T^{-1} bounded) inter-implications for an operator and its dual.

This systematic approach started in Taylor and Halberg [25], with the corresponding (first) state diagram for everywhere defined bounded operators on a normed linear space appearing in [24]. This diagram was extended to unbounded (and, in particular, closed) densely defined operators in Goldberg [7] and it is this diagram that is shown in Fig. 1; a complete exposition is given in [8]. These results have been extended to general locally convex Hausdorff spaces with various topologies in Krishnamurthy [18] and Krishnamurthy and Loustaunau [19]. The special case of (pre)-compact operators was investigated by Thorpe [26].

Since we will be principally concerned with Hilbert space and the usual Hilbert space adjoint (conjugate) dual operator T^* , the following lemma is important.

Lemma 2.1. *Let $X = Y = H$. Then T^* exists iff T' exists iff $D(T)$ is dense; $D(T^*) = H$ iff $D(T') = H$ iff T is bounded; state $T^* =$ state T' .*

We omit the proof, most of which is available in standard texts. At this point we should remark that (in this paper) T may be bounded without $D(T) = H$ (although we always assume $D(T)$ dense); although somewhat pathological, this case occurs in perturbation questions (e.g., $T = T_1 - T_1$, T_1 any closed unbounded operator) and cannot be ignored, and in fact sometimes gives rise to certain difficulties, e.g. concerning preservation of the essential spectrum under perturbation, etc. In this paper it will create no special difficulties, however.

Concerning the second duals, we note the following well-known facts.

FIGURE 1. T (linear) state diagram.

Y : Can't occur if Y is complete

X : Can't occur if X is complete, T closed

$X - r$: Can't occur if X is reflexive, T closed

Lemma 2.2. T''' exists if Y is reflexive and T is closeable; in particular, for $X = Y = H$, T''' and T^{**} exist for T symmetric, bounded, closed, or with numerical range $W(T)$ not the whole complex plane.

Hereafter we will denote the (smallest) closed extension of an operator by T_* .

For $X = Y = H$, we recall the definitions of the following symmetric operator subclasses.

$\{s\}$: $T \subseteq T^*$; symmetric.

$\{e.s.a.\}$: $T \in \{s\}$, $T^* = T^{**}$; essentially self-adjoint.

$\{cl.s.\}$: $T \in \{s\}$, T closed; closed symmetric.

$\{m.s.\}$: $T \in \{s\}$, T has no proper symmetric extensions; maximal symmetric.

$\{s.a.\}$: $T = T^*$; self-adjoint.

While establishing this notation, we should observe that some authors use $\{s\}$ and $\{s.a.\}$ interchangeably, a procedure causing no trouble only for bounded everywhere defined operators, when all five classes above coincide; in general, the situation is of course $\{s.a.\} \subset \{m.s.\} \subset \{cl.s.\} \subset \{s\}$, $\{s.a.\} \subset \{e.s.a.\} \subset$

$\{s\}$. Sometimes “strictly self-adjoint” is used for $\{s.a.\}$ to preclude any confusion. Also sometimes “formally self-adjoint” is used, usually to connote that some particular Green’s identity holds, most often to mean that T has a restriction $T_0 \in \{s\}$, often with $D(T_0) = C_0^\infty$, the infinitely differentiable functions of compact support in the domain over which the function space is defined.

We will also consider

$$\{n\} : T \text{ closed, } T^*T = TT^*; \text{ normal.}$$

One has the isometric ($\|Tx\| = \|x\|$ on $D(T)$) special case of $\{n\}$, namely: T isometric, $D(T) = R(T) = H$; unitary.

For the above operator classes, Fig. 2 shows the class intersections involved. Fig. 2 follows from the observation that $\{s\} \cap \{\text{bounded}\} \subset \{e.s.a.\}$, which is easily seen, plus the closed graph theorem and other elementary considerations. The $D(T) = H$ rectangle means that for T falling within the confines of the diagram (i.e., $T \in \{s\}$, bounded, or closed), $D(T) = H$ there and only there. Examples in Section 4 will show that all subrectangles shown are non-empty; the value of Fig. 2 is in the subrectangles which cannot appear.

FIGURE 2. Class diagram for (symmetric) Hilbert space operators.

In connection with $\{c.l.s.\}$, we note the treatment via the Cayley Transform $C_T = (T - i)(T + i)^{-1}$ for $T \in \{c.l.s.\}$ and H complex; then $T \in \{s.a.\}$ iff C_T is unitary iff T has defect indices $\text{codim } R(T \pm i)$ both 0. Otherwise either $T \in \{m.s.\}$ properly (i.e., $T \notin \{s.a.\}$) iff just one defect index is 0, or else both defect indices are not 0, in which case two things can happen: $T \subset T_*$, $T_* \in \{s.a.\}$ iff the indices are equal, $T \subset T_*$, $T_* \in \{m.s.\}$ properly iff the indices are unequal. In all cases $C_T \in \{\text{closed, isometric}\}$, $T \pm i \in \{s.f.\}$, (this class is to be defined immediately below), and state $C_T = \text{state } T - i = I_1 I_1$ or $III_1 I_1$.

Other operator classes, well-known for their importance in applications, are:

- $\{\text{n.s.}\} : T \text{ closed, } R(T) \text{ closed; normally solvable.}$
 $\{\text{s.F.}\} : T \in \{\text{n.s.}\}, \text{ either } \dim N(T) < \infty \text{ or } \text{codim } R(T) < \infty; \text{ semi-Fredholm.}$
 $\{\text{F}\} : T \in \{\text{n.s.}\}, \text{ both } \dim N(T) < \infty \text{ and } \text{codim } R(T) < \infty; \text{ Fredholm.}$

Here we assume X and Y to be Banach spaces, since applications occur in non-Hilbert spaces also; there is no particular connection between $\{\text{n}\}$ and $\{\text{n.s.}\}$. An important characteristic of $\{\text{s.F.}\}$ is that index $(T) = \dim N(T) - \text{codim } R(T)$ is then defined and a rather complete dual and stability theory is possible (*e.g.*, see [16]); the subclass $\{\text{F}\}$ is distinguished by index $(T) < \infty$, and for integral operators was the beginning of the dual theory, sometimes known as the Fredholm Alternative (*e.g.*, see [21]). Some of the dual and index theory extends to $\{\text{n.s.}\}$ (*e.g.*, see [8]) and beyond (*e.g.*, see [10, 11]).

The theory of normal operators $\{\text{n}\}$ has been extended to several larger classes recently, and it seems appropriate to obtain their state diagrams in this paper also. In the following definitions it is assumed that all operators are bounded with domain H ; the definitions are, by now, relatively standard.

- $\{\text{q.n.}\} : T(T^*T) = (T^*T)T; \text{ quasi-normal.}$
 $\{\text{sub.n.}\} : T \text{ has a normal extension on a larger } H\text{-space; subnormal.}$
 $\{\text{h.n.}\} : \text{the commutator } [T^*, T] \equiv T^*T - TT^* \geq 0; \text{ hyponormal.}$
 $\{\text{co.sub.n.}\} : T^* \in \{\text{sub.n.}\}; \text{ cosubnormal.}$
 $\{\text{co.h.n.}\} : [T^*, T] \leq 0; \text{ cohyponormal.}$
 $\{\text{s.n.}\} : \text{either } [T^*, T] \geq 0 \text{ or } [T^*, T] \leq 0; \text{ seminormal.}$
 $\{\text{l.normaloid}\} : \text{each } p(T) = \alpha T + \beta \in \{\text{normaloid}\}; \text{ linearly normaloid.}$
 $\{\text{normaloid}\} : \text{numerical radius } w(T) = \|T\|.$
 $\{\text{convexoid}\} : \text{the convex hull of the spectrum } CH\sigma(T) = \text{cl } W(T), \text{ the closure of the numerical range.}$
 $\{\text{spectraloid}\} : \text{spectral radius } r(T) = w(T).$

These operator classes all generalize one or more properties of $\{\text{n}\}$ (here considered bounded with domain H); one can deduce the following relationships.

$$\begin{aligned}
 (2.1) \quad \{\text{n}\} &\subset \{\text{q.n.}\} \subset \{\text{sub.n.}\} = \{\text{co.sub.n.}\}^* \subset \{\text{h.n.}\} = \{\text{co.h.n.}\}^* \\
 &\subset \{\text{s.n.}\} = \{\text{h.n.}\} \cup \{\text{co.h.n.}\} \subset \{\text{l.normaloid}\} \\
 &\subset \{\text{normaloid}\} \cap \{\text{convexoid}\} \\
 &\subset \{\text{normaloid}\} \cup \{\text{convexoid}\} \subset \{\text{spectraloid}\}.
 \end{aligned}$$

Showing some of the inclusions to be proper is nontrivial. Definitions of some of the symbols involved are herewith supplied. $W(T) =$ all complex numbers (Tx, x) , $x \in H$, $\|x\| = 1$. $w(T) = \sup |\lambda|$, $\lambda \in W(T)$. $\sigma(T) =$ all complex numbers λ such that $T - \lambda \notin \text{state } I_1$. $r(T) = \sup |\lambda|$, $\lambda \in \sigma(T)$. One always has $r(T) \leq w(T) \leq \|T\|$ and $CH\sigma(T) \subseteq \text{cl } W(T)$. It can be shown that $T \in \{\text{normaloid}\}$ iff $r(T) = w(T) = \|T\|$ iff $\|T^n\| = \|T\|^n$ for all $n \geq 1$.

The class {q.n.} was introduced in Brown [3], {sub.n.} in Halmos [12], {h.n.} in Berberian [1], {s.n.} in Halmos [13], {l.normaloid} in Hildebrandt [15], {normaloid} by Wintner [30]; for more information, the other above definitions, and related important work, see Halmos [14] and Putnam [20] and the references therein, especially Stampfli [22, 23] and Hildebrandt [15].

In a somewhat different direction (unbounded operators) the following operators have been of interest.

$$\{\text{f.n.}\} : T \text{ closed, } D(T) \subseteq D(T^*), \|Tx\| = \|T^*x\| \text{ on } D(T); \text{ formally normal.}$$

These operators include the essential ingredients of both {cl.s.} and {n}, as seen from

$$\begin{aligned} (2.2) \quad \{\text{s.a.}\} &= \{\text{s}\} \cap \{D(T) = D(T^*)\} \\ &= \{\text{m.s.}\} \cap \{\text{n}\} \subset \{\text{cl.s.}\} \cup \{\text{n}\} \subset \{\text{f.n.}\}. \end{aligned}$$

Unlike {sub.n.}, there exist $T \in \{\text{f.n.}\}$ which do not possess a normal extension on a larger Hilbert space; see Biriuk and Coddington [2], Coddington [5].

3. State diagrams. In developing state diagrams, one is faced with two limitations. First, some of the desired examples can be elusive; for this reason, in Section 4 where the proof of Theorem 3.1 is completed by supplying examples, we freely use (several times) simple operators. In compensation, however, we list a few additional more interesting (unbounded) examples. Second, there is the question of “proper” state diagrams (*e.g.*, what exactly is the state diagram of $T \in \{\text{f.n.}\}$, $T \notin \{\text{cl.s.}\}$); when one observes the large number of proper subclass state diagrams that could be considered, it is clear that such a program could be pursued almost indefinitely, and certainly beyond a point of diminishing returns. Indeed, often a significant effort has been involved in the past in proving that a whole subclass is proper (*e.g.*, giving an example $T \in \{\text{h.n.}\}$, $T \notin \{\text{sub.n.}\}$, see [14]); the proper state diagram procedure requires that this be done for each possible state of the proper subclass. Since it seems, roughly, that not very many states are theoretically eliminated from the proper subdiagrams, the procedure thus consists largely of constructing many somewhat complicated examples. As a compromise, however, it does seem worth-while to consider properness with respect to the most important subclass {s.a.} deleted; thus hereafter “p.” is to denote $T \notin \{\text{s.a.}\}$.

For expository brevity it is convenient to list some diagrams which will occur frequently.

$$3d. \quad I_1 I_1, II_2 II_2, III_3 III_3.$$

$$4d. \quad I_1 I_1, I_3 III_1, III_1 I_3, III_3 III_3.$$

$$7d. \quad I_1 I_1, I_3 III_1, II_2 II_2, II_3 III_2, III_1 I_3, III_2 II_3, III_3 III_3.$$

$$7_{\bullet} . \quad I_1 I_1 , II_1 I_1 , II_2 II_2 , III_1 I_3 , III_2 II_2 , III_2 III_3 , III_3 III_3 .$$

$$5_1 . \quad I_1 I_1 , II_2 II_2 , III_1 I_3 , III_2 II_3 , III_3 III_3 .$$

$$5_u . \quad I_1 I_1 , I_3 III_1 , II_2 II_2 , II_3 III_2 , III_3 III_3 .$$

The d connotes diagonal, s symmetric {s}, l lower-triangular, and u upper-triangular, as will be seen.

Theorem 3.1. *The following state diagrams are obtained.*

$$\{s\} : 7_{\bullet} .$$

$$\{e.s.a\} : 3d + II_1 I_1 + III_2 III_3 .$$

$$\{cl.s.\}, \{q.n.\}, \{sub.n.\}, \{h.n.\}, \{f.n.\} : 5_1 .$$

$$\{m.s.\} : 5_1 - III_1 I_3 .$$

$$\{s.a.\}, \{n\} : 3d .$$

$$\{cl. isometric\}, \{Cayley Transforms\} : I_1 I_1 , III_1 I_3 .$$

$$\{n.s.\}, \{s.F.\}, \{F\} : 4d .$$

$$\{co.sub.n.\}, \{co.h.n.\} : 5_u .$$

$$\{s.n.\}, \{l.normaloid\}, \{normaloid\}, \{convexoid\}, \{spectraloid\} : 7d .$$

For $\{p.s.\}, \{p.e.s.a.\}, \{p.cl.s.\}, \{p.m.s.\}$, delete $I_1 I_1$ from each parent class state diagram; also (clearly) all of the other state diagrams remain the same after deleting $\{s.a.\}$ from the operator classes.

Fig. 3 shows the state diagram for $T \in \{s\}$, 7d was shown on Fig. 1, and the other smaller state diagrams are easily displayed.

Proof. Although one can proceed more directly, we first wish to make some useful tautological, and other, observations.

Lemma 3.2. *For all linear T ,*

$$(i) \quad \{state\ T = state\ T'\} = 3d .$$

$$(ii) \quad \{state\ T = state\ T''\} = 7d .$$

Proof. Part (i) is immediate from Fig. 1, and (ii) follows from an inspection of the diagrams for T and T' . Part (i), which characterizes operator classes possessing *diagonal* state diagrams, does not depend on closedness, completeness, reflexivity, etc; it even remains true for the LM and LS state diagrams of [19] for general locally convex Hausdorff spaces (part (ii) does not). By part (ii), just knowing that $T = T^{**}$ for T closed in a Hilbert space is sufficient alone to restrict T to this *diagonal-symmetric* diagram; examples (e.g., from Section 4) show that the diagram is fully achieved by T closed, $X = Y = H$. More generally, by Lemma 2.2, T'' exists when Y is reflexive and T is closeable.

FIGURE 3. T (symmetric) state diagram.

Lemma 3.3. *Let T be closed, X reflexive, Y complete, $N(T) = 0$ iff $N(T') = 0$. Then T is restricted to diagram 3d.*

Proof. Clear from Lemma 3.2(i). Since $N(T) = N(T^*)$ for $T \in \{n\}$, this shows that $\{n\}$ is restricted to diagram 3d; for bounded everywhere defined $\{n\}$, this was proved in [24], and as in the bounded case, it shows that normal operators have no residual spectrum. Obviously $\{s.a.\}$ is restricted to diagram (3d), as was observed in [8]. Easy examples (*e.g.*, in Section 4) show that these diagrams are fully achieved.

To complete our consideration of not necessarily Hilbert space operators, we next consider $T \in \{n.s.\}$, X and Y Banach spaces. $T' \in \{n.s.\}$ also, (*i.e.*, $R(T')$ is closed), and by applying to both T and T' the fact that $T \in \{n.s.\}$ implies a positive minimum modulus (see [8]) which in turn requires T^{-1} continuous whenever T^{-1} exists, we arrive at the *diagonal-symmetric* 4d diagram restricting $\{n.s.\}$; examples of section 4 will show that $\{F\}$ attains the full diagram. Note that $T \in \{n.s.\}$, $T \notin \{s.F.\}$ occurs only at $III_3 III_3$.

That $\{cl.isometric\}$ is restricted as stated is only incidentally observed because of the important subclass $\{Cayley\ Transforms\}$ when $X = Y = H$.

Turning now to the Hilbert space operators, we first consider $T \in \{s\}$, for which we have the well-known behavior

$$(3.1) \quad T \subseteq T_c \equiv T^{**} \subseteq T^*,$$

from which we have the consequence

$$(3.2) \quad R(T) \subseteq R(T^{**}) \subseteq R(T^*)$$

and all three operators have dense domains. Using these relations, Lemma 2.1 and a consideration of the inverses of the three operators, it follows that $\{s\}$ is limited to the diagram shown in Fig. 3.

A symmetric transformation has the property that T and T_* have the same adjoint. This extends somewhat in the sense that for $T \in \{s\}$ possessing any self-adjoint extension T_* , then state $T = \text{state } T_*$ and state $T^* = \text{state } T_*$ whenever possible, *i.e.*,

Lemma 3.4. *Let $T \in \{s\}$, $T_* \in \{s.a.\}$, $T \subseteq T_*$, $T \notin III_1I_3$ or III_2II_3 . Then state $T^* = \text{state } T_*$.*

Proof. After elementary considerations it remains to consider only $T \in \{\text{cl.s.}\}$, $T \notin \{\text{m.s.}\}$, in which case we have $T \subset T_* \subset T^*$. From this situation and the state diagram for $\{s\}$, $T \in II_2II_2$ restricts T_* to II_2II_2 ; for $T \in III_3$, T_* cannot be $1 - 1$, so $T_* \in III_3$. For the two cases not included in Lemma 3.4, it can be seen (*e.g.*, by varying the boundary conditions of T_{24} of Section 4) that for $T \in III_1$, T_* has access to all three $\{s.a.\}$ states; $T \in III_2$ would have T_* restricted to II_2 or III_3 , however.

Lemma 3.4 shows clearly that $\{e.s.a.\}$ is restricted as stated in Theorem 3.1. For $T \in \{s\}$, $T \in II_1I_1$, T is necessarily $\{p.e.s.a.\}$ because $T_* \in I_1I_1$; for $T \in \{e.s.a.\}$, $T \in III_2III_3$ is clearly $\{p.e.s.a.\}$ necessarily.

By Fig. 1 and Fig. 3, $\{\text{cl.s.}\}$ is clearly restricted to 5_1 . However, $\{\text{m.s.}\}$ cannot occur in III_1I_3 . This can be seen in two ways: first, 0 here is a *regular point* (in the residual spectrum), which by Calkin [4], (see also Krein and Krasnosel'skii [17]) requires that the defect indices be equal; alternately, the pair (T, T^*) is *potentially solvable* (*i.e.*, $T \subset T^*$, T^{-1} bounded, T^* surjective) and since T^* is closed and the setting is a Hilbert space, by Visik [27], (*e.g.*, see [8]), the pair (T, T^*) is *solvable* (*i.e.*, there exists T_* such that $T \subseteq T_* \subseteq T^*$, T_*^{-1} bounded, T_* surjective. Clearly T_* is a self-adjoint extension of T).

Considering now the normal-like operators, since $T \in \{\text{h.n.}\}$ is equivalent to $\|T^*x\| \geq \|Tx\|$, it follows easily that $\{\text{h.n.}\}$, and therefore $\{\text{q.n.}\}$ and $\{\text{sub.n.}\}$ by (2.1), are restricted to 5_1 ; examples will show $\{\text{q.n.}\}$, and therefore $\{\text{sub.n.}\}$ and $\{\text{h.n.}\}$, attain this diagram. The adjoints of $\{\text{sub.n.}\}$, namely $\{\text{co.sub.n.}\}$, are thus restricted to and attain 5_u ; by (2.1) therefore $\{\text{s.n.}\}$ and the larger classes $\{\text{l.normaloid}\}$, $\{\text{normaloid}\}$, $\{\text{convexoid}\}$, and $\{\text{spectraloid}\}$ all obtain diagram 7d. . Since $T^* 1 - 1$ implies $T 1 - 1$ for $T \in \{\text{f.n.}\}$, $\{\text{f.n.}\}$ is restricted to 5_1 also, and attains 5_1 independent of $\{s.a.\}$ from examples for $\{p.\text{cl.s.}\}$ and $\{p.n.\}$, due to (2.2). That I_1I_1 must be deleted from $\{p.s.\}$, and therefore from $\{p.e.s.a.\}$, $\{p.\text{cl.s.}\}$, and $\{p.m.s.\}$, is clear; that it is sufficient to delete only I_1I_1 will be shown by examples, as will $\{p.n.\}$ having diagram 3d. . Clearly then all the other diagrams remain the same without the subclass $\{s.a.\}$.

4. Examples. The following examples will be used to complete the proof of Theorem 3.1 and the assertions of Fig. 2; accordingly the properties relevant for these purposes are listed with each example, even when obvious. Properties shown in parentheses ($\cdot$) are not necessary to this end, but are stated to distinguish certain examples or for informative reasons. All Hilbert spaces involved are infinite dimensional.

Examples T_{29} and T_{39} were kindly supplied by J. Weidmann and T. Kato, respectively; they show {c.l.s.} and {e.s.a.} operators possessing 0 in the specific part III_2 of their residual spectrums. Examples similar to T_{29} have been useful in other contexts, (see Weidmann [29], also [6]).

EXAMPLES

- T_1 : $T = I$, $D(T) = H$, $I_1 I_1$, {s.a.}, unitary, bounded.
 T_2 : $T = I$, $D(T)$ properly dense in H , $II_1 I_1$, {p.e.s.a.}.
 T_3 : $T = 0$, $D(T) = H$, $III_3 III_3$, {s.a.}, ({n.s.}, not {s.F.}).
 T_4 : $T = 0$, $D(T)$ properly dense in H , $III_3 III_3$, {p.e.s.a.}.
 T_5 : $2T_1$, $(I_1 I_1)$, {s.a.}, not unitary.
 T_6 : iT_2 , bounded, not {s}, not closed.
 T_7 : unilateral right shift, $D(T) = l_2$, $T(e_n) = e_{n+1}$, $n \geq 1$, $III_1 I_3$, {q.n.}, Cayley Transform, {F}.
 T_8 : left shift, $D(T) = l_2$, $T(e_n) = e_{n-1}$, $n \geq 2$, $T(e_1) = 0$, {F}, $(T_s = T^*)$.
 T_9 : bilateral shift, $D(T) = l_2(\pm)$, $T(e_n) = e_{n+1}$, $n = 0, \pm 1, \pm 2, \dots$, unitary, not {s.a.}.
 T_{10} : C_{T_1} , $I_1 I_1$, Cayley Transform.
 T_{11} : $T(e_n) = n^{-1}e_n$, $D(T) = l_2$, $II_2 II_2$, {s.a.}.
 T_{12} : T_{11} restricted to $M_1 = \{\text{all finite linear combinations of } \{e_n\}\}$, $II_2 II_2$, {p.e.s.a.}.
 T_{13} : $T_{11} T_7 T_8$, $III_3 III_3$, {s.a.}, {F}.
 T_{14} : coord. op. in q.m., $H = L_2(-\infty, \infty)$, $D(T) = \{u \in H, t \cdot u(t) \in H\}$, $Tu = t \cdot u(t)$, $II_2 II_2$, {s.a.}, (unbounded, 0th order).
 T_{15} : mom. op. in q.m., $H = L_2(-\infty, \infty)$, $D(T) = \{u \in H, u \text{ a.c.}, u' \in H\}$, $Tu = -iu'(t)$, $II_2 II_2$, {s.a.}, (unbounded, 1st order).
 T_{16} : T_{14} restricted to $M_2 = \{u \in C^\infty, t^n u^{(m)}(t) \rightarrow 0 \text{ as } |t| \rightarrow \infty, m, n \geq 0\}$, $II_2 II_2$, {p.e.s.a.}, (unbounded, 0th order).
 T_{17} : T_{15} restricted to M_2 , $II_2 II_2$, {p.e.s.a.}, (unbounded, 1st order).
 T_{18} : Heisenberg c. op., $H = l_2$, $D(T) = M_1$, $T(e_n) = 2^{-1/2}[(n-1)^{1/2}e_{n-1} + n^{1/2}e_{n+1}]$, $n \geq 1$, $II_2 II_2$, {p.e.s.a.}, (unbounded matrix operator).
 T_{19} : Heisenberg m. op., $H = l_2$, $D(T) = M_1$, $T(e_n) = -i2^{-1/2}[(n-1)^{1/2}e_{n-1} - n^{1/2}e_{n+1}]$, $n \geq 1$, $II_2 II_2$, {p.e.s.a.}.
 T_{20} : n -particle operator (nonrelativistic, noncreative, nondestructive, Coulomb potential),

$$H = L_2(R^{3n}), \quad D(T) = C_0^\infty(R^{3n}), \quad Tu = -\Delta u - \sum_{k=1}^n c r_k^{-1} u + \sum_{1 \leq k \leq l} r_{kl}^{-1} u,$$

Δ the $3n$ -dimensional Laplacian, r_k the 3-dim. radii, r_{kl} the interaction

- relative radii, $n \geq 1$, c any real constant, II_2II_2 , {p.e.s.a.}, (unbounded, 2nd order).
- T_{21} : Schrödinger op., T = closure of T_{20} (for $n = c = 1$, the representation of a (hydrogen) atom of one electron and charge 1 on the nucleus), II_2II_2 , {s.a.}, (unbounded, 2nd order).
- T_{22} : Dirichlet op., $H = L_2(0, 1)$, $D(T) = \{u \in H, u' \text{ a.c.}, u'' \in H, u(0) = u(1) = 0\}$, $Tu = -u''(t)$, I_1I_1 , {s.a.}, (unbounded).
- T_{23} : T_{22} restricted to $C^2(0, 1)$, II_1I_1 , {p.e.s.a.}, (unbounded).
- T_{24} : T_{22} restricted to $D(T_{22}) \cap \{u'(0) = u'(1) = 0\}$, III_1I_3 , {p.c.l.s.}.
- T_{25} : $T_{22} - \pi^2$, III_3III_3 , {s.a.}, (unbounded).
- T_{26} : $T_{23} - \pi^2$, III_3III_3 , {p.e.s.a.} (unbounded).
- T_{27} : iT_{22} , (I_1I_1) , {p.n.}, closed, unbounded, not {s}, {F}.
- T_{28} : $H = L_2(0, \infty)$, $D(T) = \{u \in H, u \text{ a.c.}, u' \in H, u(0) = 0\}$, $Tu = -iu'(t)$, II_2II_2 , {p.m.s.}.
- T_{29} : $H = L_2(0, \infty)$, $Tu = -u'' + (q - 1)u$, $q(t)$ a particular function, locally of bounded variation, $q(t) \rightarrow 0$ as $t \rightarrow \infty$, (see [29]) $D(T) = \{u \in H, u' \text{ a.c.}, Tu \in H, u(0) = u'(0) = 0\}$, III_2II_3 , {c.l.s.}.
- T_{30} : iT_{28} , (II_2II_2) , closed, unbounded, not {s}.
- T_{31} : $T_{28} \times T_{24}$, $\{III_2II_3\}$, {p.m.s.}.
- T_{32} : $T_{28} \times T_3$, $\{III_3III_3\}$, {p.m.s.}.
- T_{33} : T_{28} restricted to C_0^∞ , (II_2II_2) , {s}, unbounded, not closed.
- T_{34} : $D(T) = l_2$, $\{\lambda_n\}$ a set of points dense in a compact nonreal subset K of the complex plane, $T(e_n) = \lambda_n e_n$, $n \geq 1$, $0 \notin K$, I_1I_1 , {p.n.}.
- T_{35} : T_{34} except $0 \in K$, $0 \notin \{\lambda_n\}$, II_2II_2 , {p.n.}.
- T_{36} : T_{35} except $0 \in \{\lambda_n\}$, III_3III_3 , {p.n.}.
- T_{37} : iT_{15} , II_2II_2 , {p.n.}, (unbounded).
- T_{38} : iT_{25} , III_3III_3 , {p.n.}, (unbounded).
- T_{39} : $T = P - I$, P the projection of H onto $\{sp u_0\}$, u_0 any prechosen vector, $u_0 \in H$, $\|u_0\| = 1$, $D(T)$ any dense subspace of H such that $D(T) \cap u_0$ is empty, III_2III_3 , {p.e.s.a.}.
- T_{40} : $T_7 \times T_{11}$, III_2II_3 , {q.n.}.
- T_{41} : T_{40}^* , II_3III_2 , {co.sub.n.}.
- T_{42} : T_{25} restricted to $u \in D(T_{22})$, $u \neq c \sin \pi t$, $c \neq 0$, III_2III_3 , {p.e.s.a.}, (unbounded).

That these examples suffice for Theorem 3.1 can be seen by selecting (some alternate choices are available) as follows. {p.e.s.a.} : T_2 , T_4 , T_{12} , T_{39} . {p.m.s.} : T_{28} , T_{31} , T_{32} . {p.c.l.s.} : T_{24} , T_{29} . {s.a.} : T_1 , T_3 , T_{11} . {p.n.} : T_{34} , T_{35} , T_{36} . {Cayley Transforms} : T_7 , T_{10} . {F} : T_7 , T_8 , T_{13} , T_{27} . {q.n.} : T_7 , T_{40} . {co.sub.n.} : T_8 , T_{41} . That is, this list, plus subclass inclusions, provides the needed examples for all states listed as possible in Theorem 3.1.

For Fig. 2, one may use T_1 , T_2 , T_5 , T_6 , T_7 , T_9 , T_{22} , T_{23} , T_{24} , T_{27} , T_{28} , T_{30} , T_{33} , T_{34} to affirm the nonempty subrectangles; we omit the detailing of which goes where.

The (all unbounded) examples not yet mentioned were included either be-

cause of their interest for physics and/or to illustrate that unbounded operators can fill state diagrams. That is, $T_{14} - T_{21}$, T_{25} , T_{26} , T_{37} , T_{38} , T_{42} exhibit that {s.a.}, {p.n.}, and {p.e.s.a.} all achieve their full diagrams properly in the sense of being unbounded; to complete this assertion, append T_{22} , T_{23} , T_{27} to this list. Examples T_{31} and T_{42} were included as alternates to T_{29} and T_{39} .

We conclude with justifications for the non-obvious properties of the examples. It is well-known (*e.g.*, see [21]) that T_7 is the Cayley Transform of the elementary symmetric transformation. It is easy to see that $R(T_{12}) = M_1$; similarly M_1 is an invariant subspace of T_{18} and T_{19} , so that T_{12} , T_{18} , T_{19} are in II_2II_2 . The 1 - 1 ness of T_{14} and T_{15} is immediate, and then non-continuous inverses place them in II_2II_2 ; similarly with T_{16} and T_{17} , by looking at their adjoints. All of these operators, as well as T_{20} and T_{21} , are in II_2II_2 , once it is known that 0 is in their continuous spectrum; that this is so for T_{20} in the generality stated (*e.g.*, no restriction on the charge c) is due to Weidmann [28]. That $T_{23} \in II_1I_1$ and {p.e.s.a.} follows from Lemma 3.4 and the fact that T_{22} has the well-known (first eigenvalue of the vibrating string) $\lambda_1 = \|T_{22}^{-1}\|^{-1} = \pi^2$. Restricting T_{22} more (than T_{23}) by boundary conditions and less by regularity, T_{24} is closed, {s}, and retains the bounded inverse, but by the additional boundary conditions $R(T_{24})$ is orthogonal to constants. That perturbation by a bounded operator preserves the {s.a.} and {p.e.s.a.} properties verifies T_{25} and T_{26} . For T_{27} , T_{37} , T_{38} we need to recall that $T \in \{n\}$ iff $T = A + iB$ where $A \in \{s.a.\}$, $B \in \{s.a.\}$, and A and B have commuting resolutions of the identity. T_{28} has defect indices 0 and 1 and T_{28}^* is 1 - 1, which places T_{28} in II_2II_2 . The demonstration that T_{29} is in III_2II_3 goes as follows: T_{29} is closed, {s}, and $D(T_{29}^*) = \{u \in H, u' \text{ a.c.}, Tu \in H\}$; the results of Weidmann [29] show that T_{29}^* is not 1 - 1, whereas T_{29} is clearly 1 - 1, so that T_{29} is in one of the two states III_1I_3 or III_2II_3 ; it may then be shown that T_{29}^{-1} is not continuous. Concerning T_{31} , T_{32} , T_{40} , we can use the "lattice" procedure of Goldberg and Thorp [9] for constructing examples near the upper right corner of a state diagram. If $T^1 : X_1 \rightarrow Y_1$ is in state $A_e B_e$, and $T^2 : X_2 \rightarrow Y_2$ is in state $C_e D_e$, then $T^3 = T^1 \times T^2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$, $T^3(x_1, x_2) = (T^1 x_1, T^2 x_2)$, is in state $E_e F_e$, where $E = \max(A, C)$, $e = \max(a, c)$, $F = \max(B, D)$, $f = \max(b, d)$, for any suitable product norm; that is, the direct product operator assumes the highest common state numbers. One can say more, since with the topology given by the inner products $\langle (x_{11}, x_{12}), (x_{21}, x_{22}) \rangle_3 = \langle x_{11}, x_{21} \rangle_1 + \langle x_{12}, x_{22} \rangle_2$ when $X_1 = Y_1 = H_1$, $X_2 = Y_2 = H_2$, one has $(H_1 \times H_2)^* = H_1 \times H_2$ and $(T^3)^* = ((T^1)^*, (T^2)^*)$, and one can verify easily that {s.a.}, {n}, etc., are preserved in the same manner as the state; that is, T^3 will assume the weakest possible common property. Thus T_{31} cannot have a self-adjoint extension (or else T_{28} would), and is clearly closed, {s}. T_{32} is placed in III_3III_3 by T_3 but cannot have a self-adjoint extension. T_{40} , as a product of {n} and {q.n.}, becomes only {q.n.}. The construction of {n} operators from points dense in a compact set K is standard (*e.g.*, see [24]) and then $\sigma(T) = K$; the inverse properties of T_{34} , T_{35} , and T_{36} then follow from the spectral radius theorem. $T_{39} \in III_2III_3$ because T_{39} is 1 - 1 but $T_{39}^* = P^* - I$ has eigenfunction u_0 ; T_{42} is similarly constructed.

REFERENCES

- [1] S. K. BERBERIAN, *Introduction to Hilbert space*, Oxford University Press, N. Y., 1961.
- [2] G. BIRIUK & E. A. CODDINGTON, Normal extensions of unbounded formally normal operators, *J. Math. Mech.*, **13** (1964) 617–637.
- [3] A. BROWN, On a class of operators, *Proc. Amer. Math. Soc.*, **4** (1953) 723–728.
- [4] J. W. CALKIN, Symmetric transformations in Hilbert space, *Duke J. Math.*, **7** (1940) 504–508.
- [5] E. A. CODDINGTON, Formally normal operators having no normal extensions, *Canadian J. Math.*, **17** (1965) 1030–1040.
- [6] A. GALBRAITH, E. J. MCSHANE & G. PARRISH, On the solutions of linear second-order differential equations, *Proc. Nat. Acad. Sci. U. S. A.*, **53** (1965) 247–249.
- [7] SEYMOUR GOLDBERG, Linear operators and their conjugates, *Pac. J. Math.*, (1959) 69–79.
- [8] SEYMOUR GOLDBERG, *Unbounded Linear Operators*, McGraw-Hill, N. Y., 1966.
- [9] SEYMOUR GOLDBERG & E. O. THORP, The range as range space for compact operators, *J. Reine Angew. Math.*, **211** (1962) 113–115.
- [10] K. GUSTAFSON, A perturbation lemma, *Bull. Amer. Math. Soc.*, **72** (1966) 334–338.
- [11] K. GUSTAFSON, Doubling perturbation sizes for linear operators, (to appear).
- [12] P. R. HALMOS, Normal dilations and extensions of operators, *Summa Brasil.*, **2** (1950) 125–134.
- [13] P. R. HALMOS, Commutators of operators, *Amer. J. Math.*, **74** (1952) 237–240.
- [14] P. R. HALMOS, *A Hilbert space problem book*, D. Van Nostrand, Princeton, N. J., 1967.
- [15] S. HILDEBRANDT, Ueber den numerischen Wertebereich eines Operators, *Math. Annalen*, **163** (1966) 230–247.
- [16] T. KATO, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1966.
- [17] M. KREIN & M. A. KRASNOSELSKII, Fundamental theorems on the extension of Hermitian operators and some of their applications to the theory of orthogonal polynomials and the moment problem, *Uspehi. Mat. Nauk*, **2** (1947) 60–107.
- [18] V. KRISHNAMURTHY, On the state diagram of a linear operator and its adjoint, *Math. Annalen*, **141** (1960) 153–160.
- [19] V. KRISHNAMURTHY & J. O. LOUSTAUNAU, On the state diagram of a linear operator and its adjoint in locally convex spaces I, *Math. Annalen*, **164** (1966) 176–206.
- [20] C. R. PUTNAM, *Commutation Properties of Hilbert Space Operators and Related Topics*, Springer-Verlag, Berlin, 1967.
- [21] F. RIESZ & B. SZ.-NAGY, *Functional Analysis*, Frederick Ungar, N. Y., 1955.
- [22] J. G. STAMPFLI, Hyponormal operators, *Pac. J. Math.*, **12** (1962) 1453–1458.
- [23] J. G. STAMPFLI, Hyponormal operators and spectral density, *Trans. Amer. Math. Soc.*, **117** (1965) 469–476.
- [24] A. E. TAYLOR, *Introduction to Functional Analysis*, Wiley, N. Y., 1958.
- [25] A. E. TAYLOR & C. J. A. HALBERG, JR., General theorems about a bounded linear operator and its conjugate, *J. Reine Angew. Math.*, **198** (1957) 93–111.
- [26] E. O. THORP, The relation between a compact linear operator and its conjugate, *Amer. Math. Monthly*, **66** (1959) 764–769.
- [27] M. I. VISIK, On general boundary problems for elliptic differential equations, *A.M.S. Transl. Ser. 2*, **24** (1963) 107–172.
- [28] J. WEIDMANN, On the continuous spectrum of Schrödinger operators, *Comm. Pure Appl. Math.*, **19** (1966) 107–110.
- [29] J. WEIDMANN, Zur Spectraltheorie von Sturm-Liouville-Operatoren, *Math Zeitschr.*, **98** (1967) 268–302.
- [30] A. WINTNER, Zur Theorie der beschränkten Bilinearformen, *Math Zeitschr.*, **30** (1929) 228–282.

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